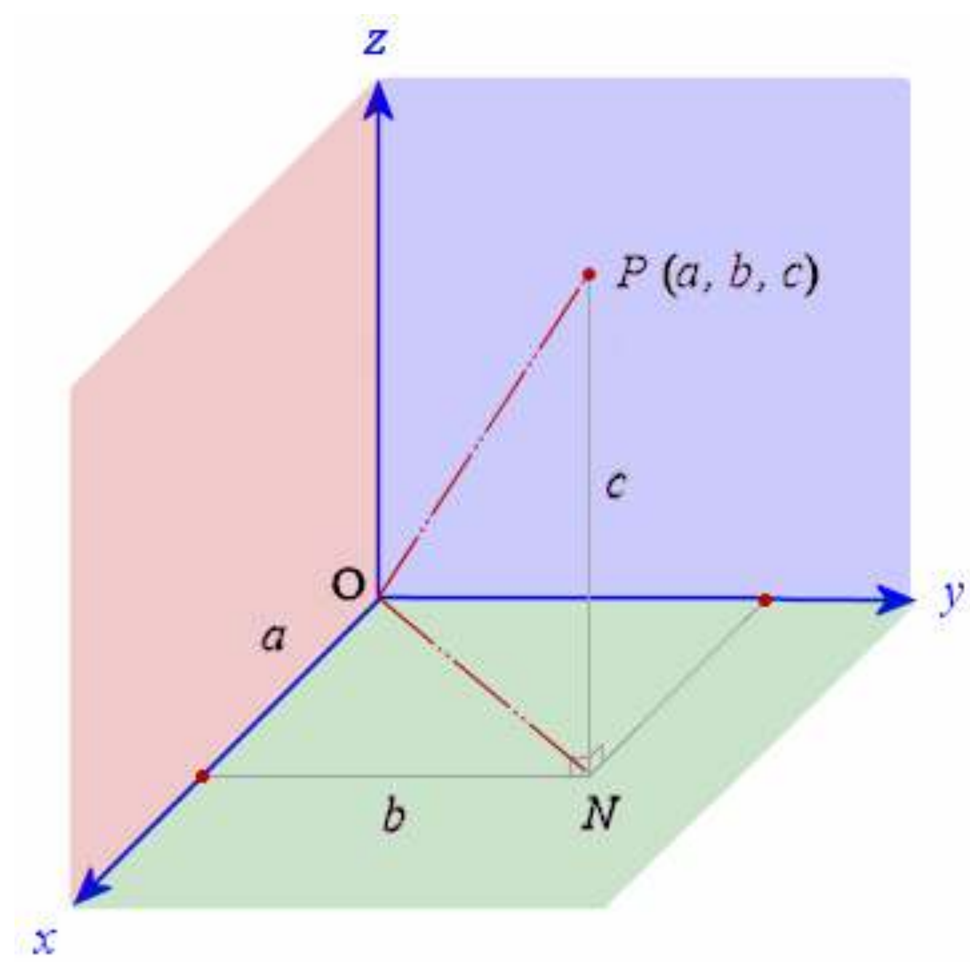
$\int$ Calc. III Supplementary Notes – Fall 2019 – Patrick Heslin

1. 3D - Coordinates:

Definition: As a set: $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

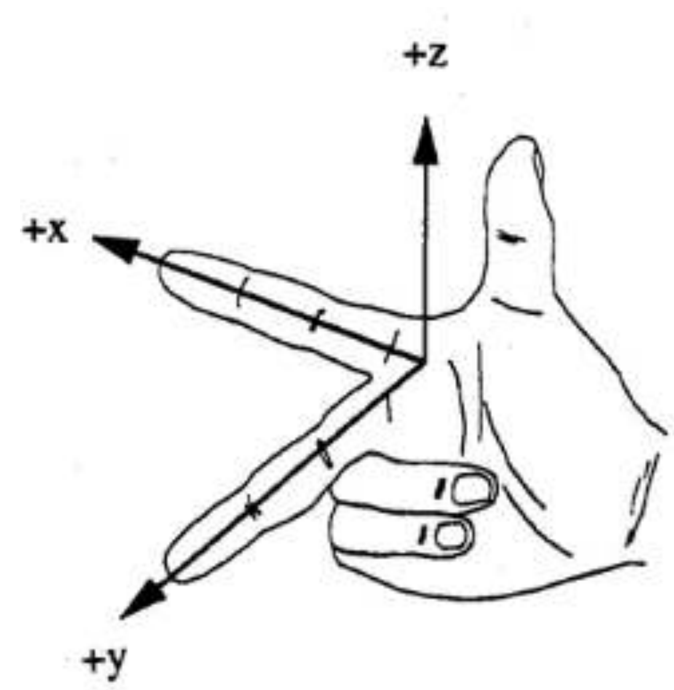
Visually:

"$\mathbb{R}^3 =$"



Remark: Think of $\mathbb{R}^3$ as the "inside" of an infinitely large "box". Similarly, think of $\mathbb{R}^2$ (the "xy-plane") as an infinitely large "sheet".

Right-Hand Rule:



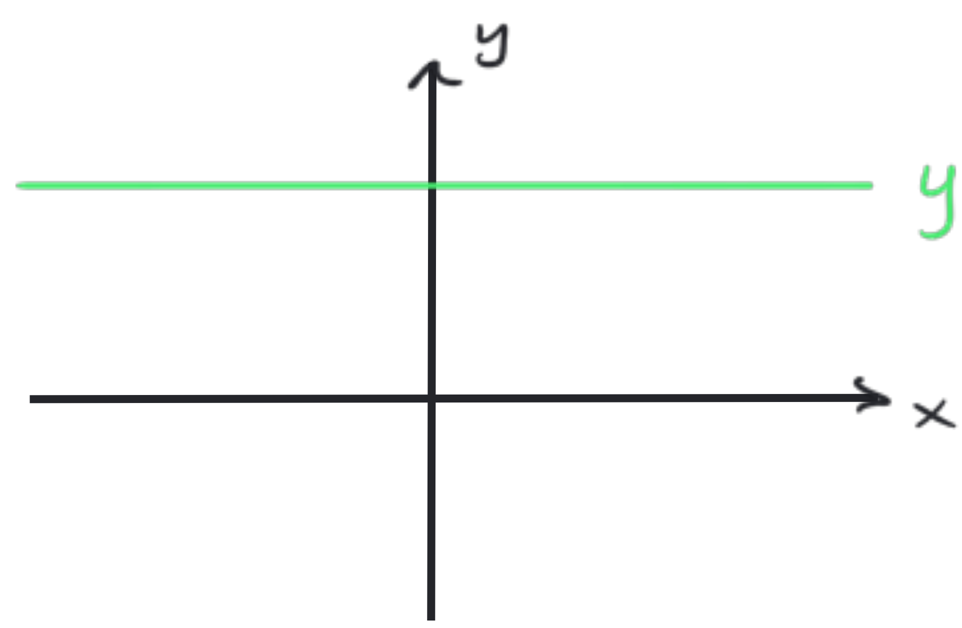
Question: If you were asked to draw $y=3$, what would that look like?

Answer: It depends.

↳ In $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$, this is all pairs (x,y) such that $y=3$.

i.e. $(1,3)$, $(\pi,3)$, $(671,3)$, ...

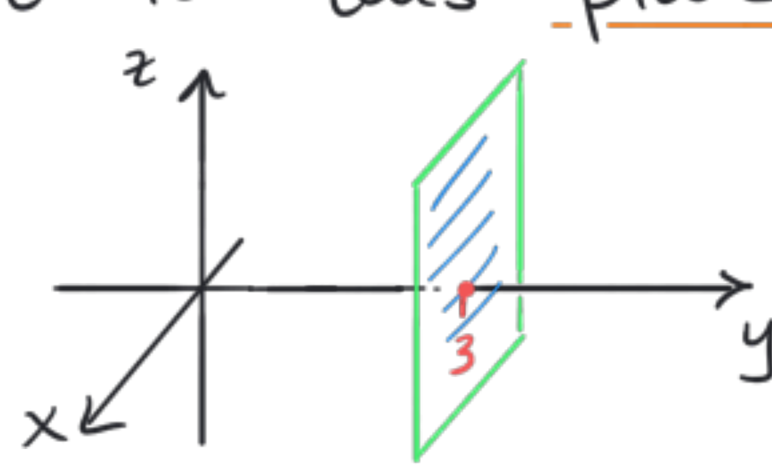
As a picture, it's this green line:



↳ In $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$, this is all (x,y,z), such that $y=3$.

i.e. $(1,3,1)$, $(\pi,3,9)$, ...

