8. Sequences

Definition: A sequence $\{a_1, a_2, a_3, \dots\} = \{a_n\}_{n\in\mathbb{N}}$ is an ordered list of numbers.

Remark: The nth term of the sequence is the nth number on the list. On the list above, a_1 = 1st term, a_2 = 2nd term, etc.

Examples:

(1) $\{1, 2, 3, 4, 5, \dots\}$, $a_n = n$

(2) $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$, $a_n = 1/n$

(3) $\{3, 5, 7, 9, 11, \dots\}$, $a_n = 2n+1$

Remark: We sometimes index our sequences in different ways. e.g. $\{a_0, a_1, a_2, \dots\}$ or $\{a_{13}, a_{14}, a_{15}, \dots\}$.

Remark: Some sequences have patterns, some do not.

Example: If I keep rolling a fair six-sided die repeatedly, I generate a sequence of numbers which has no pattern.

Definition: We say a sequence is explicitly defined if we can give a formula for the nth term:

$$a_n = f(n).$$

Examples:

(1) $\{1, 4, 9, 16, \ldots\} = \{a_n\}$, $a_n =$

(2) $\{1, -1, 1, -1, \ldots\}$, $a_n =$

Non-example (Fibonacci Sequence):

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

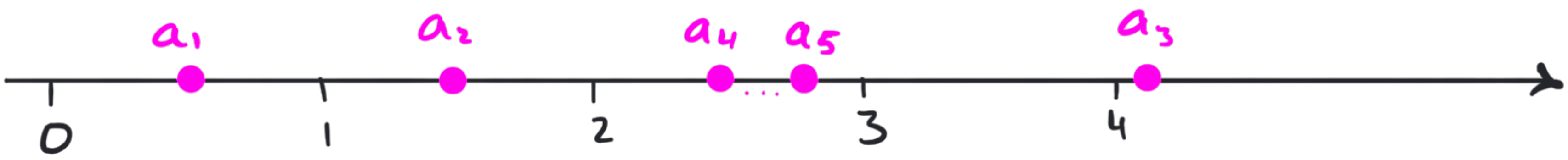
Remark: Below we show 3 different ways to algebraically represent a sequence:

$\{a_1, a_2, \ldots, a_n, \ldots\}$ $\qquad \{a_n\}_{n=1}^{\infty}$ $\qquad a_n = f(n)$

