11. Comparison Tests :

Goal : To develop tools to tell us if a given series $\sum a_n$ will converge or diverge.

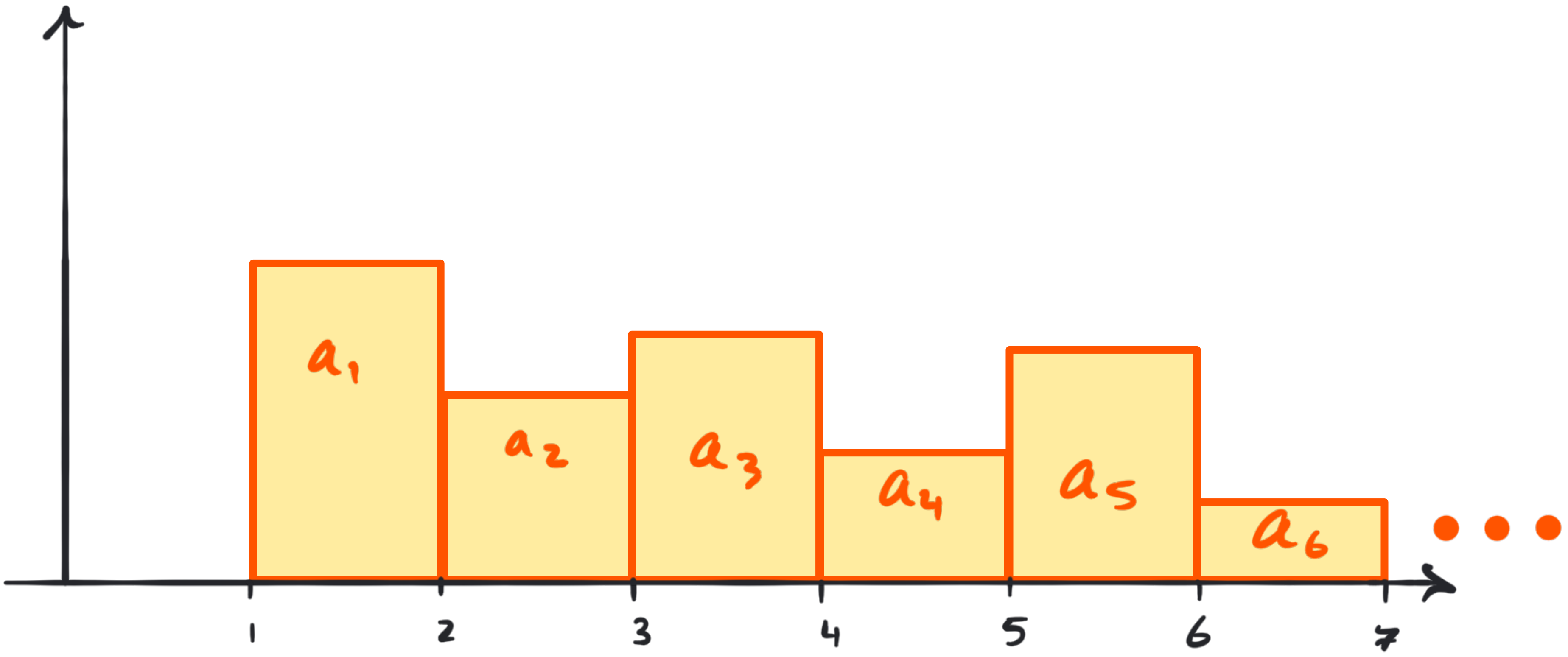
Theorem : (Comparison Test)

