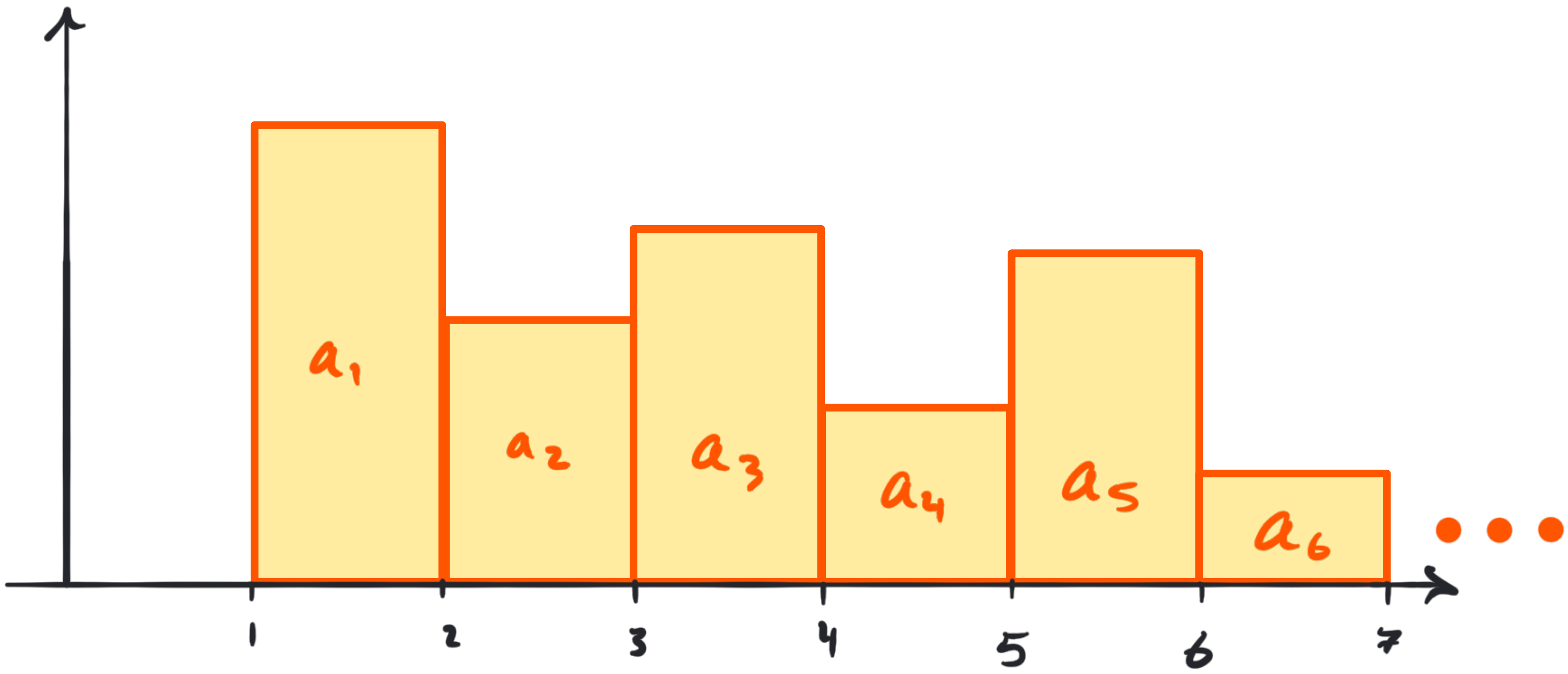
10. Integral Test

Remark: For a sequence of positive terms, $\{a_n\}$, we can think of the corresponding series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

as an area:



with the boxes of width 1 and height a_n.

