5. Trigonometric Substitution

Goal: To be able to handle integrals involving expressions like:

$$\sqrt{a^2 - x^2} \qquad a^2 + x^2 \qquad \sqrt{x^2 - a^2}$$

Inspiration:

$\sqrt{a^2 - x^2}$: $x = a\sin\theta$: $-\pi/2 \le \theta \le \pi/2$

$a^2 + x^2$: $x = a\tan\theta$: $-\pi/2 < \theta < \pi/2$

$\sqrt{x^2 - a^2}$: $x = a\sec\theta$: $\theta \in [0, \pi/2) \cup [\pi, \frac{3\pi}{2})$

Example : $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\, dx$

Important Tip: Draw the triangle for your substitution.

Examples:

1) $$\int \frac{dx}{x^2\sqrt{9-x^2}}$$

2) $$\int \frac{dx}{\sqrt{x^2+4}}$$

3) $$\int \frac{dx}{x^2\sqrt{x^2-25}}$$

Completing the square:

Sometimes we can convert integrals involving $\sqrt{\ }$'s of quadratic functions into one of the cases we now know how to deal with.

Examples:

1) $$\int \frac{dx}{\sqrt{x^2 - 4x + 8}}$$

2) $$\int_4^6 \frac{dx}{\sqrt{x^2 - 6x + 8}}$$

Special Examples :

1) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

2) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

3) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$

Exercises:

1) $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

2) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\, dx$

3) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\, dx$

4) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\, dx$

5) $\displaystyle\int_2^3 \frac{dx}{(x^2-1)^{3/2}}$

6) $\displaystyle\int_0^{1/2} x\sqrt{1-4x^2}\, dx$

7) $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\, dx$