## Recap: Logarithms and Indeterminate Forms

$$\boxed{\log_a(b) = c \iff b = a^c}$$

Examples:

$\log_3(9) =$

$\log_5(25) =$

$\log_{25}(5) =$

$\log_2(2^x) =$

$\log_2(4^x) =$

$\log_{10}(10^{\text{chair}}) =$

Laws of logs:

1) $\log_a(bc) = \log_a(b) + \log_a(c)$

2) $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$

3) $\log_a(b^c) = c\log_a(b)$

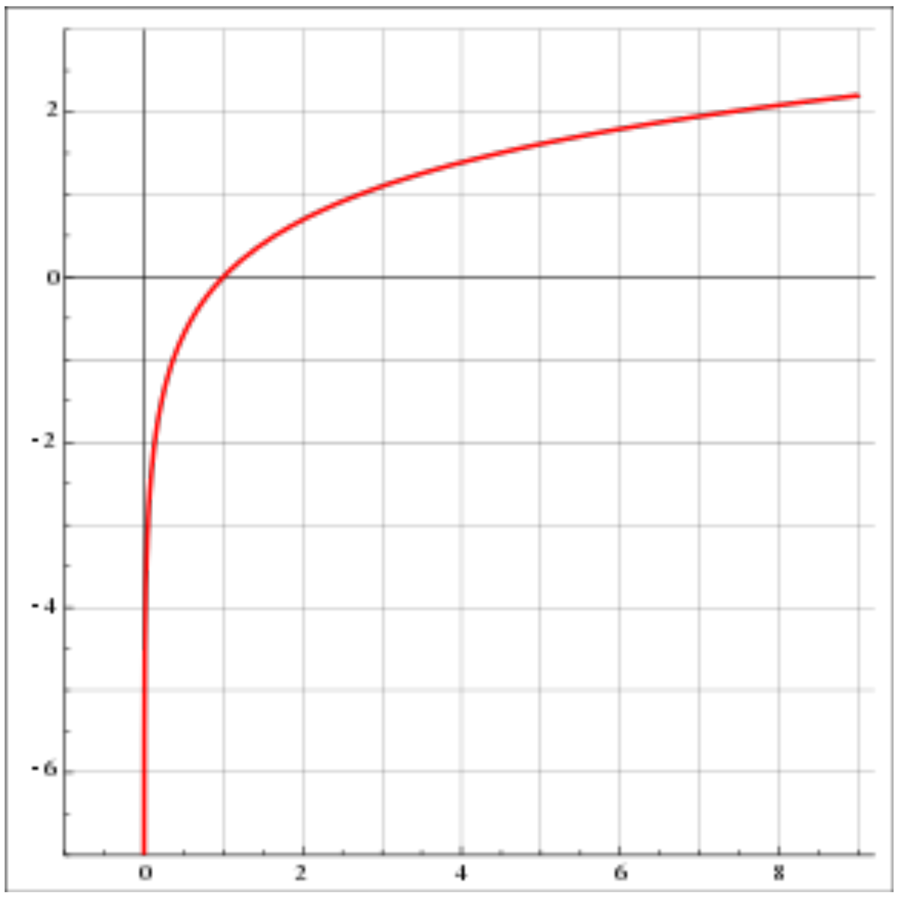
Remark:

## The Natural Logarithm:

$$\log_e(x) = \ln(x) := \int_1^x \frac{1}{t}\, dt$$

- Defined for $x > 0$. Why?

