16. Taylor and Maclaurin Series

Definition: We say a function f has a power series representation at a if there are $\{c_n\}$ and an $R > 0$ such that:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for all } |x-a| < R$$

Question: Given a function and a real number a, can we find a power series representation, centred at a, for that function?

Answer: We can try.

Attempt: Let's say we could. So:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Definition: Given a smooth function f and a real number a, we define the **Taylor Series for f at a** to be:

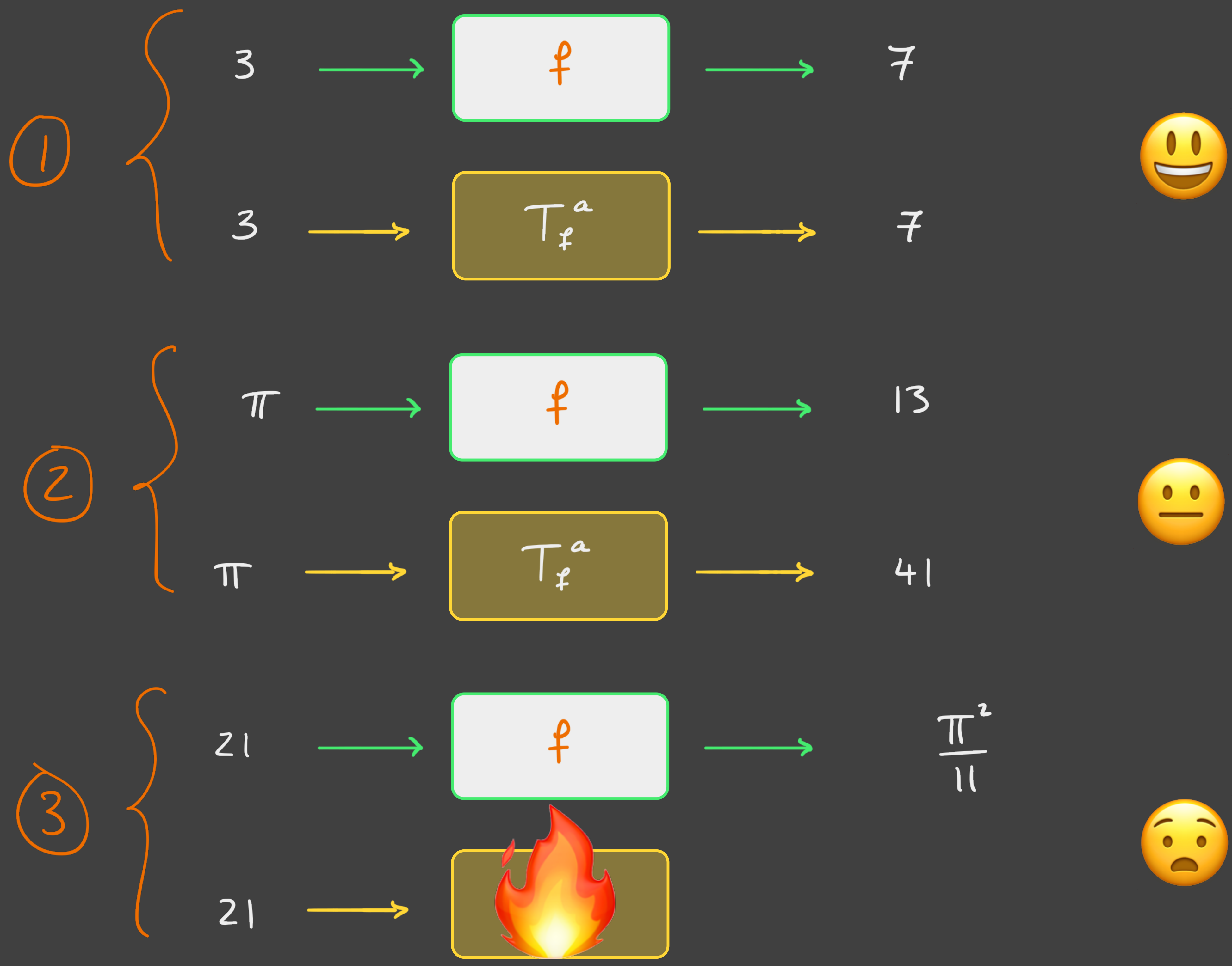
$$\boxed{T_f^a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Definition: Given a smooth function f, the **MacLaurin Series for f** is defined to be:

$$\boxed{M_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n}$$

Moral:

Potential Scenarios :



Examples :

Find the MacLaurin Series for :

1. e^x

2. $\sin(x)$

Question: When does $f(x) = T_f^a(x)$?