As a picture, it is this plane:



Remark: Equations in $\mathbb{R}^2$ usually lead to curves.

Equations in $\mathbb{R}^3$ usually lead to surfaces.

Distance Formula: For $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ in $\mathbb{R}^3$:

$$\boxed{d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}}$$

Exercises: 1) Draw all points in $\mathbb{R}^2$ that are 1 unit from the origin. Find an equation describing this set.

2) Do the same in $\mathbb{R}^3$.

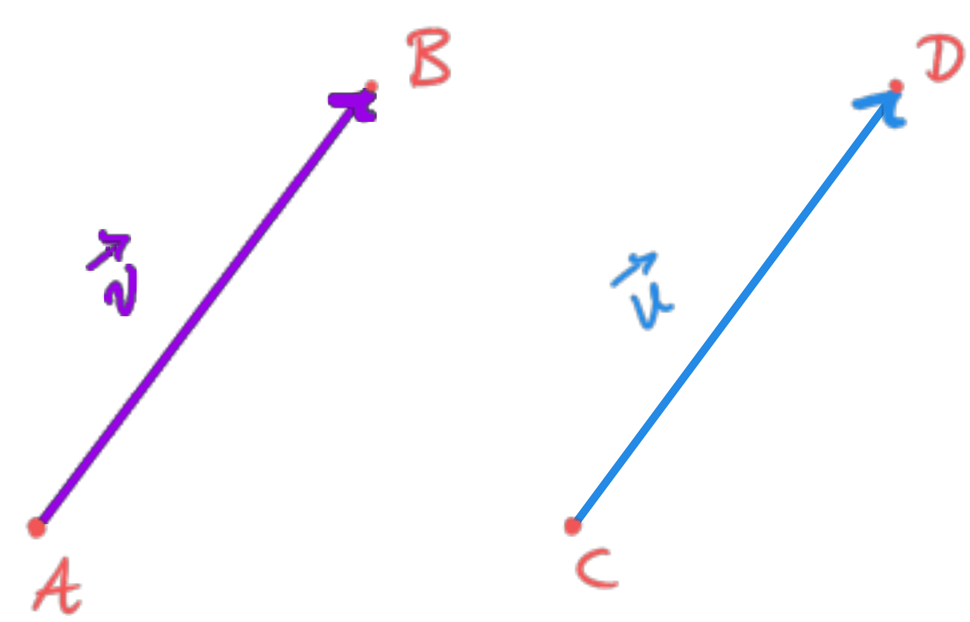
§2. Vectors :

When scientists talk about 'vectors', they are referring to something which has a 'magnitude' (or 'length') and a 'direction'.

Vectors can be denoted in multiple ways :

$$\vec{v}\ ,\ \mathbf{v}\ ,\ \underline{v}\ ,\ \ldots$$

Fig. 1 :



Consider the above vectors.

$\vec{v}$ moves from the point A to the point B.

We express this by writing $\vec{v} = \overrightarrow{AB}$.

Notice $\vec{u} = \overrightarrow{CD}$ has the same length and direction as $\vec{v}$.

We actually identify these vectors and write $\vec{v} = \vec{u}$.

Remark: The zero vector $\vec{0}$ has zero length and is the only vector with no specific direction.

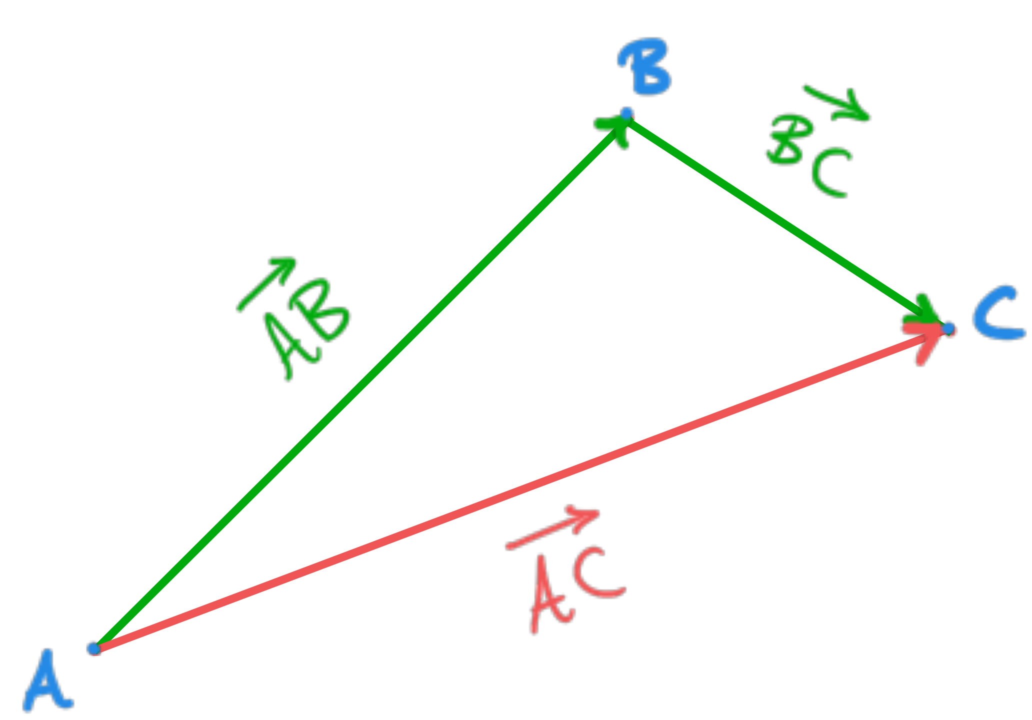
- Imagine a particle moves from a point A to a point B.

So its displacement vector is given by $\vec{AB}$.

Suppose the particle then moves from B to C.

This displacement vector is given by $\vec{BC}$.

Fig. 2:



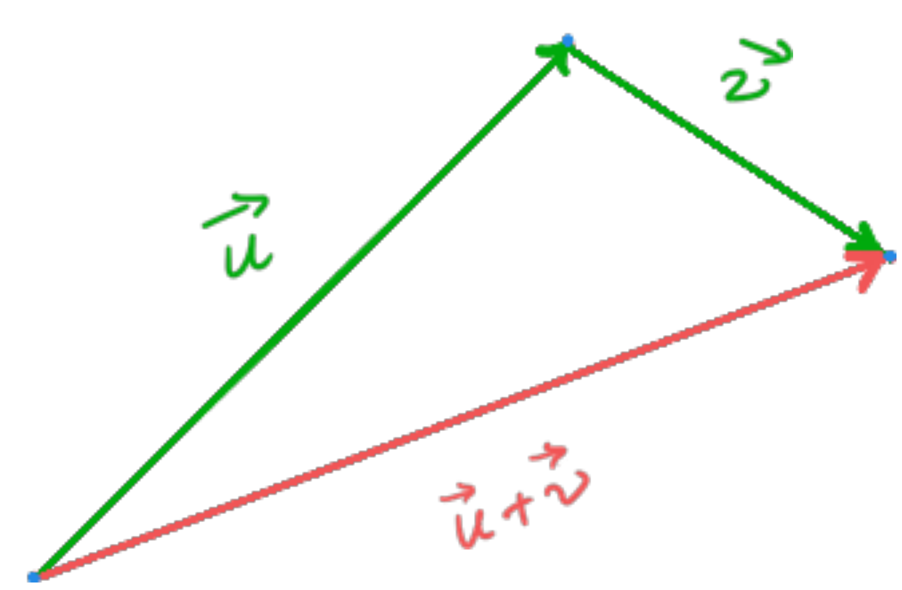
The resulting red vector $\vec{AC}$ is called the sum of $\vec{AB}$ and $\vec{BC}$ and we write:

$$\boxed{\vec{AC} = \vec{AB} + \vec{BC}}$$

Definition: (Vector Addition)

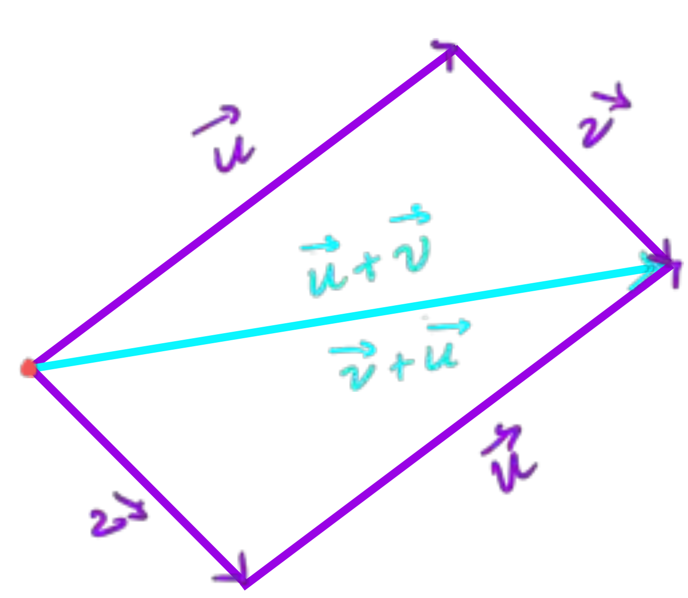
If $\vec{u}$ and $\vec{v}$ are vectors, positioned such that the initial point of $\vec{v}$ coincides with the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

Fig. 3: (Triangle Law)



We can see by symmetry that $\vec{u}+\vec{v} = \vec{v}+\vec{u}$:

Fig. 4: (Parallelogram law)



Question: What should we get if we add $\vec{v}$ and $\vec{v}$?

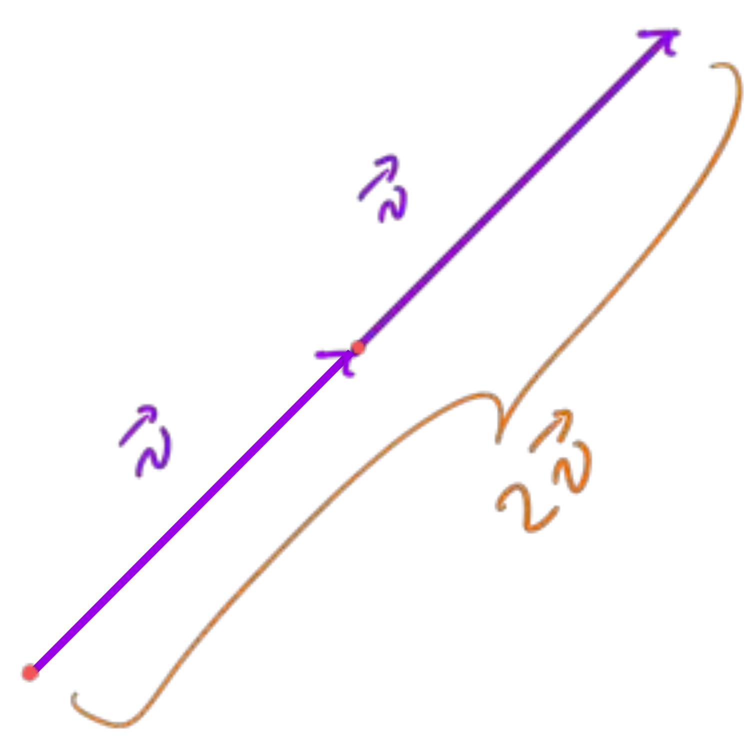
Algebraically, we should want: $\vec{v} + \vec{v} = 2\vec{v}$.

But what does "$2\vec{v}$" mean?

What does it mean to multiply a vector by a real number?

The answer is motivated by the geometry:

Fig. 5:



Direction? No change.

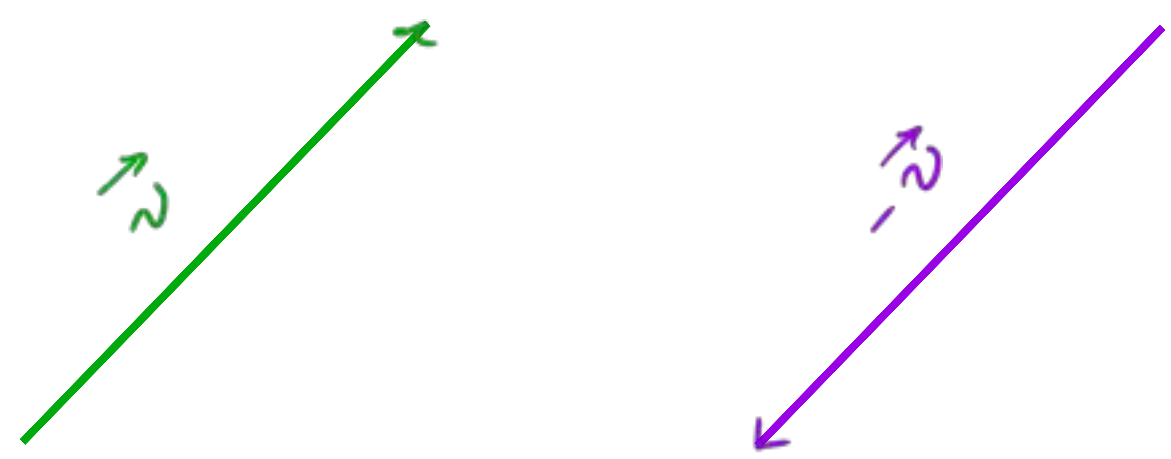
Length? Twice as long.