Intuition:

Theorem: (Integral Test)

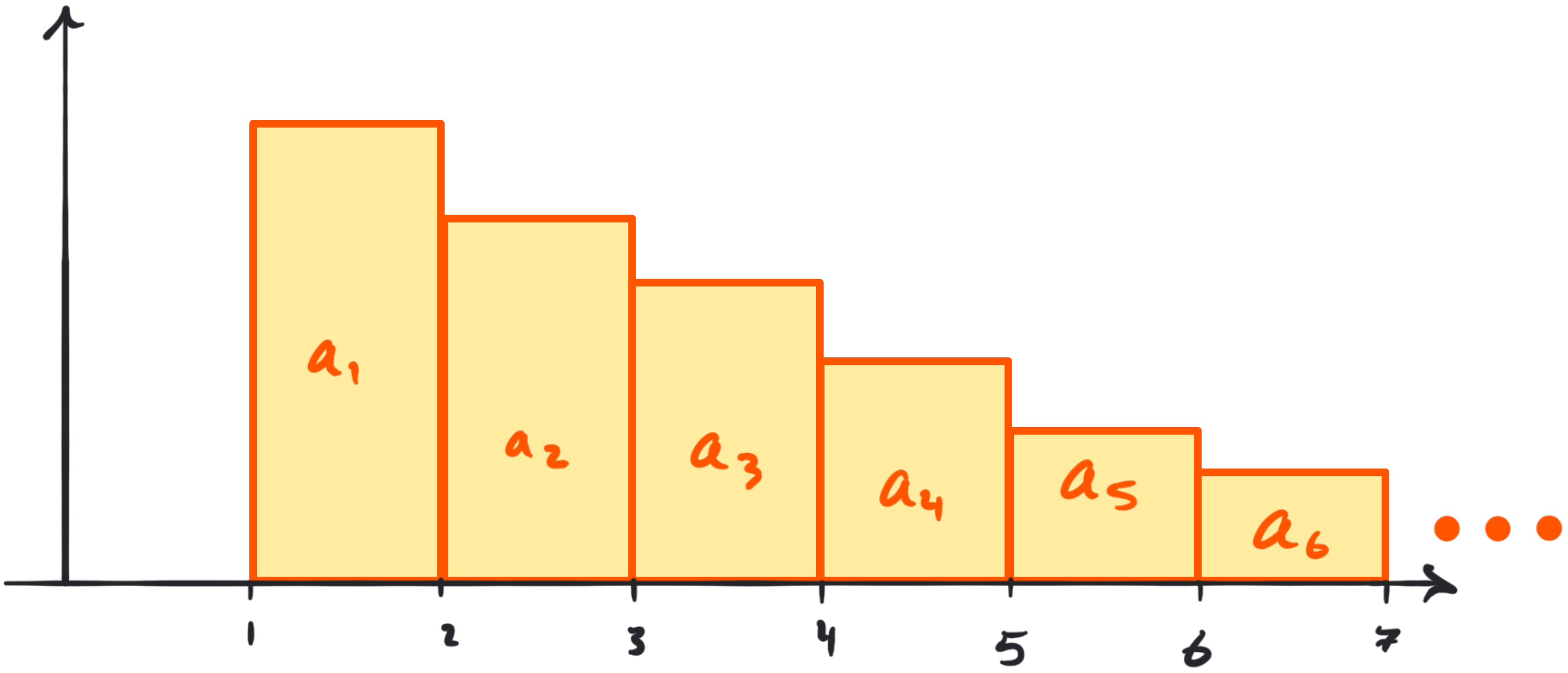
If $f(x)$ is a positive, decreasing, continuous function on $[1,\infty)$ with $f(n) = a_n$, then:

① If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.

