18. Differential Equations

Motivation :

"A system is said to be deterministic if its entire future course and past are uniquely determined by its state at the present time.

Informally speaking, a differential equation gives the local law of evolution of a process and the task of the theory of ordinary differential equations is to reconstruct the past and predict the future of the process from a knowledge of this local law of evolution." - Paraphrased from V. Arnold, Ordinary Differential Equations MIT Press (1973).

Separable Equations :

Goal : To develop a method for solving certain differential equations and investigate applications.

Definition : A separable differential equation is an equation of the form :

$$\boxed{\frac{dy}{dx} = f(x)\,g(y)}$$

Method : Provided that $g(y) \neq 0$, the above equation can be solved by computing

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(x)\,dx}$$

and solving for y as a function of x .

Remark (IVP) :

Examples:

1) Solve $\frac{dy}{dx} = \frac{y^2}{x^2}$.

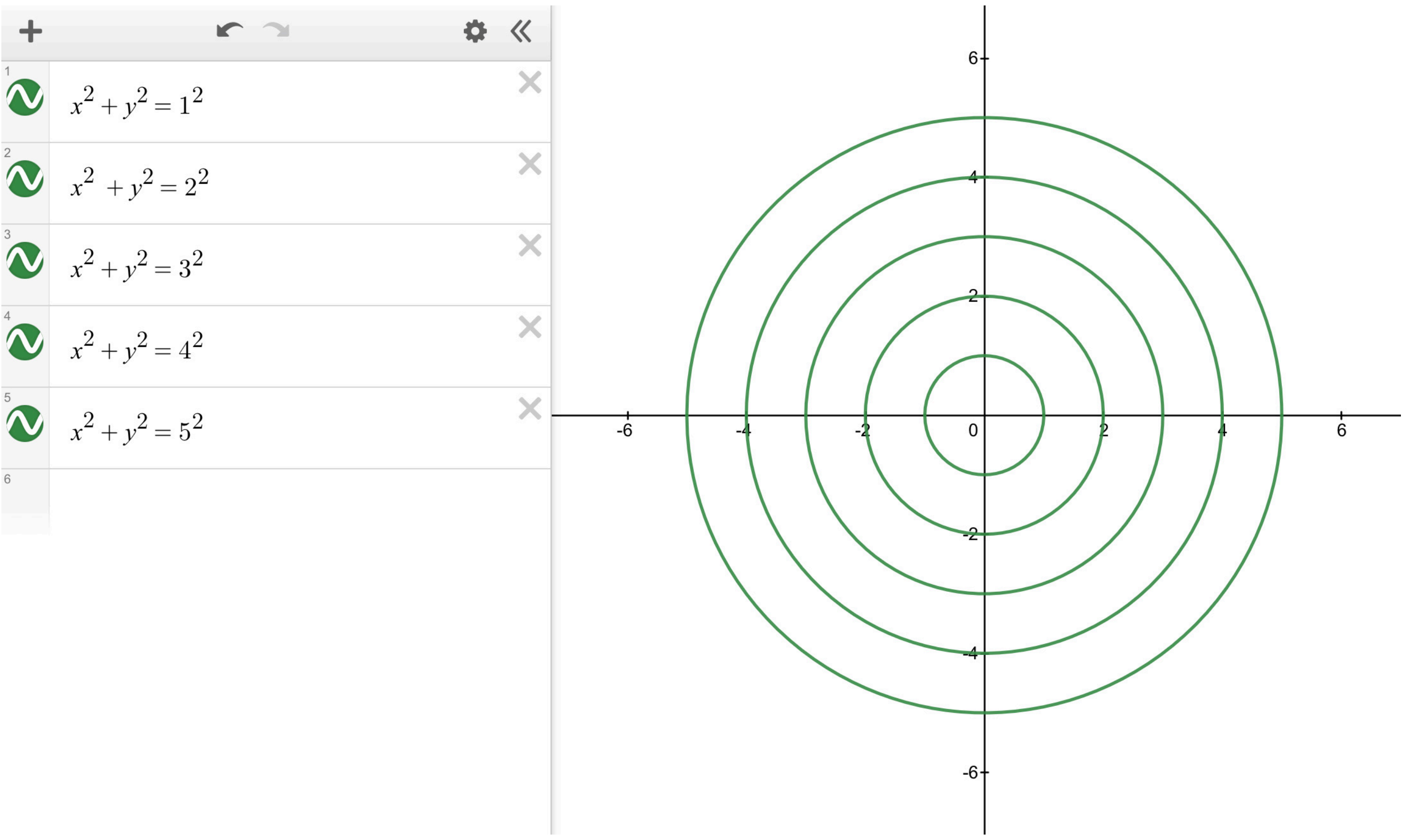
2) Find the unique solution to the above so that $y(0) = 2$.

3) $\frac{dy}{dx} = \frac{6x^2}{2y + \cos(y)}$

Orthogonal Trajectories:

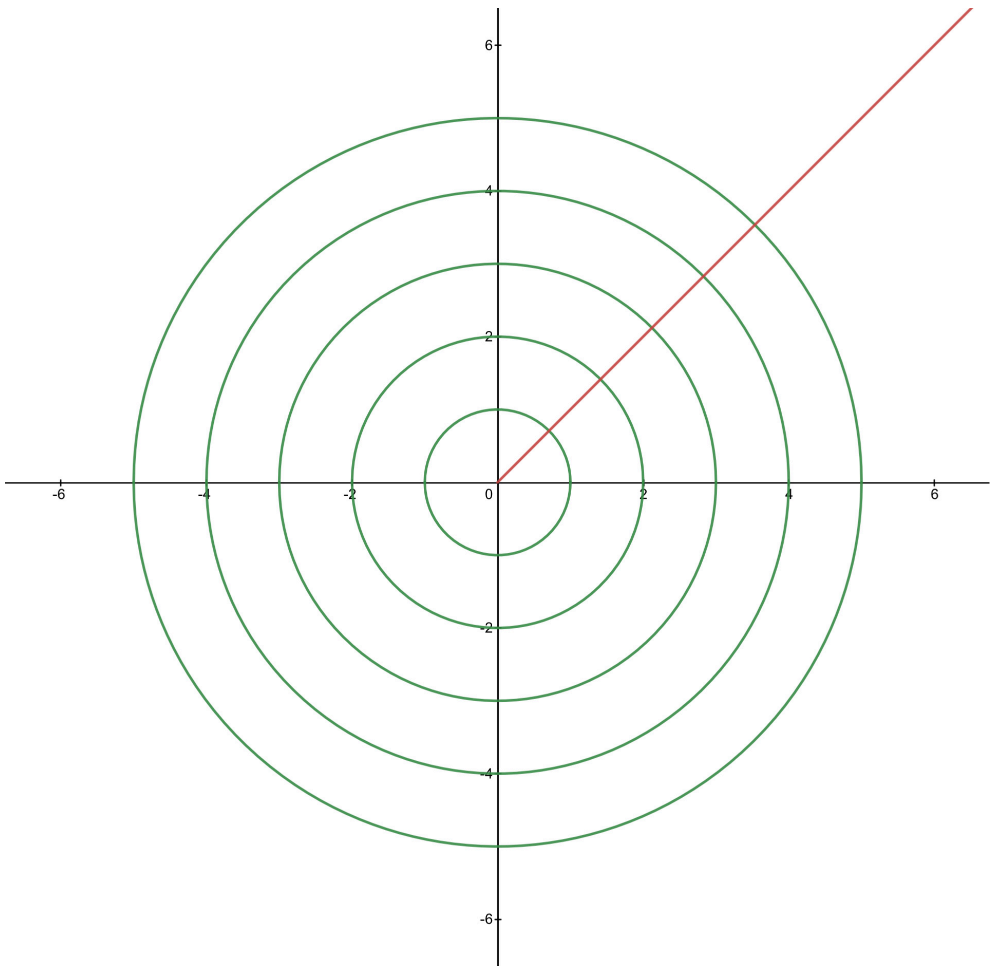
Example: The equation $x^2 + y^2 = k^2$ describes a family of curves (concentric circles).

For each value of k we get a different curve in the family.



Definition: Given a family of curves, an orthogonal trajectory for the family is a curve that intersects every curve in the family orthogonally.

Example:



Very Important Remark: If the slope of a member of the family of curves was given by $\frac{dy}{dx} = g(x,y)$, then the slope of the orthogonal trajectory would be given by $-\frac{1}{g(x,y)}$.

Question: Given a family of curves, can we find orthogonal trajectories?

Method:

1) Use the equation for the family of curves to get an expression for $\frac{dy}{dx}$. This is usually done by implicit differentiation. Remove the parameter k from the expression by substitution if necessary.

2) If $\frac{dy}{dx} = g(x,y)$, the equation describing the orthogonal trajectories is given by

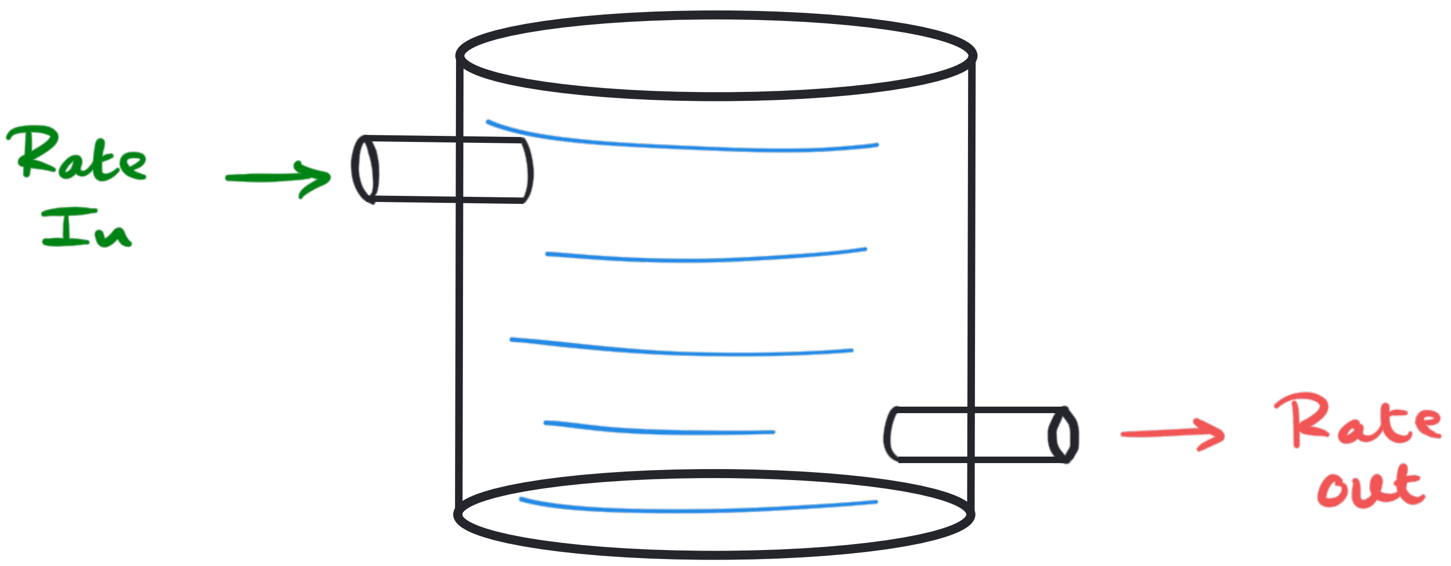
$$\frac{dy}{dx} = \frac{-1}{g(x,y)}.$$

3) Solve this differential equation to find the equations for the orthogonal trajectories.

Example: Find the orthogonal trajectories to the family of curves $x = ky^2$.

Mixing Problems:

In these problems a container begins with an initial concentration of a certain substance. A solution of a, usually different, concentration is pumped into the container at a certain rate in L/min. The solution is kept thoroughly mixed and pumped out at the same rate in L/min.



Key Point:

$$\boxed{\text{Rate of change of amount} = \text{Rate in} - \text{Rate Out}}$$

Method: Let $A(t)$ = Amount of substance in tank @ time t.

1) Rate In = (Concentration of Incoming Mixture)(Flow Rate of Incoming Mixture)

2) Rate Out = (Concentration of Outgoing Mixture)(Flow Rate of Outgoing Mixture)

3) Concentration of Outgoing Mixture $= \dfrac{A(t)}{\text{Vol (Tank)}}$

4) $\dfrac{dA}{dt}$ = Rate In − Rate Out

5) Solve, remembering to take care if $A(t)$ is increasing or decreasing and using the intial condition to get the particular solution.

Example: A vat at the Guinness brewery has a volume of 500 gallons and contains beer with a strength of 3% ethanol.

Fireball whiskey with a strength of 33% ethanol is pumped into the vat at a rate of 2 gallons a minute. The mixture is kept thoroughly mixed and drains at the same rate.

What is the percentage ethanol of the mixture in the vat after an hour?

Exercises :

1) Solve the differential equation $xy + \frac{dy}{dx} = 100x$.

2) Solve the I.V.P. $\frac{dy}{dx} = \frac{y \sin x}{y^2 + 1}$, $y(0) = 1$.

3) Solve the I.V.P. $\frac{dy}{dx} = ye^x$, $y(0) = 2$.

4) Find the orthogonal trajectories to the family of curves $y = Kx^3$.

5) A tank initially contains 80 litres of water with 12 Kg of salt dissolved in it. A solution with 0.3 Kg of salt is pumped into the tank at a rate of 5 litres per minute. The solution is kept thoroughly mixed and drains at the same rate. How much salt is in the tank after an hour?