17. Applications of Taylor Series :

(I) Approximations :

Definition : Given a function f, a real number a and a positive integer K, we define the **Kth Taylor Polynomial for f at a** to be :

$$\boxed{\begin{aligned} T_f^{a,K}(x) &= \sum_{n=0}^{K} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \dots + \frac{f^{(K)}(a)}{K!}(x-a)^K \end{aligned}}$$

and the **Kth Remainder for the Taylor Series** :

$$\boxed{R_K(x) = f(x) - T_f^{a,K}(x)}$$

Example : $f(x) = e^x$. Find $T_f^{1,4}(x)$.

Taylor's Theorem :

If $|f^{(k+1)}(x)| \leq B_k$ for all $|x-a| \leq r$, then

$$\boxed{|R_k(x)| \leq \frac{B_k}{(k+1)!} |x-a|^{k+1}}$$

for all $|x-a| < r$.

Remark : This helps you measure how accurate of an approximation "$f(x) \approx T_f^{a,k}(x)$" is.

Example : Use Taylor's Theorem to find a k such that $T_f^{1,k}(x)$ will approximate $f(x) = e^x$ on the interval $[0,2]$ to an accuracy of $\frac{1}{10}$.

(B) Approximating Integrals

Example : Approximate $\int_0^1 e^{-x^2}\,dx$ to an accuracy of $\frac{1}{1000}$.

(c) Evaluations :

Examples :

① Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\pi^{2n+1}}{2^{2n+1}}$

② Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)\,2^{n+1}}$.

(D) Limits :

Examples:

① $$\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2}$$

② $$\lim_{x \to 0} \frac{\sin(x^3) - x^3 + \frac{x^9}{3!}}{x^{15}}$$

Exercises:

1. Let $f(x) = e^x$.

   a. How accurate is $T_f^{1,4}(x)$ at approximating e^x on $[-4,4]$?

   b. Find an interval around 0 where $T_f^{1,4}(x)$ approximates e^x to within $\frac{1}{1000}$.

2. Let $g(x) = \cos x$.

   a. Find the 3rd Taylor polynomial for g at $a = \pi/2$.

   b. Use A.S.E.T. to determine the maximum possible error for this approximation on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

3. Let $h(x) = \sqrt{1+x}$. Find $T_h^{0,3}(x)$.

   Use this to approximate $\displaystyle\int_{-1/10}^{1/10} \sqrt{1+x}\, dx$.

4. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\pi^{2n}}{36^n}$

5. $\displaystyle\lim_{x \to 0} \frac{\ln(1-x^2) - x^2 - \frac{x^4}{2}}{x^6}$