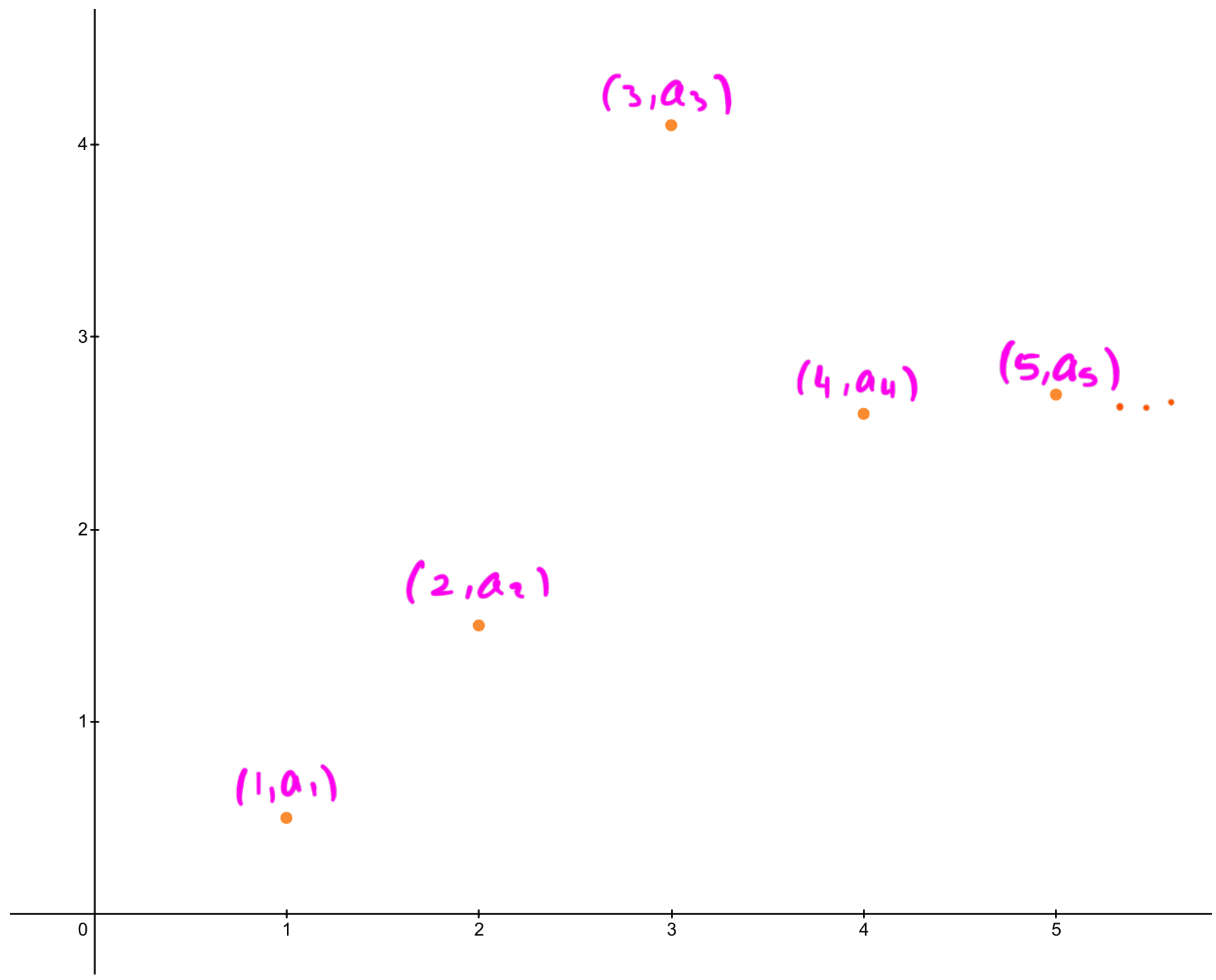
e.g. $\{\frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\}$ $\qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \frac{n}{n+1}$

Visualising Sequences :

① On the numberline :



② On a graph :



Important Remark : One aspect of a sequence we we are often concerned with is its long term behaviour. In particular, does it start to 'settle'?

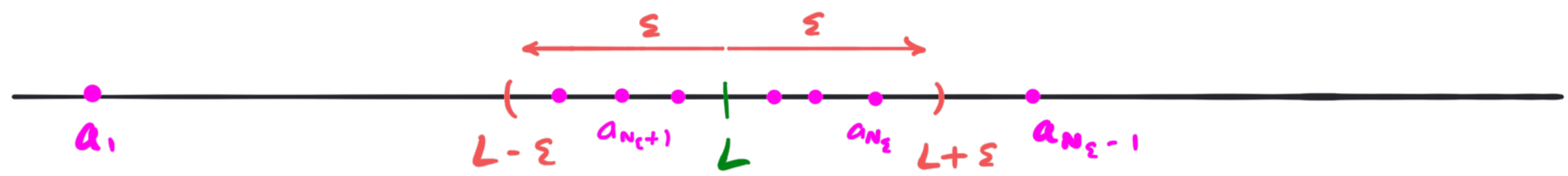
Definition: We say a sequence $\{a_n\}$ converges and has a limit $L \in \mathbb{R}$, written:

$$\boxed{\lim_{n \to \infty} a_n = L}$$

if for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ we have

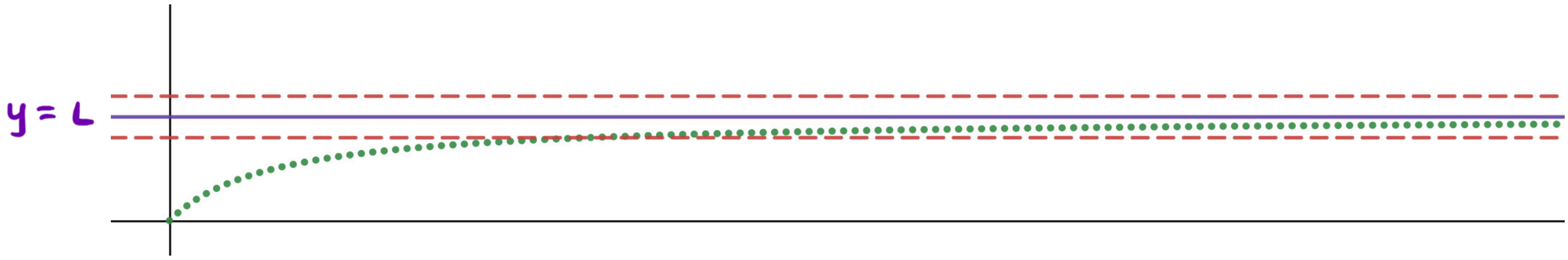
$$|a_n - L| < \varepsilon.$$

Numberline Translation: We say $\lim_{n \to \infty} a_n = L$ if, when we look at L on the numberline, no matter how small of a gap we put around L, eventually our sequence terms are trapped inside that gap.



All terms after a_{N_ε} trapped in here.

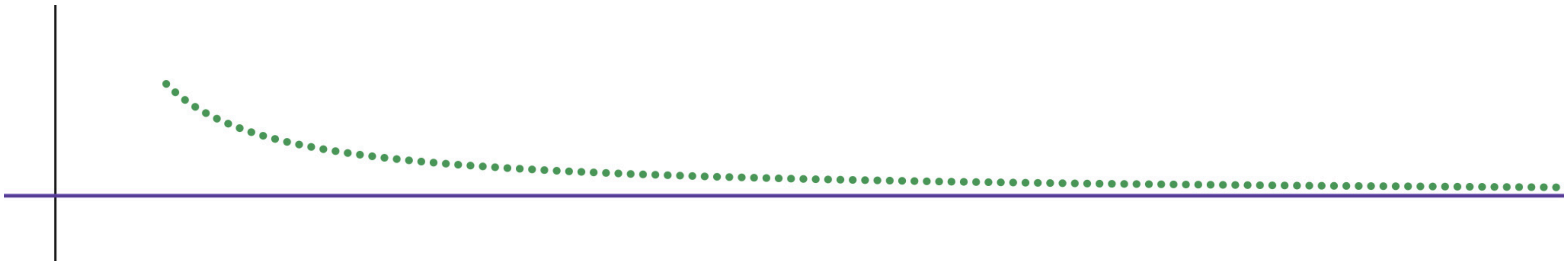
Graph Translation: We say $\lim_{n\to\infty} a_n = L$ if, when we look at the horizontal line $y = L$, no matter how small of a gap we put around this line, eventually the sequence becomes trapped inside this gap.



Example: $\{1, 1/2, 1/3, \ldots\} = \{1/n\}_{n=1}^{\infty}$

Numberline:

Graph:



$$\lim_{n\to\infty} \frac{1}{n} =$$

Definition: We say $\boxed{\lim_{n\to\infty} a_n = \infty}$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n > M$.

Translation: No matter how big of a number you pick, eventually the sequence surpasses that number and remains larger.

Examples:

(1) $\{1, 2, 3, \ldots\} = \{n\}$

(2) $\{e, e^2, e^3, \ldots\} = \{e^n\}$

(3) $\{1, 2, 6, 24, \ldots\} = \{n!\}$

Remark: • Alternatively (and equivalently) we may consider a sequence $\{a_n\}$ to be a function

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

where, to reconcile this with our previous notions, our 'terms' are $a_n = f(n)$.

e.g. $f : \mathbb{N} \to \mathbb{R}$; $f(n) = 1/n$.

- Sometimes our function f is clearly just a restriction of a function $f : [1, \infty) \to \mathbb{R}$.

i.e "If we replace 'n' with 'x', our formula for $f(x)$ makes sense for any $x \in [1, \infty)$".

e.g. $f(x) = 1/x$ makes sense for $x = 1/2$, $x = \pi$.

But $f(x) = x!$ doesn't.

Theorem: If $a_n = f(n)$, where f extends to $[1, \infty)$, and $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$.

Remark: This is extremely useful as it allows us to use tools like L'Hôpital to find limits of sequences.

