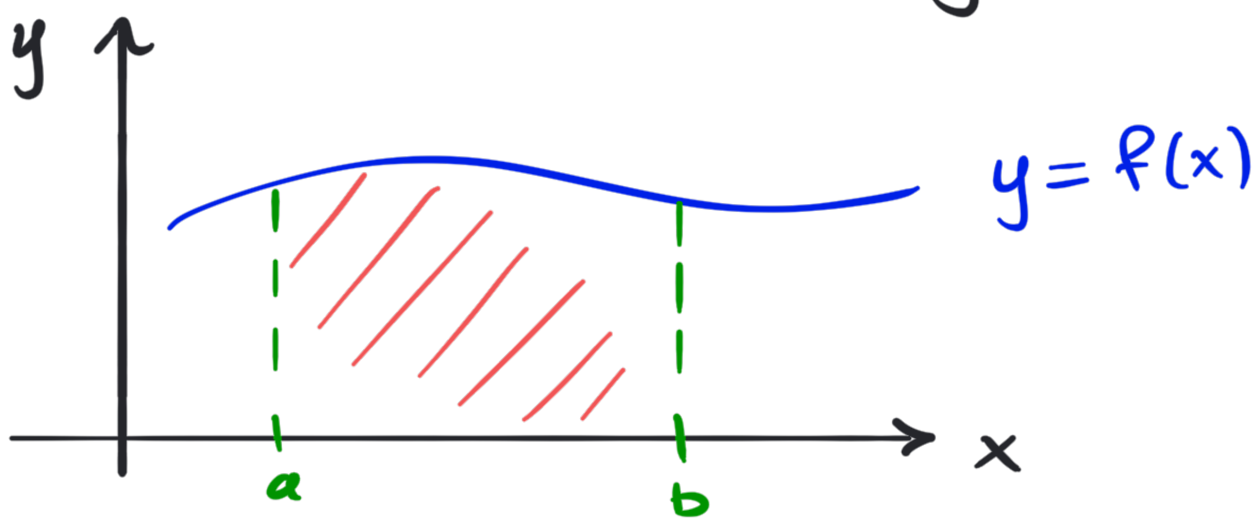
3. Improper Integrals:

We are familiar with definite integrals:



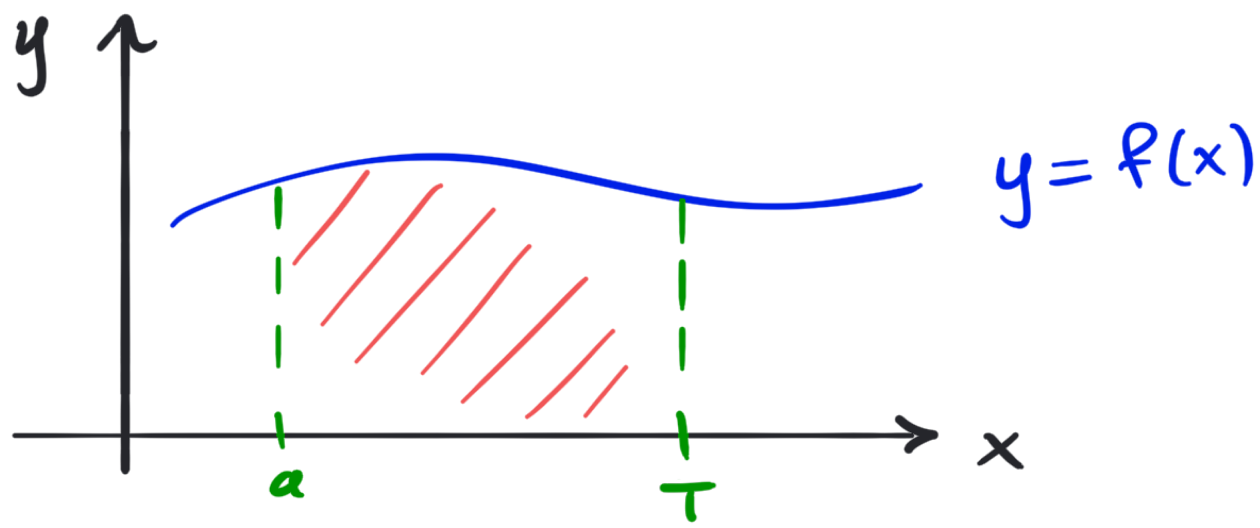
$$= \int_a^b f(x)\,dx$$

Remark:

We now want to extend this notion.

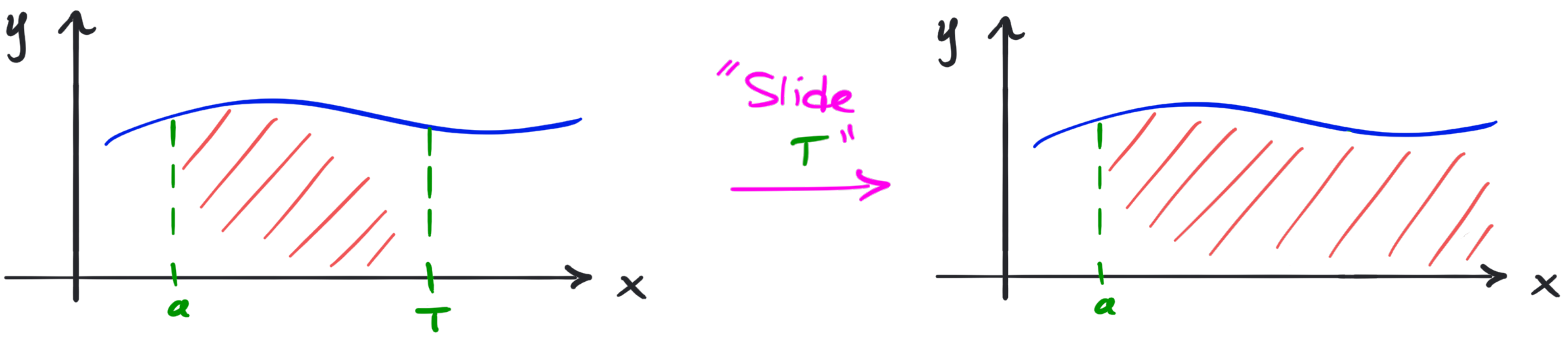
Case I: Infinite Intervals

If $\int_a^T f(x)\,dx$ exists for every $T \geq a$:



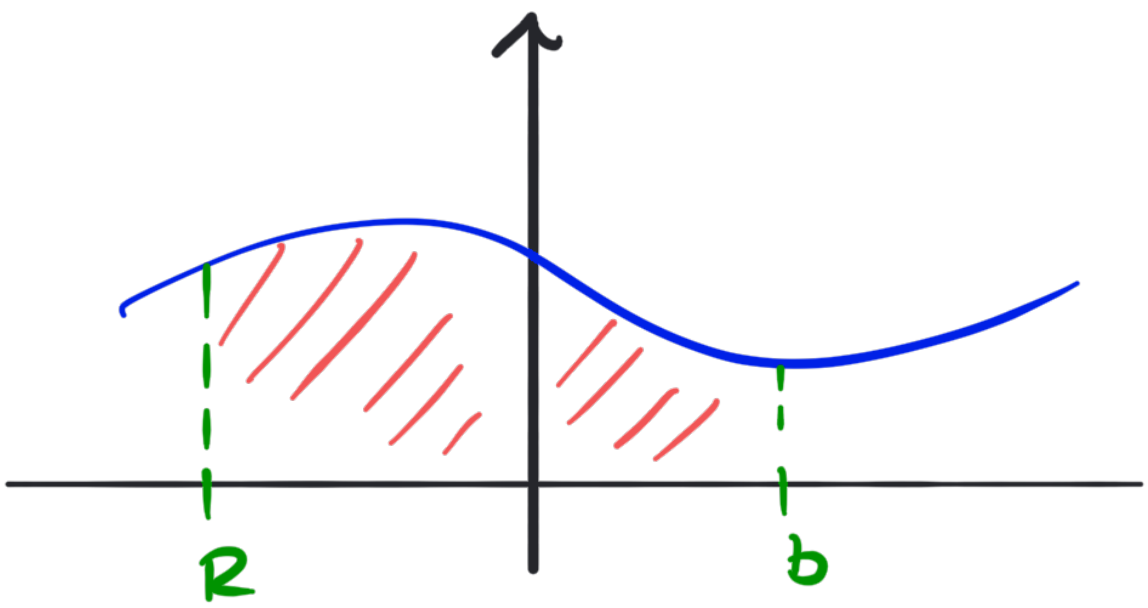
We define:

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx}$$



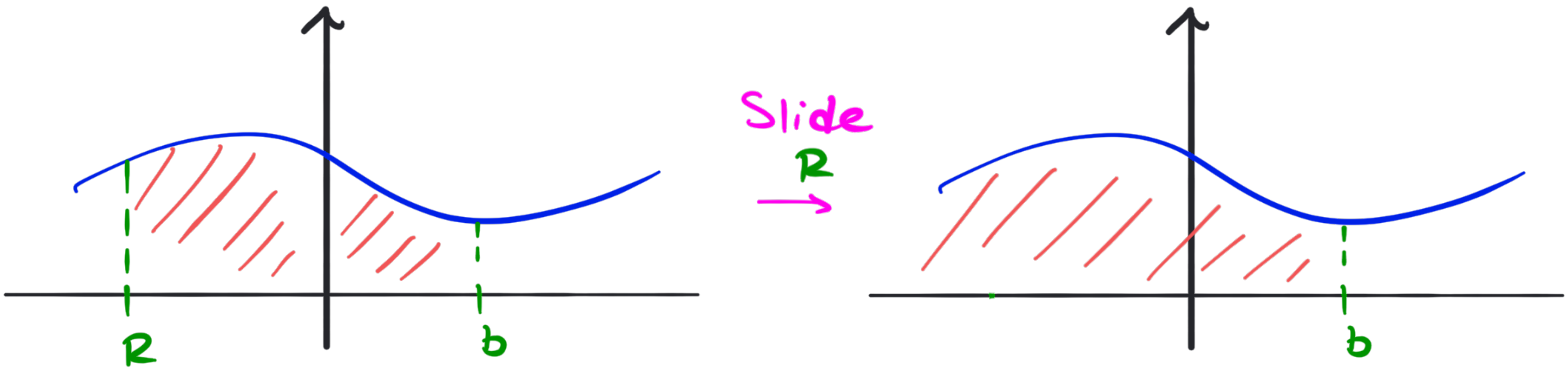
$$= \int_a^\infty f(x)\,dx$$

- Similarly if $\int_R^b f(x)\,dx$ exists for all $R \leq b$:



We define :

$$\boxed{\int_{-\infty}^{b} f(x)\,dx := \lim_{R\to-\infty} \int_{R}^{b} f(x)\,dx}$$



Definition : We say $\int_{a}^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ are convergent if the corresponding limit exists and is finite.

Otherwise, we say they are divergent.

Examples:

1) $\int_0^\infty e^{-x} \, dx$

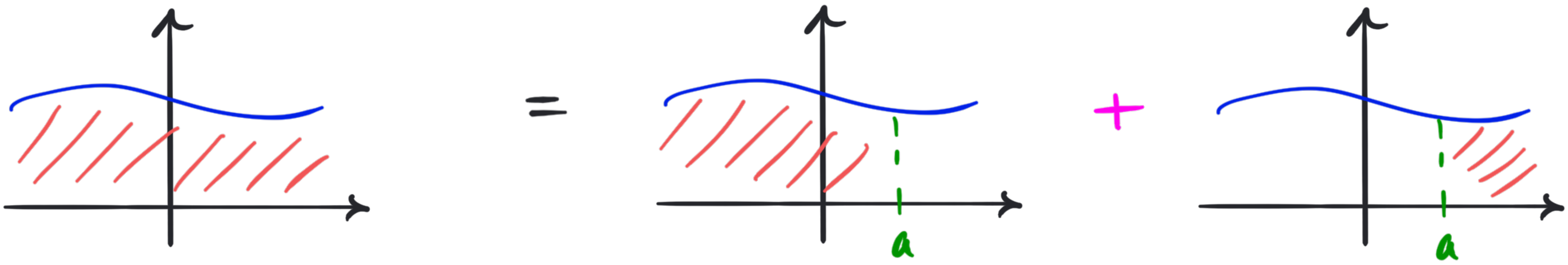
2) $\int_1^\infty \frac{1}{x} \, dx$

3) $\int_1^\infty \frac{1}{x^2} \, dx$

Definition: If for any value of a, both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define:

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx}$$

Picture:



Example: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Theorem:

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $\boxed{p > 1}$ and

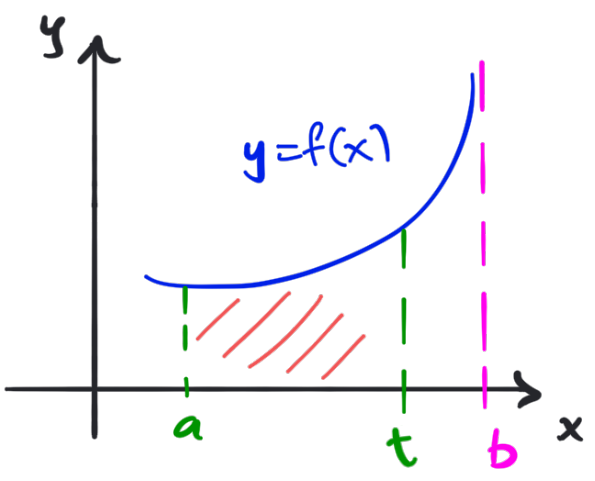
divergent if $\boxed{p \le 1}$

Proof:

Case II : Discontinuities

i) If f is continuous on $[a,b)$ but not necessarily continuous at b, then

$$\boxed{\int_a^b f(x)\,dx := \lim_{t \to b^-} \int_a^t f(x)\,dx}$$



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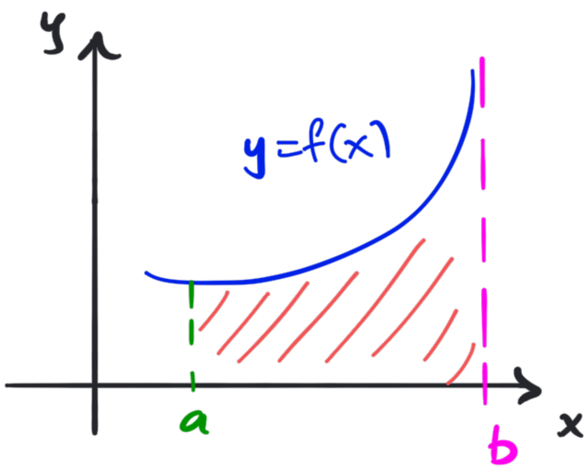
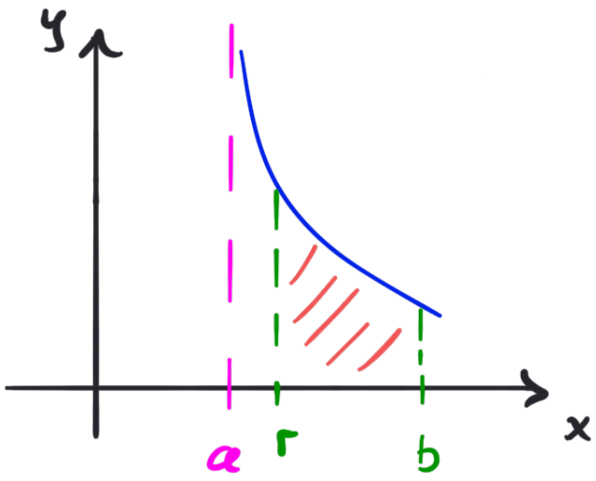
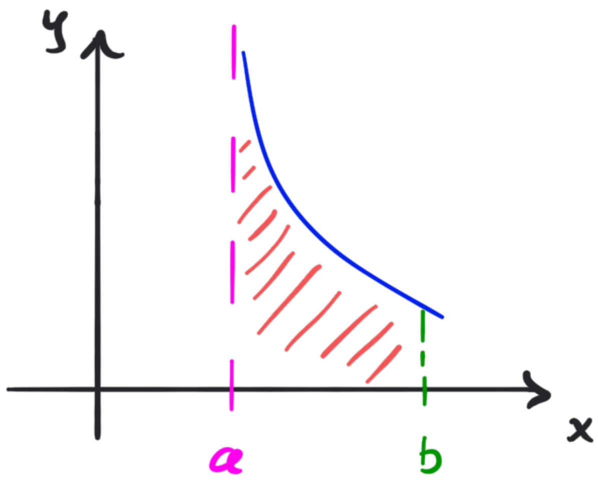


