14. Power Series

Definition: A Power Series centred at 0 is a function of the form

$$\boxed{\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots}$$

where x is a variable and the $\{c_n\}$ are constants called the coefficients of the series.

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots$

Very Important Remark: Power series may converge for some x and diverge for others.

Example continued:

$x = 1$:

$x = 2$:

Question: For what values of x will $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

Definition: A Power Series centred at a is a function of the form

$$\boxed{\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots}$$

where x is a variable and the $\{c_n\}$ are constants called the coefficients of the series.

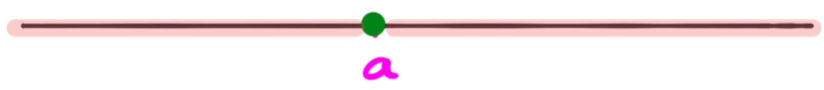
Example : $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n3^n}(x-1)^n$

Remark : A power series centred at a will always converge at $x = a$:

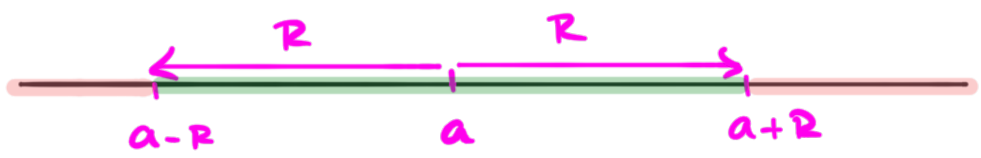
Theorem: $\sum_{n=0}^{\infty} C_n (x-a)^n$ will converge either:

① For all $x \in \mathbb{R}$:

② Only at $x = a$:



③ On an interval of width $2R$ centred at a:



Extremely Important Remark:

The power series may or may not converge at the endpoints of the interval. You have to check these points individually.

Definitions :

① The value R is called the **Radius of Convergence : R.O.C.**

② The collection of values for which $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is called the **Interval of Convergence I.O.C.**

Remark :

Example : What is the R.O.C. and I.O.C. of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$?

Method : Given a power series centred at a $\sum_{n=0}^{\infty} c_n (x-a)^n$, we will almost always find its R.O.C. & I.O.C. using the Ratio Test:

① Here $a_n = c_n (x-a)^n$.

② So, for $x \neq a$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n}\right|$$

$$= \left|\frac{c_{n+1}}{c_n}\right| |x-a|$$

③ $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| |x-a|$$

④ By the Ratio Test, if

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right||x-a| \begin{cases} <1, & \sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges absolutely} \\ =1, & \text{inconclusive} \\ >1, & \sum_{n=0}^{\infty} c_n(x-a)^n \text{ diverges} \end{cases}$$

This will give you convergence for $|x-a| < R$ where R is the R.O.C.

Remark: R may be ∞ or 0.

Tip:

⑤ To find I.O.C. you need to check each endpoint $x = a - R$ and $x = a + R$ individually.

Example: Find R.O.C. and I.O.C. of

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} (x-\pi)^n$$

Exercises :

1. $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1}$$

3. $$\sum_{n=0}^{\infty} \frac{1}{(n+1)\,4^n} (x+2)^n$$

4. $$\sum_{n=0}^{\infty} n!\,(x+1)^n$$

5. $$\sum_{n=0}^{\infty} \frac{\cos(1/n)}{(n+1)^2} (x+e)^n$$