9. Series

We understand what it means to sum up a finite collection of numbers.

e.g. $\{1,2,3\}$: $1+2+3=6$

Goal : Develop a notion of 'summing up' an infinite sequence of numbers : $\{a_1, a_2, \dots\} = \{a_n\}$.

Intuition :

Definition : For a sequence $\{a_n\}_{n=1}^{\infty}$, we define the Nth partial sum :

$$S_N := \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N$$

i.e. add up the first N terms of the sequence.

Example: $a_n = n$.

$S_1 =$ $S_2 =$

In general, $S_N =$

Definition: A series $\sum_{n=1}^{\infty} a_n$ is a limit of partial sums:

$$\sum_{n=1}^{\infty} a_n := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$$

Notation: We sometimes write $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$.

Examples:

(1) $\sum_{n=1}^{\infty} n =$

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n =$

Example: (Geometric Series)

$$\boxed{\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \text{Diverges} & \text{if } |r| \geq 1 \end{cases}}$$

Proof:

Laws of Series

If $\sum a_n$ and $\sum b_n$ are convergent series,

with c a constant, then:

(1) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

(2) $\sum c \cdot a_n = c \sum a_n$

Examples:

(1) $\sum_{n=0}^{\infty} \frac{2}{3^n}$

(2) (A lie)

$$0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} (1-1)$$

(3) (Telescoping)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

Exercise: Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12}$

Example (Harmonic Series)

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$S_1 =$

$S_2 =$

$S_4 =$

$S_8 =$

Similarly $S_{2^n} > \frac{n+2}{2}$.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ ________________ .

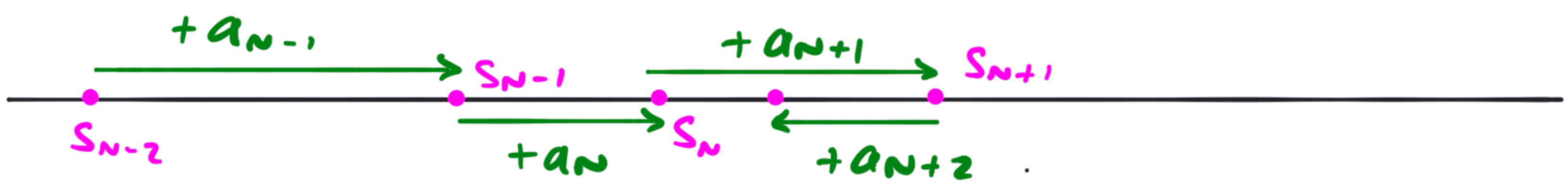
Theorem:

If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

Intuition:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_N = a_1 + a_2 + \cdots + a_N$$
$$S_{N+1} = a_1 + a_2 + \cdots + a_N + a_{N+1}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N$$



For the Pink Dots S_N to 'settle',

the length of the Green Arrows a_n

must __________.

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Example: $\sum_{n=1}^{\infty} \frac{2n^2+1}{3n^2+n+2}$

Very very important remark:

The converse is not true.

i.e. $\lim_{n\to\infty} a_n = 0$ does not guarantee $\sum a_n$ converges

Example: $\lim_{n\to\infty} \frac{1}{n} = 0$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Exercises :

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^{2n}}$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + (-1)^n}{3^n}$

(3) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1} 2^n}{9^{n-1}}$ ← Harder Algebra (write out first few terms)

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{e^n + n}{2e^n}$

(5) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n}{n+3} - \frac{5(n+1)}{n+4} \right)$

(6) $\displaystyle\sum_{n=2}^{\infty} \frac{4^n + (-3)^n}{5^n}$ ← Be careful. n starts at 2.

(7) If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, does $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ diverge ?