1. Integration by Substitution:

We know how to deal with integrals where we can 'write down' an answer.

E.g. $\int 2x \, dx = x^2 + C$

$$\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = 1$$

"Golden Rule of Integration":

If you come across an integral where you can't 'write down' an answer, see if one part differentiates to another.

Why? Most integration techniques come from a 'twin' concept in differentiation.

Indefinite Integrals:

If $u(x)$ is a function of x, then:

$$\boxed{\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C}$$

OR

$$\boxed{\int f'(u(x))\,\frac{du}{dx}\,dx = \int f'(u)\,du = f(u(x)) + C}$$

Proof:

Examples:

1) $\int 3x^2 \cos(x^3)\, dx$

2) $\int x^3 \sqrt{x^4+5}\, dx$

3) $\int e^{\cos w} \sin w \, dw$

Linear substitutions:

"Linear substitutions cost very little."

Examples:

1) $\int \sin(4x+1)\,dx$

2) $\int e^{3x}\,dx$

Moral:

Definite Integrals:

$$\boxed{\int_a^b f'(g(x))\, g'(x)\, dx = f(g(b)) - f(g(a))}$$

OR

$$\boxed{\int_a^b f'(u)\, \frac{du}{dx}\, dx = \int_{u(a)}^{u(b)} f'(u)\, du = f(u(b)) - f(u(a))}$$

Remark:

Examples:

1) $\displaystyle\int_0^2 x e^{x^2}\, dx$

2) $\displaystyle\int_0^{\pi/4} \tan\theta\, d\theta$

3) $\displaystyle\int_{1}^{3} \frac{1}{5-x}\,dx$

Exercises:

1) $\int \frac{x}{1+x^2}\,dx$

2) $\int x\cos(x^2+1)\,dx$

3) $\int 25e^{-t/5}\,dt$

4) $\int_0^1 \frac{e^x}{1+e^x}\,dx$

5) $\int_0^{\pi/4} \cos(4x)\,dx$

6) $\int_1^4 \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

7) $\int_0^2 \frac{2x}{(1+x^2)^2}\,dx$

8) (Harder) $\int \sqrt{1+\sqrt{x}}\,dx$

9) (Harder) $\int (x+7)\sqrt[3]{3-2x}\,dx$