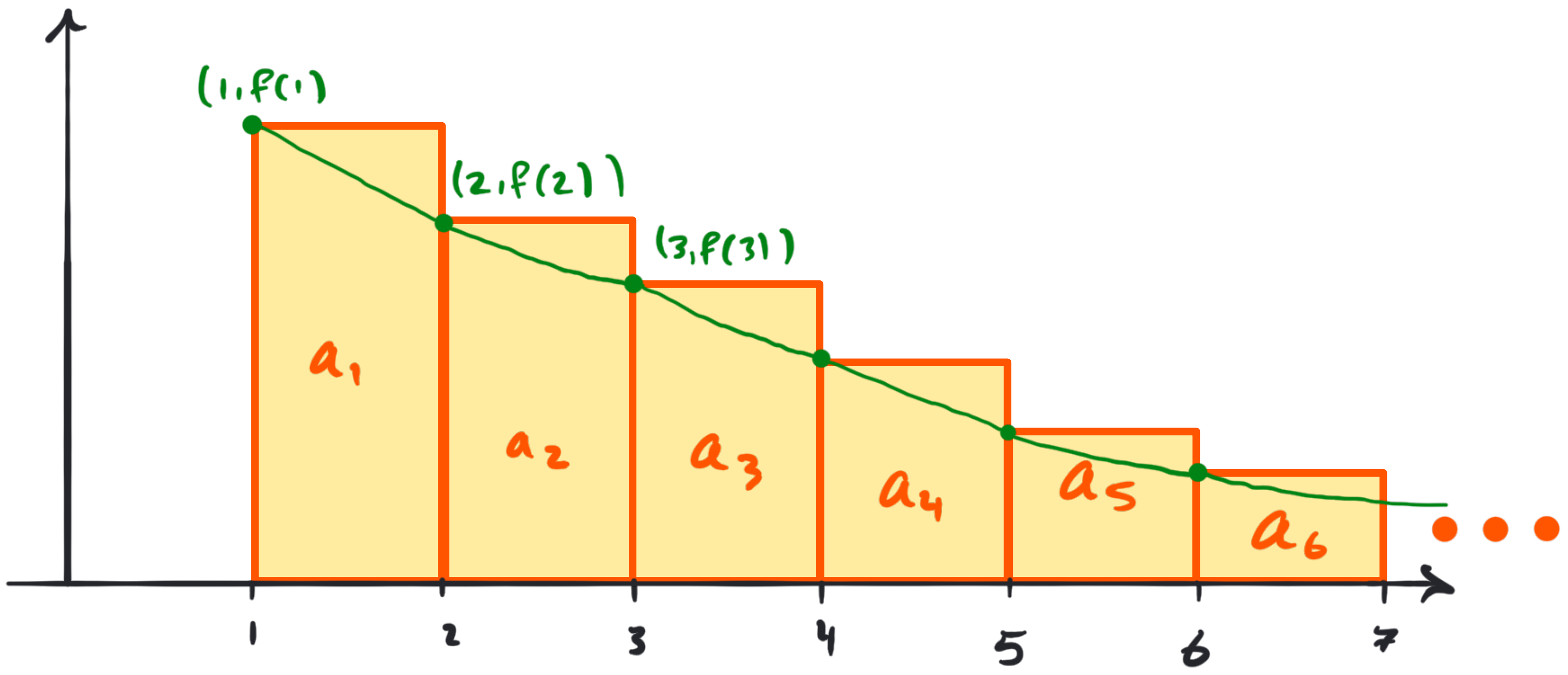
② If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Proof:

As $f(x)$ is decreasing, a_n must be decreasing.