Definition: Given a function f, a real number a and a positive integer k, we define the **kth Taylor Polynomial for f at a** to be:

$$\boxed{\begin{aligned} T_f^{a,k}(x) &= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}}$$

and the **kth Remainder for the Taylor Series**:

$$\boxed{R_k(x) = f(x) - T_f^{a,k}(x)}$$

Theorem: If $\lim_{k\to\infty} R_k(x) = 0$ then $f(x) = T_f^a(x)$.

Taylor's Theorem :

If $|f^{(k+1)}(x)| \leq B_k$ for all $|x-a| \leq r$, then

$$\boxed{|R_k(x)| \leq \frac{B_k}{(k+1)!} |x-a|^{k+1}}$$

for all $|x-a| < r$.

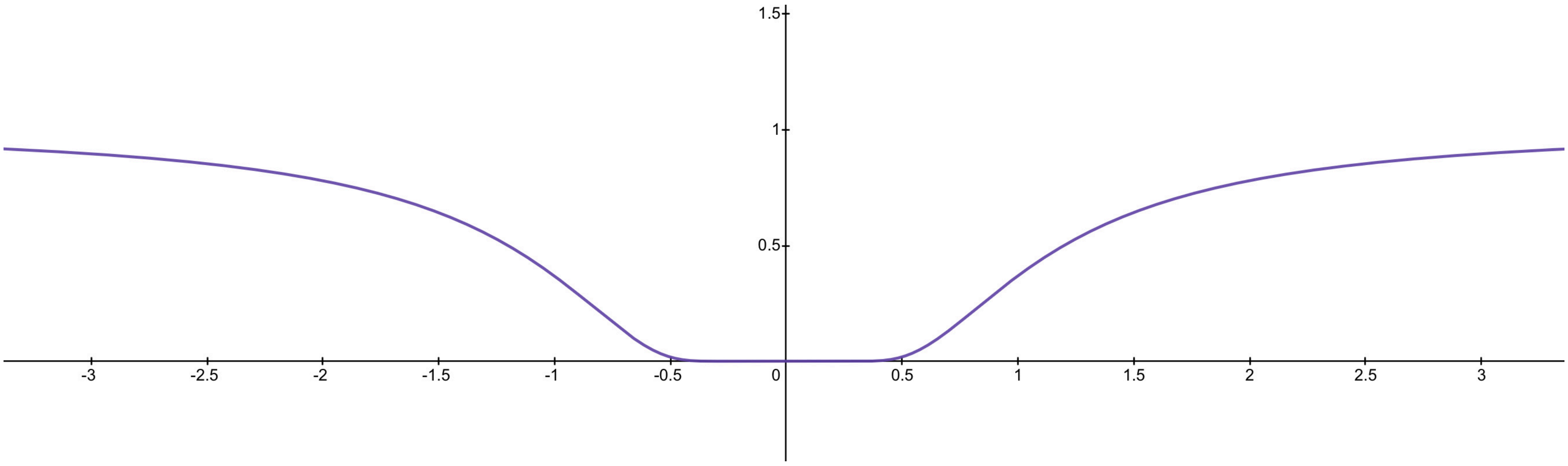
Examples :

1. Show that $\sin(x)$ agrees with its Maclaurin on $(-\infty, \infty)$.

2. Show that e^x agrees with its Maclaurin on $(-\infty, \infty)$.

A weird example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Famous Taylor Series :

1) $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$, if $|w| < 1$.

2) $\ln|1-w| = -\sum_{n=0}^{\infty} \frac{1}{n+1} w^{n+1}$, if $|w| < 1$.

3) $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$, for any w in $(-\infty, \infty)$.

4) $\sin(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$, for any w in $(-\infty, \infty)$.

5) $\cos(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n}$, for any w in $(-\infty, \infty)$.

Method of Substitution (revisited):

Examples: Find Taylor Series for the following functions at the given centres and state where they agree.

1. $f(x) = \cos\left(\frac{x^2}{2}\right)$, $a = 0$

2. $g(x) = \ln\left|1 + \frac{x^3}{3}\right|$, $a = 0$.

Exercises:

1. Find the Taylor Series for e^x at $a = 1$.

2. Find the MacLaurin Series for $\cos(x)$. Where do they agree?

3. Use your answers to the above to compute the limit:

$$\lim_{x \to 0} \frac{\cos(x^5) - 1}{x^{10}}$$

4. Find a power series representation for e^{-x^2}. Where is it valid?

5. Use your answer to the above and A.S.E.T. to estimate $\int_0^1 e^{-x^2}\,dx$ to within $\frac{1}{1000}$.