12. Alternating Series

Remark: Our previous lectures on the Integral Test, the Comparison Test and the Limit Comparison Test only apply to Series with positive terms $a_n \geq 0$.

Definition: An Alternating Series is a series of the form:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

where $b_n \geq 0$.

Remark: These series are called alternating because

Examples :

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{3^{n+1}}$

(3) $\displaystyle\sum_{n=2}^{\infty} (-1)^{n} \frac{n^2}{5n^2+1}$

Theorem: (Alternating Series Test)

If the alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \dots \quad , \quad b_n \geq 0$$

satisfies:

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

Proof: If we consider the partial sums:

$S_1 =$

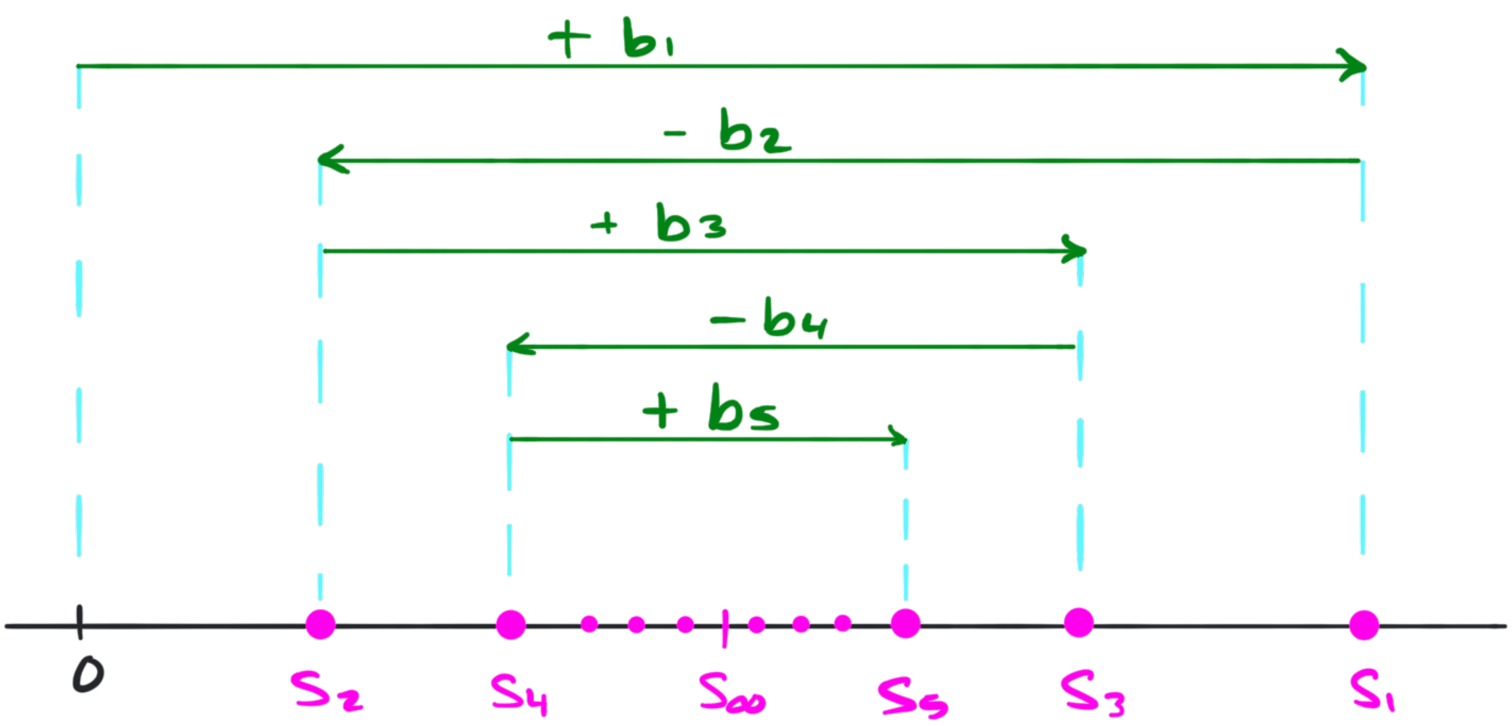
$S_2 =$

$\vdots$

$S_N =$

$S_{N+1} =$

and follow these on a numberline:



As the b_ns decreasing, change sign and shrink to 0, the pink dots get "trapped" in smaller and smaller gaps.

i.e $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n = S_\infty$ converges.

$\boxtimes$

Remarks:

(1) We only need the b_ns to 'eventually' be decreasing.

(2) If $b_n = f(n)$ and $f(x)$ 'makes sense', then

$f'(x) < 0 \implies \{b_n\}$ is decreasing.

Examples:

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n!}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} 2n^2}{n^2+1}$

Remark : $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ ________________ .

$\sum_{n=1}^{\infty} \frac{1}{n}$ ________________ .

Definition : A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if $\sum a_n$ converges and $\sum |a_n|$ diverges.