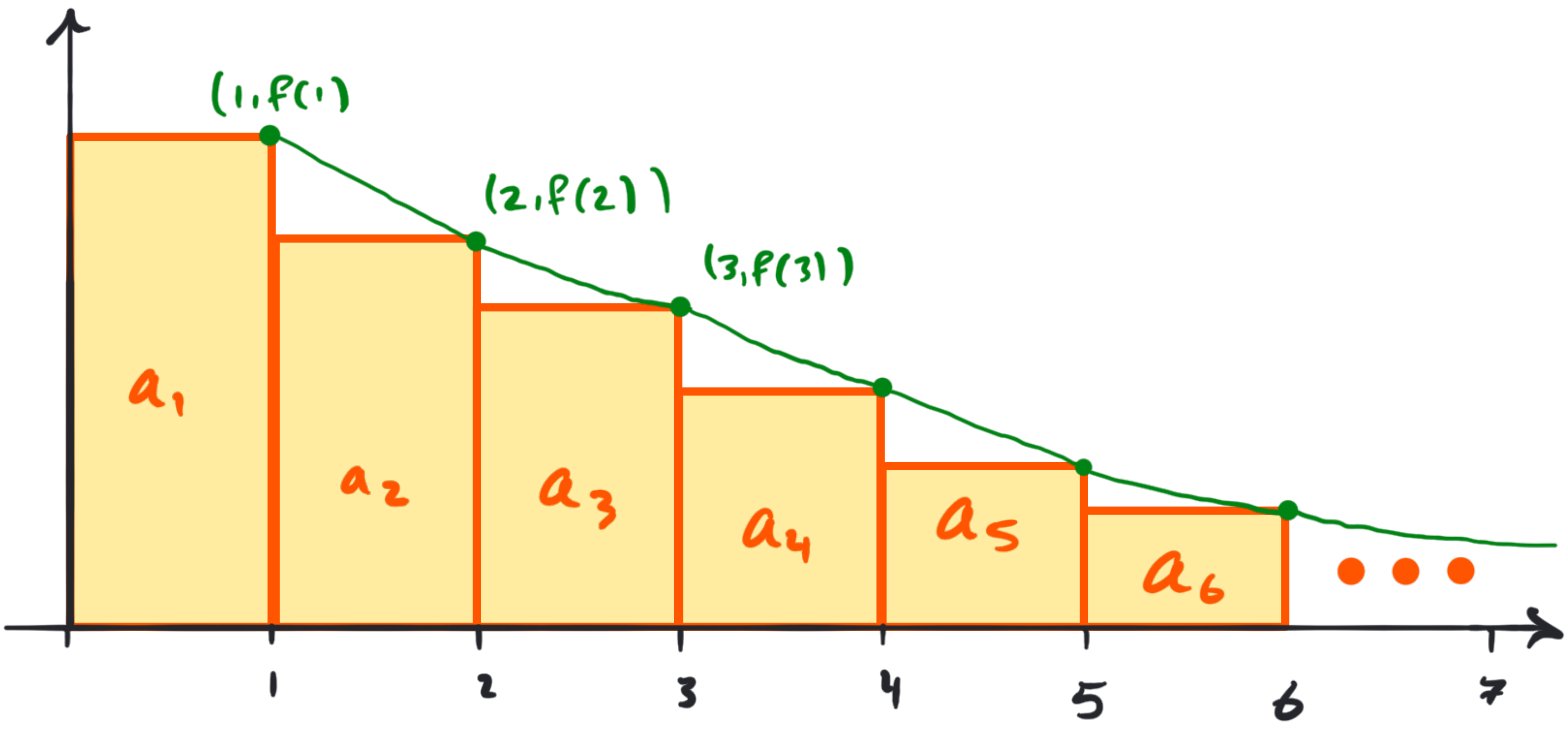
If we super impose the graph $y = f(x)$ onto this it must look like :



So :

$$\text{Area under graph} \leq \sum_{n=1}^{\infty} a_n$$

But , if we alter our perspective a little :



We see :

$$\sum_{n=2}^{\infty} a_n \leq \text{Area under graph}$$

$$\text{Area under graph} = \int_1^{\infty} f(x)\,dx$$

Hence , if :

① $\int_1^{\infty} f(x)\,dx$ converges , then $\sum_{n=1}^{\infty} a_n$ converges

② $\int_1^{\infty} f(x)\,dx$ diverges , then $\sum_{n=1}^{\infty} a_n$ diverges.

$\boxtimes$

Very Important Example: (p-series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$\boxed{\text{Converges}}$ if $\boxed{p > 1}$

$\boxed{\text{Diverges}}$ if $\boxed{p \leq 1}$

Proof: