13. Ratio & Root Test

Definition: A series $\sum a_n$ is called **Absolutely Convergent** if $\sum |a_n|$ converges.

Examples:

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n}$

Definition: A series is called **conditionally convergent** if it is convergent, but not absolutely convergent.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

Theorem: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

i.e. $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges.

"Proof":

Examples:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Theorem: (Ratio Test)

For a series $\sum a_n$, with $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, then

$$\text{if } \begin{cases} L < 1 & \Rightarrow \sum a_n \text{ is absolutely convergent.} \\ L = 1 & , \text{ the test is inconclusive} \\ L > 1 & \Rightarrow \sum a_n \text{ diverges} \end{cases}$$

Remarks:

1. If $L = \infty$, $\sum a_n$ diverges.
2. This test is useful for series with terms involving exponentials and factorials.

Proof:

Examples :

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n!}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Theorem: (Root Test)

For a series $\sum a_n$, with $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$, then

$$\text{if } \begin{cases} L < 1 \Rightarrow \sum a_n \text{ is absolutely convergent.} \\ L = 1 \text{, the test is inconclusive} \\ L > 1 \Rightarrow \sum a_n \text{ diverges} \end{cases}$$

Remarks:

1. If $L = \infty$, $\sum a_n$ diverges.
2. This test is useful for series of the form $\sum_{n=1}^{\infty} (b_n)^n$.

Proof :

Examples:

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{2n}{n+1}\right)^n$

(2) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

(3) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{n}\right)^n$

Exercises:

(1) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+n}{2n^2+1} \right)^n$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n}$

(3) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n+1} \right)^n$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

(5) Which of the following converge conditionally?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln(n)}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(6) Show $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n)}{n^2+1}$ converges by showing it is absolutely convergent.

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$