ii) If f is continuous on $(a,b]$ but not necessarily continuous at a, then

$$\boxed{\int_a^b f(x)\,dx := \lim_{r \to a^+} \int_r^b f(x)\,dx}$$

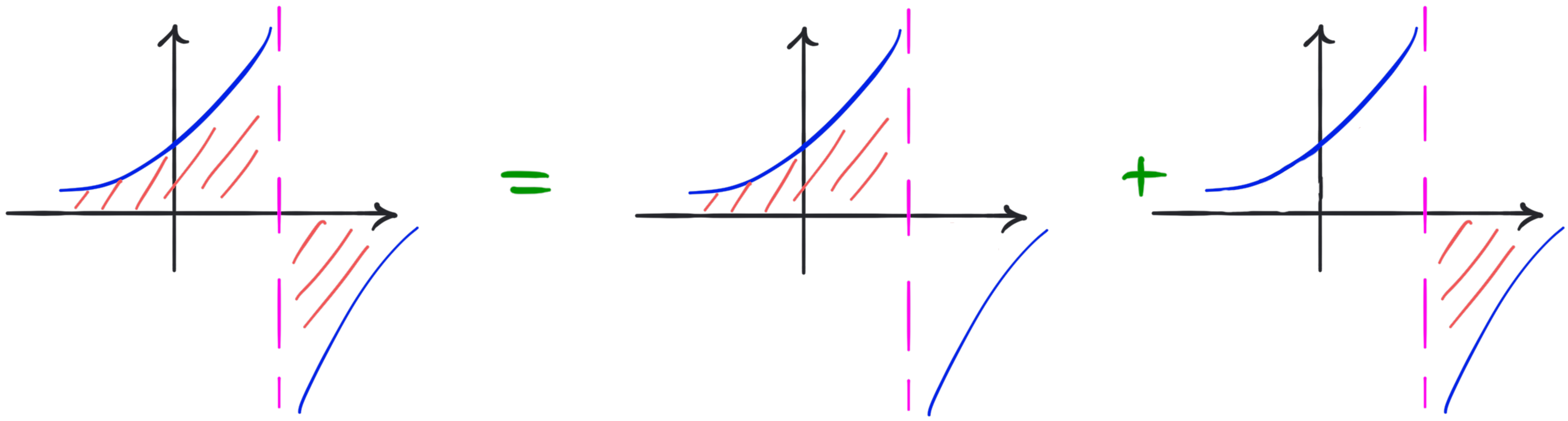




iii) If f has a discontinuity at c,

where $a < c < b$, and both

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$



Remark :

Example: $\displaystyle\int_{-2}^{2} \frac{1}{x^2}\,dx$

Theorem :

$\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $\boxed{p < 1}$ and divergent if $\boxed{p \geq 1}$

Proof :

Comparison Test for Integrals :

Theorem : If f and g are continuous functions with $g(x) \geq f(x) \geqslant 0$, then :

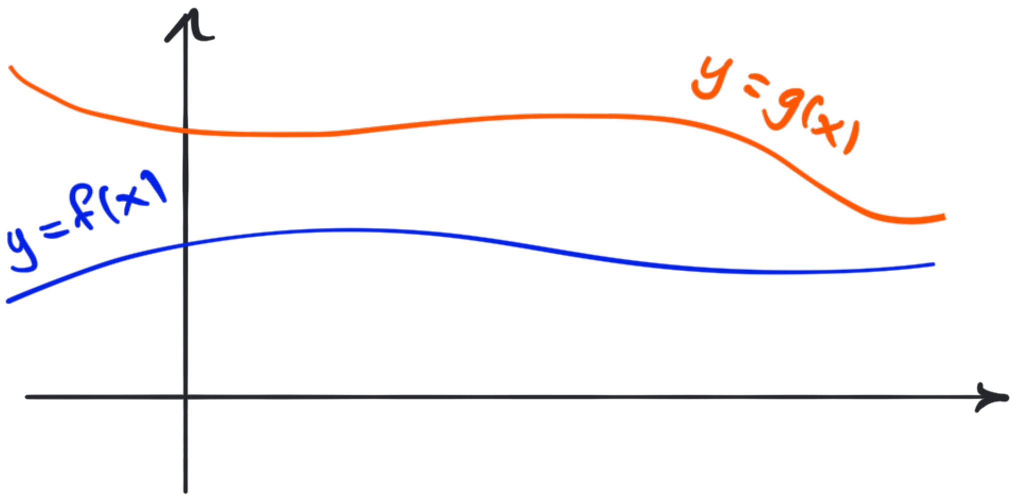
i) If $\int_a^\infty g(x)\,dx$ is convergent,

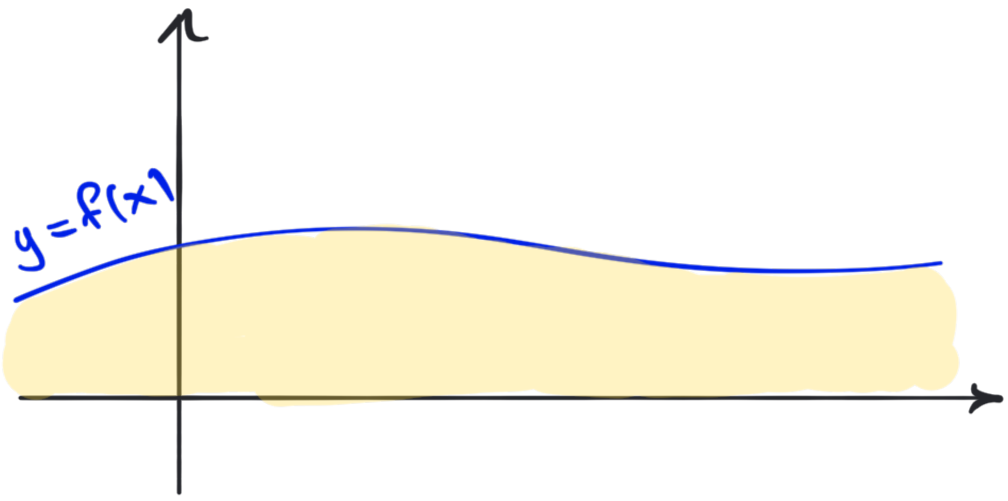
then $\int_a^\infty f(x)\,dx$ is converges.

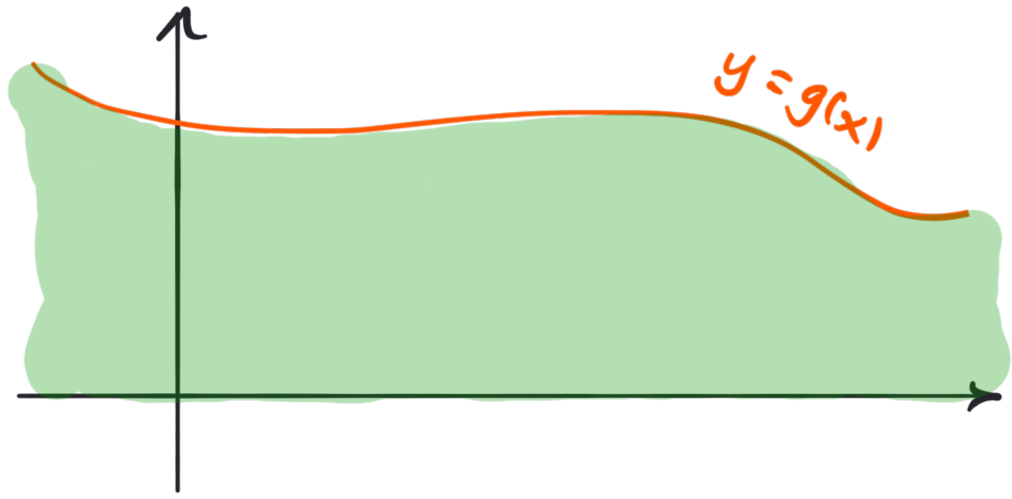
ii) If $\int_a^\infty f(x)\,dx$ is divergent

then $\int_a^\infty g(x)\,dx$ diverges.

Picture :







Moral :

Similarly:

Theorem: If f and g are functions with $g(x) \geq f(x) \geq 0$, then:

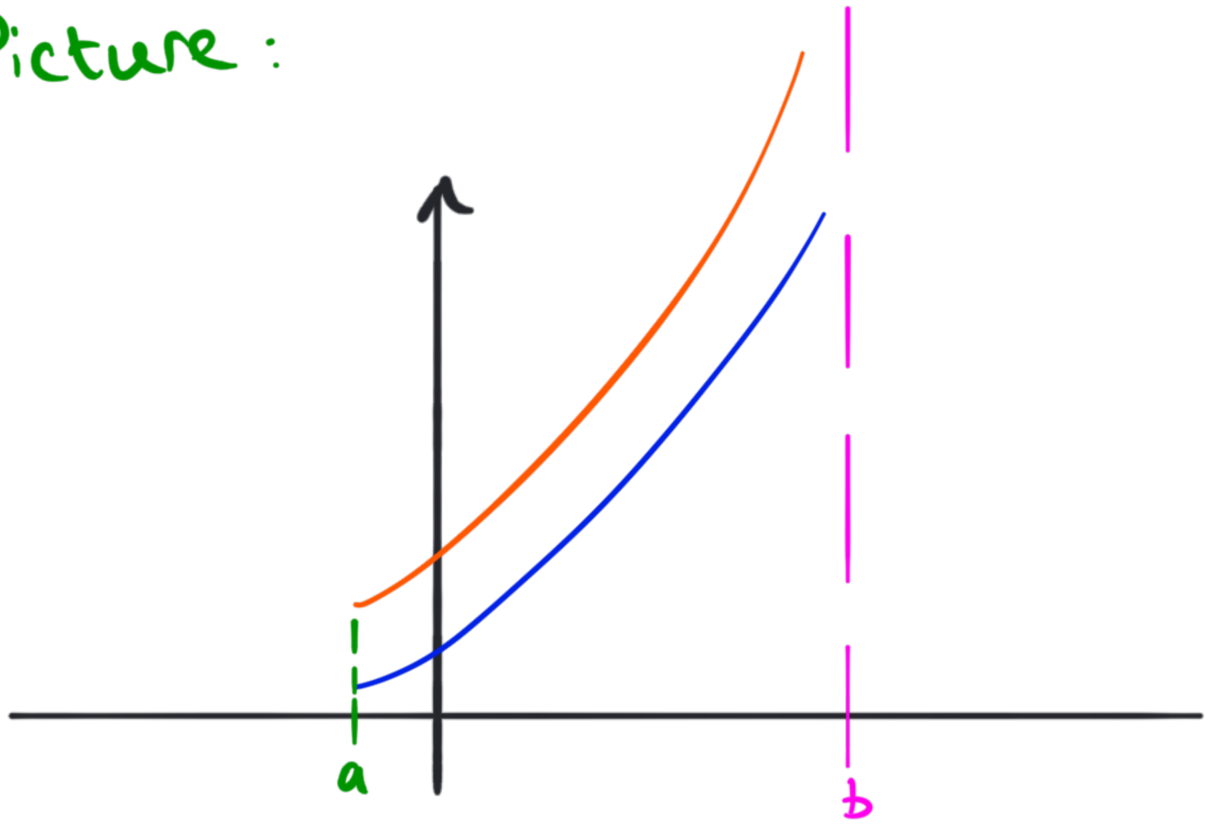
i) If $\int_a^b g(x)\,dx$ is convergent,

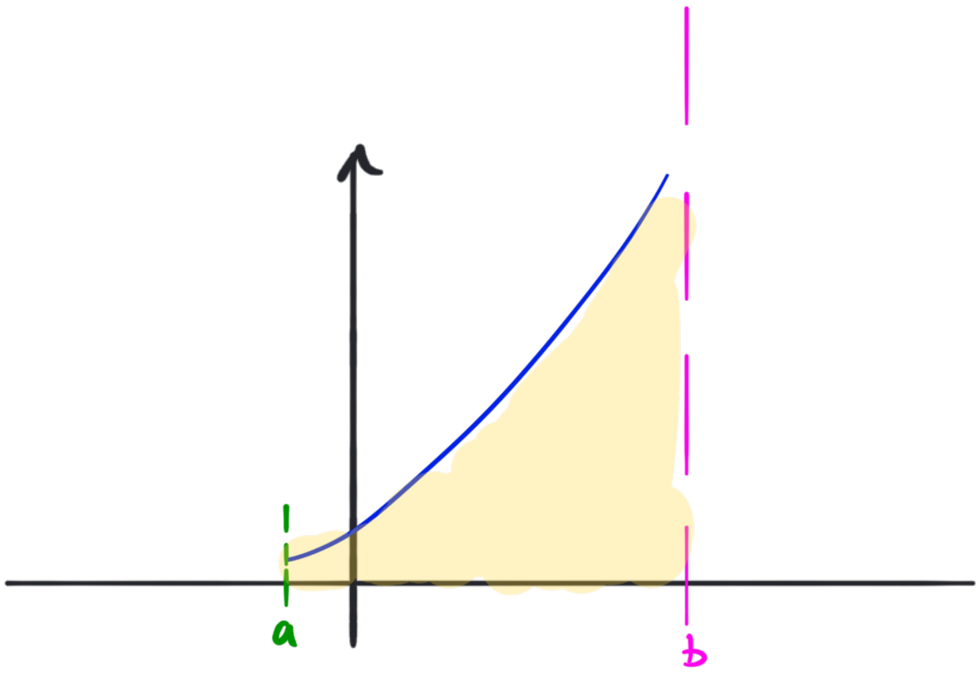
then $\int_a^b f(x)\,dx$ is converges.

