15. Representations of Functions as Power Series

Important Point: The same function can have many different representations of varying 'success'.

Ex:

Goal: To represent certain functions as power series and determine where these representations are valid.

A Very Important Example: (Geometric Series)

$$f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \quad ; \quad f(x) = \frac{1}{1-x}$$

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n , \quad \text{for } |x| < 1}$$

Remark:

Method of Substitution :

Examples: Find a power series representation for :

1. $$g(x) = \frac{1}{1+x^3}$$

2. $$h(x) = \frac{x^3}{1+x}$$

3. $$j(x) = \frac{1}{2+x}$$

Remark :

Differentiation and Integration of Power Series:

Theorem: Assume $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$. Then, the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is differentiable on $(a-R, a+R)$ and

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + \dots}$$

$$\boxed{\begin{aligned}\int f(x)\,dx &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} \\ &= C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \dots\end{aligned}}$$

Remark: The R.O.C. for both of these power series is also R.

Moral : We can differentiate and integrate power series term by term on $(a-R, a+R)$.

Example : Find a power series representation for the function $f(x) = \frac{1}{(1-x)^2}$ and state where it is valid.

Exercises: Find P.S. representations & state where they are valid:

1. $$\frac{x^2}{4-x}$$

2. $\ln(1+x)$

3. Show $1 - 1/2 + 1/3 - 1/4 + \cdots = \ln 2$.

4. Use power series to approximate

$$\int_0^{1/10} \frac{1}{1+x^7}\, dx$$

to an accuracy of $\frac{1}{1000}$.

5. (i) $\tan^{-1}(x)$

(ii) Calculate $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{(\sqrt{3})^{2n+1}(2n+1)}$.