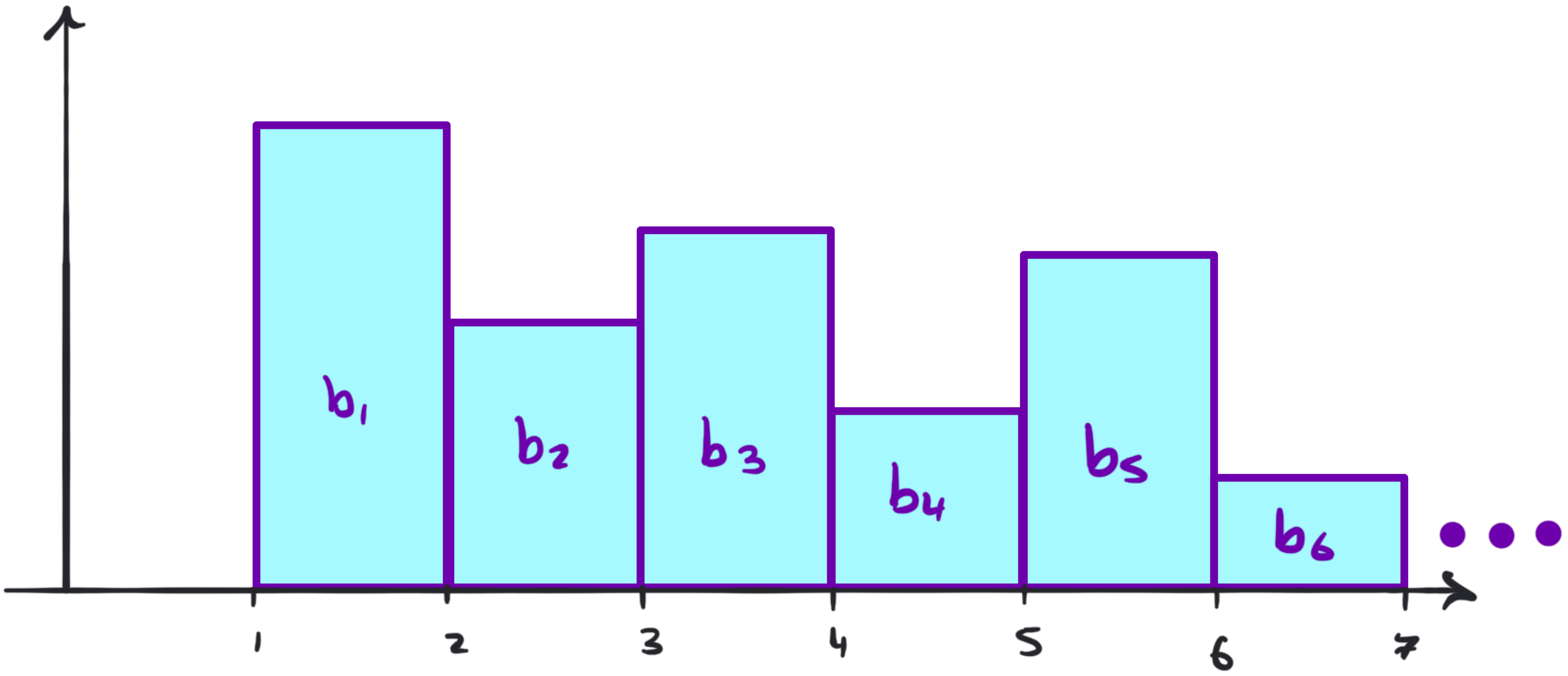
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms such that for all n : $0 \leq a_n \leq b_n$.

Then :

1. If $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

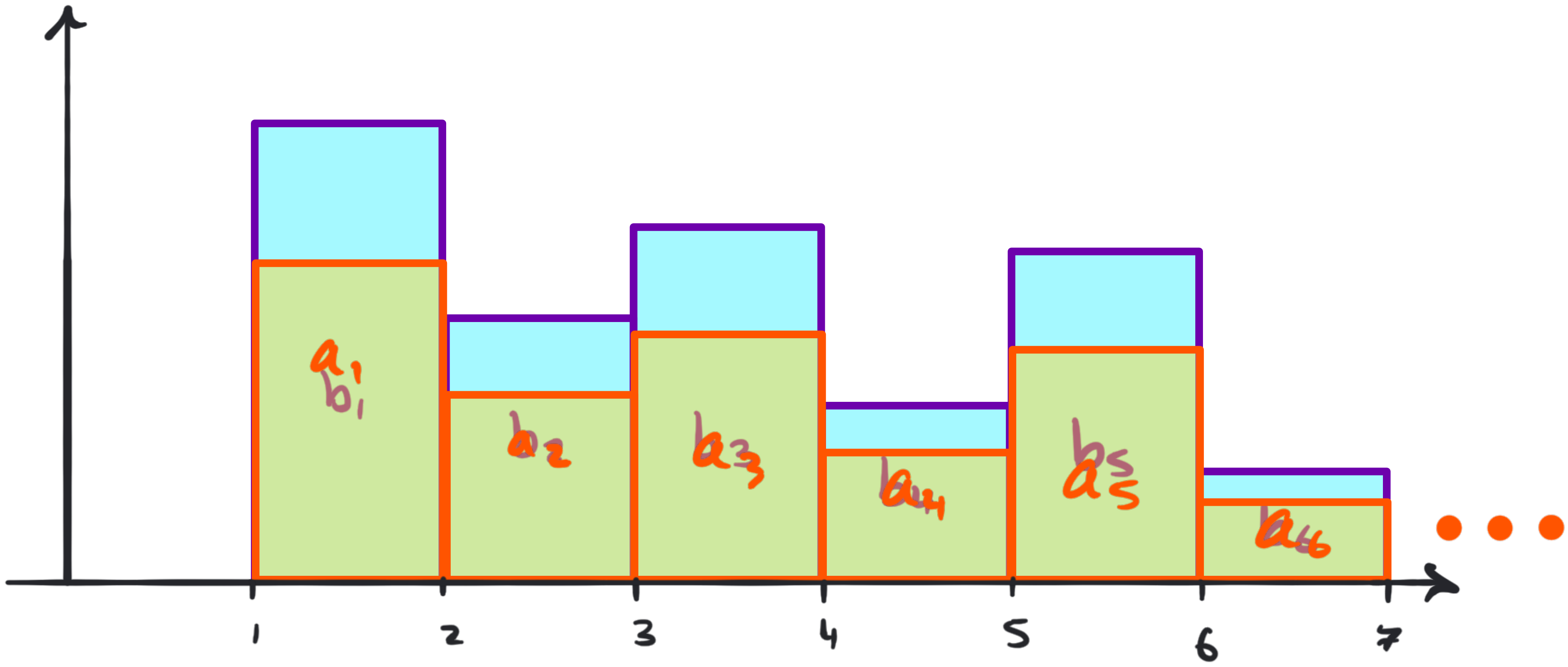
Proof : As a_n and b_n are positive, we can visualise them as areas :





Important Observation: The condition that

$0 \leq a_n \leq b_n$ means :



So $\sum a_n \le \sum b_n$.

The theorem follows. $\boxtimes$

Examples:

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Remark: When thinking about the convergence of any series, we only ever need to 'think about the tail'.

i.e.

Theorem: (Limit Comparison Test)

If $\sum a_n$ and $\sum b_n$ are series with positive terms such that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$$

Then either $\sum a_n$ and $\sum b_n$ both converge or they both diverge.

Proof: If $\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$, then, for large n:

$$a_n \approx C\, b_n$$

So $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N}^{\infty} a_n$$

$$\approx \sum_{n=1}^{N} a_n + \sum_{n=N}^{\infty} C\, b_n$$

$$= \sum_{n=1}^{N} a_n + C \sum_{n=N}^{\infty} b_n$$

$\boxtimes$

Examples:

(1) $$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n+1}$$

(2) $$\sum_{n=1}^{\infty} \frac{3^n+n^{13}+1}{\ln(n)+9^n}$$

Exercises:

(1) $$\sum_{n=1}^{\infty} \frac{n^3 + n^2 + 9}{3n^3 + n}$$

(2) $$\sum_{n=1}^{\infty} \frac{\cos^2 n + 1}{2n^3}$$

(3) $$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$$

(4) $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

(5) $$\sum_{n=1}^{\infty} \frac{2^n}{3^n n!}$$

(6) $$\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$$

(7) $$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{2n}\right) + \ln(n) + n^2}{\sqrt{n^7} + 91}$$