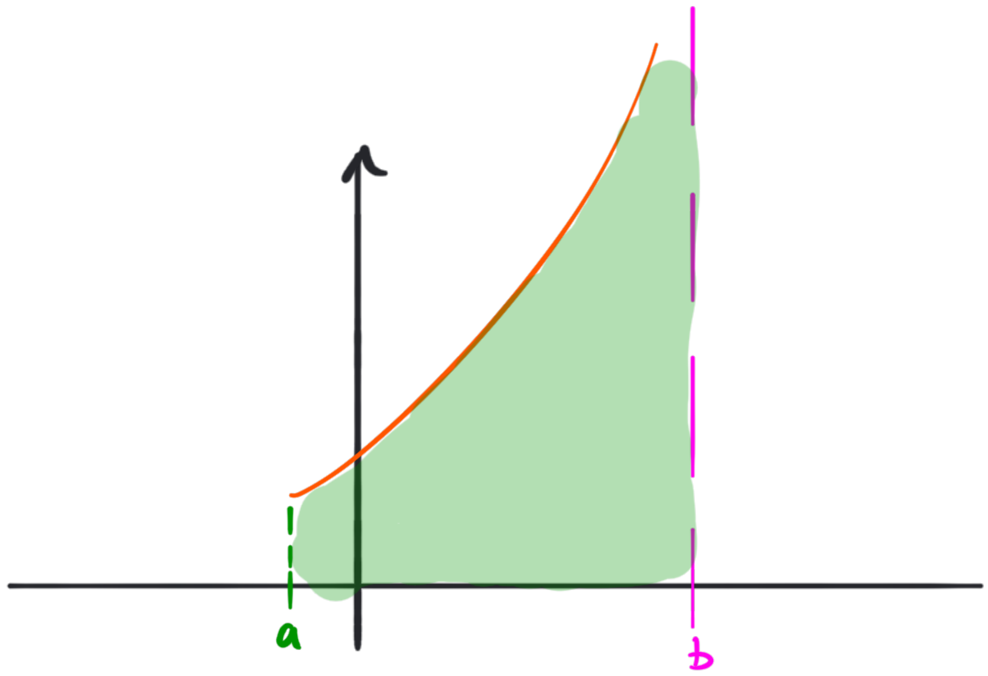
ii) If $\int_a^b f(x)\,dx$ is divergent

then $\int_a^b g(x)\,dx$ diverges.

Picture :







Moral :

Examples: Determine if the following integrals are convergent or divergent:

Tip:

1) $$\int_1^{\infty} \frac{dx}{x^2+x+1}$$

2) $$\int_1^{\infty} \frac{dx}{x-1/2}$$

3) $$\int_0^1 \frac{\cos^2 x}{\sqrt{x}}\, dx$$

4) $$\int \frac{e^{-x}}{1+\sin^2 x}\, dx$$

Exercises:

1) $$\int_{-1}^{2} \frac{1}{\sqrt[3]{x}}\,dx$$

2) $$\int_{1}^{\infty} e^{-4x}\,dx$$

3) $$\int_{2}^{\infty} xe^{-2x}\,dx$$

4) $$\int_{2}^{7} \frac{1}{5-x}\,dx$$

5) $$\int_{2}^{7} \frac{1}{\sqrt{5-x}}\,dx$$

Use the comparison test to determine if the integrals below converge / diverge:

6) $$\int_{1}^{\infty} \frac{\cos^2 x}{x^2}\,dx$$

7) $$\int \frac{x^2+9x}{x^3+4x+2}\,dx$$