4. Trigonometric Integrals

Goals: To be able to handle integrals of the form:

I) $\int \sin^n\theta \cos^m\theta \, d\theta$ for $n, m = 0, 1, 2, \ldots$

II) $\int \sec^m\theta \tan^n\theta \, d\theta$

III) $\int \sin(m\theta)\cos(n\theta) \, d\theta$

Example: $\int \sec^4\theta \tan\theta \, d\theta$

To get a better understanding of these integrals let's look at some derivatives:

$$\frac{d}{d\theta}\left(\frac{\sin^{n+1}\theta}{n+1}\right) = \sin^n\theta \cos\theta$$

$$\Rightarrow \quad \frac{\sin^{n+1}\theta}{n+1} + C = \int \sin^n\theta \cos\theta \, d\theta$$

Doing the same thing for $\frac{\cos^{n+1}\theta}{n+1}$, $\frac{\tan^{n+1}\theta}{n+1}$ & $\frac{\sec^{n}\theta}{n}$

we get :

$$\boxed{\begin{aligned}
\int \sin^n\theta \cos\theta \, d\theta &= \frac{\sin^{n+1}\theta}{n+1} + C \\
\int \cos^n\theta \sin\theta \, d\theta &= -\frac{\cos^{n+1}\theta}{n+1} + C \\
\int \tan^n\theta \sec^2\theta \, d\theta &= \frac{\tan^{n+1}\theta}{n+1} + C \\
\int \sec^n\theta \tan\theta \, d\theta &= \frac{\sec^{n}\theta}{n} + C
\end{aligned}}$$

Example : $\int \sec^4\theta \tan\theta \, d\theta =$

Remark : This can also be done using a substitution.

Moral: We can do these integrals if:

1)

2)

3)

4)

But what if it doesn't look like one of the cases above? What about:

Example : $\int \sec^7\theta \tan^3\theta \, d\theta$

Idea :

Moral :

Example : $\int \sec^7\theta \tan^3\theta \, d\theta$

Remark : This will only work if ...

If that doesn't work we can use double angle formulas:

$$\boxed{\begin{aligned} \sin^2\theta &= \tfrac{1}{2}(1 - \cos 2\theta) \\ \cos^2\theta &= \tfrac{1}{2}(1 + \cos 2\theta) \end{aligned}}$$

Example: $\int \sin^4\theta \, d\theta$

For integrals of the form $\int \sin(n\theta)\cos(m\theta)\,d\theta$ we use the addition of angles formulas:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

to give:

$$\sin(n\theta)\cos(m\theta) = \frac{1}{2}\left[\sin\left((n-m)\theta\right) + \sin\left((n+m)\theta\right)\right]$$

$$\sin(n\theta)\sin(m\theta) = \frac{1}{2}\left[\cos\left((n-m)\theta\right) - \cos\left((n+m)\theta\right)\right]$$

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}\left[\cos\left((n-m)\theta\right) + \cos\left((n+m)\theta\right)\right]$$

Remark:

$$\int \sin(k\theta)\,d\theta = -\frac{1}{k}\cos(k\theta) + C$$

$$\int \cos(k\theta)\,d\theta = \frac{1}{k}\sin(k\theta) + C$$

Example : $\int \sin 7\theta \cos 3\theta \, d\theta$

Special Examples:

1) $\int \tan\theta \, d\theta$

2) $\int \tan^2\theta \, d\theta$

3) $\int \tan^3\theta \, d\theta$

4) $\int \sec\theta \, d\theta$

5) $\int \sec^2\theta \, d\theta$

6) $\int \sec^3\theta \, d\theta$

Exercises:

1) $\int \sin^2\theta \cos^3\theta \, d\theta$

2) $\int \sin^3\theta \cos^4\theta \, d\theta$

3) $\int \tan\theta \sec^3\theta \, d\theta$

4) $\int \tan^2\theta \sec^4\theta \, d\theta$

5) $\int_0^{\pi/2} \sin^7\theta \cos^3\theta \, d\theta$

6) $\int_0^{\pi/4} \tan^3\theta \sec^3\theta \, d\theta$

7) $\int \sin(7\theta) \cos(3\theta) \, d\theta$