Laws of Limits for Sequences:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

(1) $$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

(2) $$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(3) $$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

Furthermore, if $\lim_{n\to\infty} b_n \neq 0$:

(4) $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

Examples: Determine if the following sequences converge:

(1) $\left\{ \dfrac{n^2 + 3n + 1}{2n^2 + 7} \right\}$

(2) $\left\{ \dfrac{n^7 + n^2 + 7}{n^5 + 3n^3 + 1} \right\}$

(3) $\left\{ \dfrac{n^9 + 1}{\sqrt{3n^5 + n^3}} \right\}$

Hint:

Theorem: If $\lim_{n\to\infty} a_n = L$ and g is a continuous function, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Example: $\lim_{n\to\infty} \sin(\pi/n) =$

Exercise: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n =$

Hint: $g(x) = \ln(x)$ is continuous.

Theorem: (Squeeze Theorem)

If $a_n \leq b_n \leq c_n$ for $n \geq N \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

then $$\lim_{n\to\infty} b_n = L .$$

Intuition:

Examples :

(1) $\left\{ \frac{2^n}{n!} \right\}$

(2) $\left\{ \frac{n!}{n^n} \right\}$

Hierarchy of Explosive Limits :

Faster

- **Weird Stuff :** n^n, n^{n^2+1}, ...
- **Factorials :** $n!$, $(2n)!$, ...
- **Exponentials $(a>1)$:** e^n, 3^n, ...
- **Polynomials $(\deg \geq 1)$:** n, n^3+27, ...
- **Logarithms :** $\ln(n)$, $\log_3(n)$, ...

Slower

Examples :

(1) $$\lim_{n\to\infty} \frac{n! + n^3}{e^n + \ln(n)} =$$

(2) $$\lim_{n\to\infty} \frac{n^7 + \ln(n) + 2^n}{13 + 5^n + n^{19}} =$$

Example (Geometric Sequence):

$\{1, r, r^2, r^3, \dots\}$,

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 1 & \text{if } r = 1 \\ 0 & \text{if } -1 < r < 1 \\ \text{Diverges} & \text{if } r = -1 \\ \text{Diverges} & \text{if } r < -1 \end{cases}$$

Definitions:

(1) A sequence is called increasing if $a_{n+1} > a_n$ for all n.

(2) A sequence is called decreasing if $a_{n+1} < a_n$ for all n.

(3) A sequence is called monotonic if it is increasing or decreasing.

Examples:

(1) $a_n = n$ is ______________.

(2) $a_n = 1/n$ is ______________.

Remark: If $a_n = f(n)$ extends to $f(x)$, we may use differentiation to check for monotonicity.

↳ $f'(x) > 0 \Rightarrow a_n$ increasing.

↳ $f'(x) < 0 \Rightarrow a_n$ decreasing.

Example: $\{\tan^{-1}(n)\}$

Definitions:

(1) A sequence $\{a_n\}$ is **bounded from above** if there exists a constant M such that $a_n \leq M$ for all $n \geq 1$.

(2) A sequence $\{a_n\}$ is **bounded from below** if there exists a constant m such that $a_n \geq m$ for all $n \geq 1$.

(3) A sequence $\{a_n\}$ is **bounded** if it is both bounded from above and below.

Example: $\left\{\frac{1}{n}\right\}$

Theorem: (Monotone Convergence Theorem)

Every bounded monotonic sequence converges.

Intuition :

Example : $\{\tan^{-1}(n)\}$

Alternating Sequences:

Example: $\{(-1)^n\}$

Definition: A sequence of the form $\{(-1)^n a_n\}$ where $a_n \geq 0$ is called an alternating sequence.

Example: $\left\{\frac{(-1)^n}{n}\right\}$

Theorem: An alternating sequence $\{(-1)^n a_n\}$ will converge if and only if $\lim_{n\to\infty} a_n = 0$.

Intuition:

Exercises: Find the limits of the following sequences if they exist:

(1) $\left\{ (-1)^n \cdot \dfrac{n^2}{3n^2+n+1} \right\}$

(2) $\left\{ (-1)^n \cdot \dfrac{2^n - 1}{n^2} \right\}$

(3) $\left\{ (-1)^n \cdot \dfrac{n^3+9n}{n!+7} \right\}$

(4) $\left\{ \dfrac{\tan^{-1}(n)}{n} \right\}$

(5) $\left\{ ne^{-2n} \right\}$

(6) $\left\{ n \sin\left(\frac{1}{n}\right) \right\}$

(7) $\left\{ \left(\dfrac{n}{n+1}\right)^n \right\}$ $\leftarrow$ Hard.