- Graph:




- $\lim_{x \to 0^+} \ln(x) =$

- $\ln(0) =$ $\ln(1) =$ $\ln(e) =$

- $\ln(e^x) =$ $= e^{\ln(x)}$

- $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$

- $\frac{d}{dx}\left(\ln(x^3+1)\right) =$

- In general :

$$\boxed{\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}}$$

Remark : This gives

$$\boxed{f'(x) = f(x)\,\frac{d}{dx}\left(\ln(f(x))\right)}$$

Another Remark :

Example : $f(x) = x^x \quad , \quad x > 0$

$f'(x) =$

Examples : Differentiate the following functions

1) $f(x) = \cos(x)^{\sin(x)}$

2) $f(x) = x^{e^{x}}$

**Limits:** We write $\lim_{x \to a} f(x) = L$ if "the outputs $f(x)$ approach $L$ as our inputs approach $a$."

**Remark:** This is a very rough "definition". Please refer to your Calc. I notes if you are unfamiliar with the precise definition.

**Example:** $\lim_{x \to 2} x^3 =$

**Remark:** Please review laws of limits.

**Continuity:** We say $f$ is continuous at $a \in \mathbb{R}$ if $\lim_{x \to a} f(x) = f(a)$.

**Remark:** If $f$ and $g$ are continuous functions

$$\lim_{x \to a} f(g(x)) =$$

**Example:** $\lim_{x \to 2} \sin(x^2) =$

Moral :

Very important remark :

If $f$ and $g$ are 'nice' with $g(x) > 0$:

- $\lim\limits_{x \to a} e^{f(x)} = e^{\lim\limits_{x \to a} f(x)}$

So :

- $\lim \ln(g(x)) = \ln\left(\lim g(x)\right)$

So :

## Some important limits:

- $\lim_{x \to \infty} x^p = \infty$ for $p > 0$
- $\lim_{x \to \infty} \ln(x) = \infty$
- $\lim_{x \to \infty} e^x = \infty$
- $\lim_{x \to 0^+} \frac{1}{x^p} = \infty$ for $p > 0$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to 0} \frac{1}{x}$ does not exist.
- $\lim_{x \to \pi/2^-} \tan^{-1}(x) = \infty$
- $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1$

**Definition:** An indeterminate form of type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach $0$.

**Examples:**

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x}, \quad \lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}, \quad \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

**Definition:** An indeterminate form of type $\frac{\infty}{\infty}$ is a limit of a quotient where both numerator and denominator $\to \pm\infty$.

**Examples:**

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}, \quad \lim_{x \to 0^+} \frac{x^{-1}}{\ln x}$$

**L'Hôpital's Rule:** Assume $f$ and $g$ are 'nice'.

Suppose $\lim \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then:

$$\boxed{\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}}$$

**Example:**

$$\lim_{x \to \infty} \frac{x}{e^x} =$$

**Definition:** $\lim f(x) \cdot g(x)$ is an indeterminate of form $0 \cdot \infty$ if $\lim f(x) = 0$ & $\lim g(x) = \pm \infty$.

**Example:** $\lim_{x \to \infty} x \tan(1/x)$

## Remark :

## Other types of Indeterminate Forms :

| Type | Limit | | |
|---|---|---|---|
| $0^0$ | $lim\ [f(x)]^{g(x)}$ | $lim\ f(x) = 0$ | $lim\ g(x) = 0$ |
| $\infty^0$ | $lim\ [f(x)]^{g(x)}$ | $lim\ f(x) = \infty$ | $lim\ g(x) = 0$ |
| $1^\infty$ | $lim\ [f(x)]^{g(x)}$ | $lim\ f(x) = 1$ | $lim\ g(x) = \infty$ |

**Example :** $\lim_{x \to 0^+} (1+x^2)^{1/x}$

Method:

1) Look at

$$\lim \ln\left(f(x)^{g(x)}\right) = \lim g(x)\ln(f(x))$$

2) Use L'Hôpital to find

$$\lim g(x)\ln(f(x)) = a$$

(a might be $\pm\infty$ here).

3) $$\lim f(x)^{g(x)} = \lim e^{\ln(f(x)^{g(x)})}$$

$$= e^{\lim \ln(f(x)^{g(x)})}$$

$$= e^{\lim g(x)\ln(f(x))}$$

$$= e^{a}$$

Example: $$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^{1/x}$$