Theorem : (Alternating Series Estimation Theorem)

If an alternating series $\sum (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \dots$,

with $b_n \geq 0$ satisfies :

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n \to \infty} b_n = 0$

then :

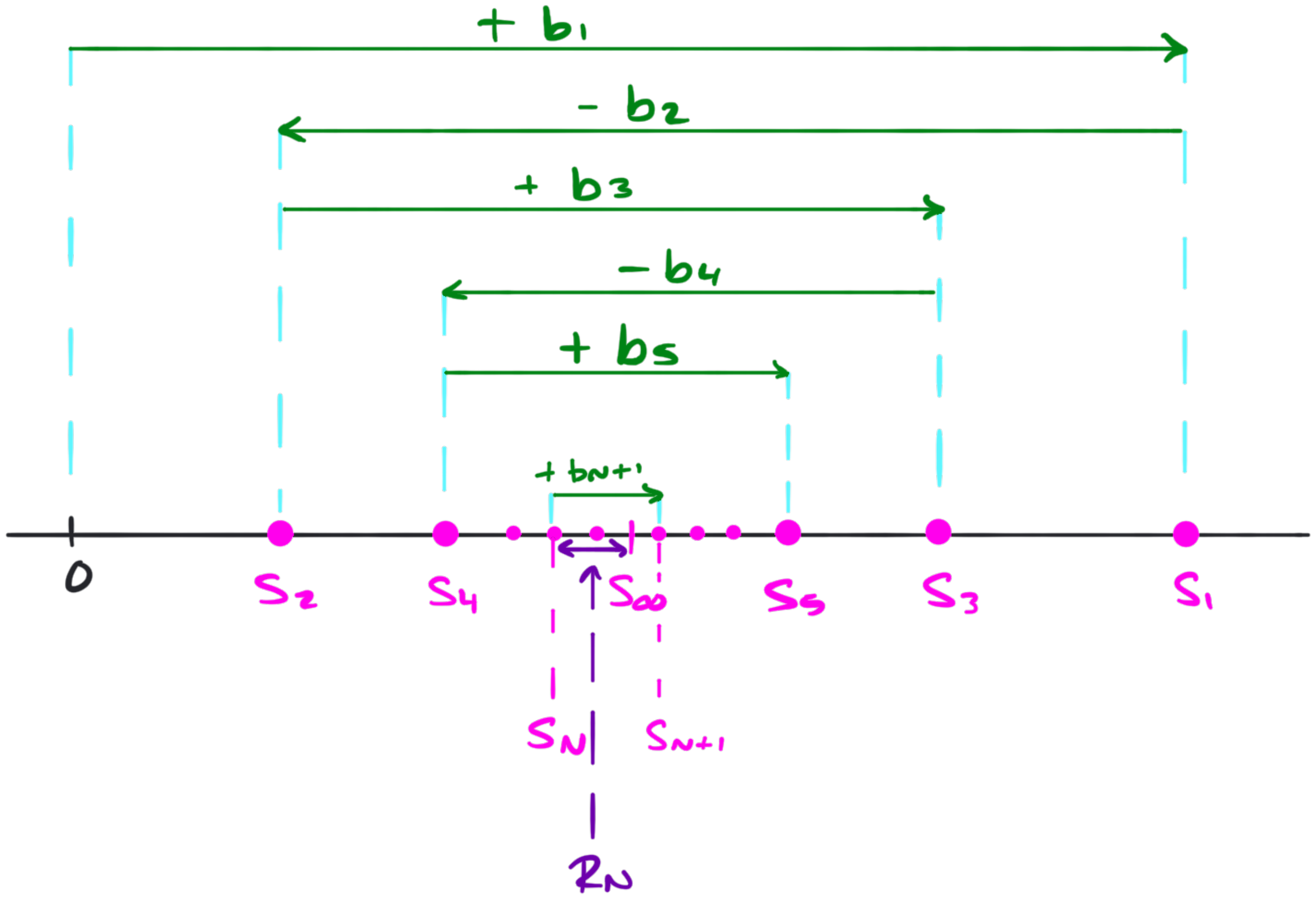
$$|R_N| = |S_\infty - S_N| \leq b_{N+1}$$

where R_N is called the Nth remainder .

Remark : R_N measures the error in the approximation:

$$\text{“}\sum_{n=1}^{\infty}(-1)^n b_n \approx \sum_{n=1}^{N}(-1)^n b_n\text{”}$$

Proof of A.S.E.T. :



Example: Estimate $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ to within $\frac{1}{10}$.

Exercises:

Determine if the following alternating series converge:

(1) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

(2) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^2}$

(3) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

(4) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$, $p > 0$

(5) $\sum_{n=1}^{\infty} (-1)^n$

(6) $\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{\sqrt{n}}$

(7) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \sqrt{n} + 1}$

(8) Use the A.S.E.T. to find an N which guarantees $S_N \approx \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ to within an accuracy of $\frac{1}{1000}$.