7. Partial Fractions

Goal: To be able to integrate rational functions:

$$\boxed{\int \frac{P(x)}{Q(x)}\,dx}$$

where $P(x)$ and $Q(x)$ are polynomials.

Preliminaries:

Definition: A polynomial is a sum of terms of the form ax^n where a is a constant and $n = 0, 1, 2, \dots$.

E.g. $p(x) = 7x^3 + 9x^2 + 13$

Definition: The degree of a polynomial is the highest power present in its expression.

E.g. $\deg(p) = 3$ for $p(x)$ above.

Definition: We say a polynomial is linear if its degree is 1. e.g. $p(x) = 3x + 7$.

Definition: A quadratic polynomial $ax^2 + bx + c$ is said to be irreducible if $b^2 - 4ac < 0$.

i.e. If it has no real roots.

i.e. If it cannot be broken up into linear factors.

E.g. $x^2 + 9$, $x^2 - 2x + 11$

Remark: It is often useful to complete the square for irreducible quadratics.

Fundamental Theorem of Algebra:

Any polynomial can be factorised into a product of linear terms and irreducible quadratic terms (with some factors possibly repeated).

Example: $x^3 - 1 = (x-1)(x^2 + x + 1)$

Definition: A **rational function** is a quotient of polynomials: $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials.

E.g. $\dfrac{x^3 + 2x - 1}{x^2 + 1}$

Definition: A rational function $\dfrac{P(x)}{Q(x)}$ is called **proper** if $\deg(P) < \deg(Q)$.

Remark: Using long division, we can express any rational function as a polynomial + a proper rational function.

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where P, Q, R & S are polynomials with $\deg(R) < \deg(Q)$.

Example :

$$\frac{x^3 + 2x - 1}{x^2 + 1}$$

Cases we can deal with :

Case 4 . $\displaystyle\int \frac{A}{(ax+b)^k}\,dx$

Examples :

i) $\displaystyle\int \frac{3}{x-5}\,dx$

ii) $\displaystyle\int \frac{13}{(3x+11)^7}\,dx$

Remark : We do these with u-sub.

Case B. $\displaystyle\int \frac{Ax + B}{(ax^2 + bx + c)^k}\, dx$

Examples:

i) $\displaystyle\int \frac{6x + 2}{3x^2 + 2x + 1}\, dx$

ii) $\displaystyle\int \frac{3}{x^2 - 2x + 10}\, dx$

iii) $\displaystyle\int \frac{2x - 3}{x^2 - 4x + 14}\, dx$

Hint: Rewrite the numerator to break this into u-sub + trig.

What if it doesn't look like that?

Examples:

i) $\displaystyle\int \frac{1}{x^2-25}\,dx$

ii) $\displaystyle\int \frac{x^3-25x+1}{x^2-25}\,dx$

Method: $\int \frac{P(x)}{Q(x)}\,dx$

① If $\deg(P) \geq \deg(Q)$, use long division to ensure we are integrating a proper rational function.

② Factorise the denominator into a product of linear and irreducible quadratic factors.

③ Use algebra to simplify the rational function into a sum of terms of the form

$$\frac{A}{(ax+b)^k} \quad ; \quad \frac{Ax+B}{(ax^2+bx+c)^k} .$$

This is called the partial fraction expansion.

④ Integrate each piece individually.

Remark: Step ③ is where the work lies.

We outline the procedure for ③ now:

Case I. $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$

where all the factors are distinct. Then:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

Examples:

i) $\dfrac{1}{x^2-9}$

ii) $\dfrac{3x^2-12x+11}{x^3-6x^2+11x-6}$

Case II. $Q(x) = (a_1x+b_1)^{\alpha_1}(a_2x+b_2)^{\alpha_2}\cdots(a_nx+b_n)^{\alpha_n}$

i.e. Q has repeated linear factors. Then, instead of the term $\dfrac{A_i}{a_ix+b_i}$, we

have:
$$\frac{A_{i,1}}{a_ix+b_i}+\frac{A_{i,2}}{(a_ix+b_i)^2}+\cdots+\frac{A_{i,\alpha_i}}{(a_ix+b_i)^{\alpha_i}}$$

where each $A_{i,j}$ may be different.

Examples:

i) $\dfrac{x+1}{x^2-4x+4}$

ii) Write down the form of the expansion of

$$\frac{x^4+2x^3+9x+1}{(x-1)(x-2)^3(x-7)^2}$$

Case III. $Q(x)$ contains irreducible quadratic factors, none of which are repeated.

i.e. $Q(x)$ has a factor $ax^2 + bx + c$ where $b^2 - 4ac < 0$. Then the expansion will have a term:

$$\frac{Ax + B}{ax^2 + bx + c} .$$

Example:

$$\frac{x^2 + x + 8}{(x-1)(x^2+9)}$$

Case IV. $Q(x)$ contains repeated irreducible quadratic factors: $(ax^2+bx+c)^k$.

Similarly to Case II, instead of the single term:
$$\frac{Ax+B}{ax^2+bx+c}$$
, we have

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Example:

Write down the form of the expansion

$$\frac{x^5+9x^2+11}{(x-3)(x^2+2x+3)^3}$$

Summary :

Factor	Contribution
$(ax+b)^n$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$
$(ax^2+bx+c)^n$	$\dfrac{A_1 + B_1 x}{ax^2+bx+c} + \dfrac{A_2 + B_2 x}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_n + B_n x}{(ax^2+bx+c)^n}$ where ax^2+bx+c is irreducible.

Exercises:

1) $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

2) $$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$$

3) $$\int \frac{2x^2-x+4}{x^3+4x}\,dx$$

4) $$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$$

5) $$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$