Question: What should $-\vec{v}$ be?

Algebraically, we should want $\vec{v} + (-\vec{v}) = \vec{0}$.

Geometrically, this means we want a vector, $-\vec{v}$, such that if we follow $\vec{v}$ and then $-\vec{v}$, our displacement is $\vec{0}$:

Fig. 6:



Direction? Opposite.

Length? Same.

This motivates our definition:

Definition: (Scalar Multiplication)

- If $c > 0$, then $c\vec{v}$ is a vector that is $|c|$ times as long as $\vec{v}$, in the same direction as $\vec{v}$.

- If $c < 0$, then $c\vec{v}$ is a vector that is $|c|$ times as long as $\vec{v}$, in the opposite direction to $\vec{v}$.

Remark: For any vector $\vec{v}$: $0 \cdot \vec{v} = \vec{0}$.

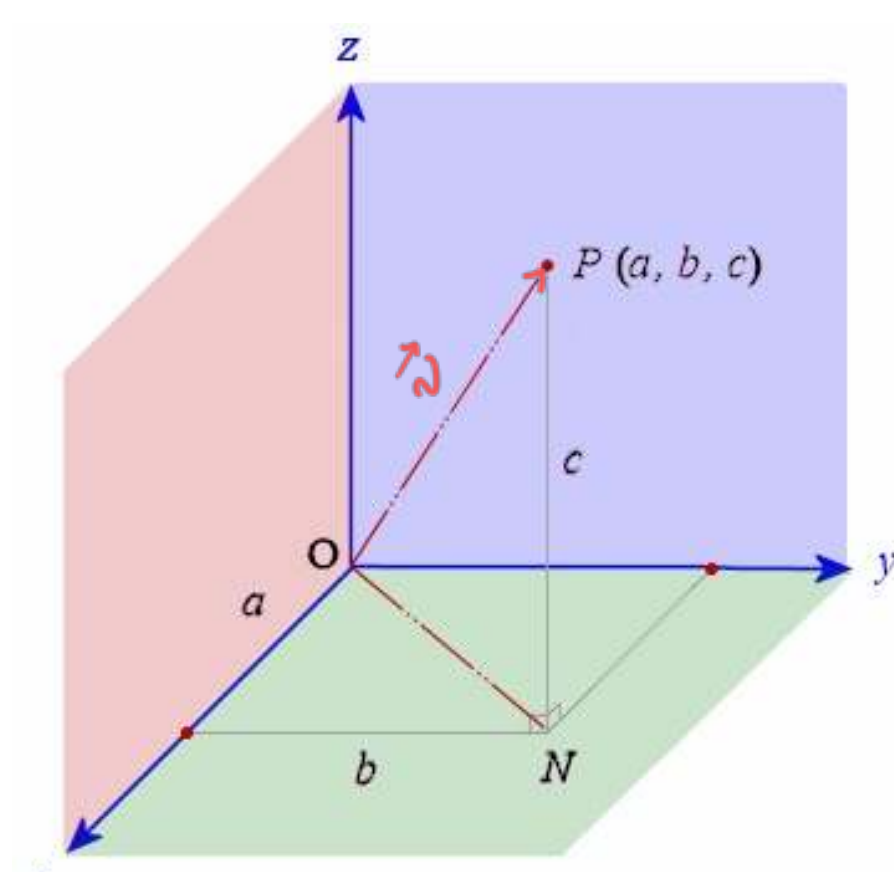
Exercise: With $\vec{a}$ and $\vec{b}$ as below, draw $\vec{a} - 2\vec{b}$.

Components:

This is a way to treat vectors algebraically.

If we take a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, based at the origin, we can write down the coordinates of the terminal point:

Fig. 7: $(\mathbb{R}^3)$



We define $\vec{v} = \vec{OP}$ to have components:

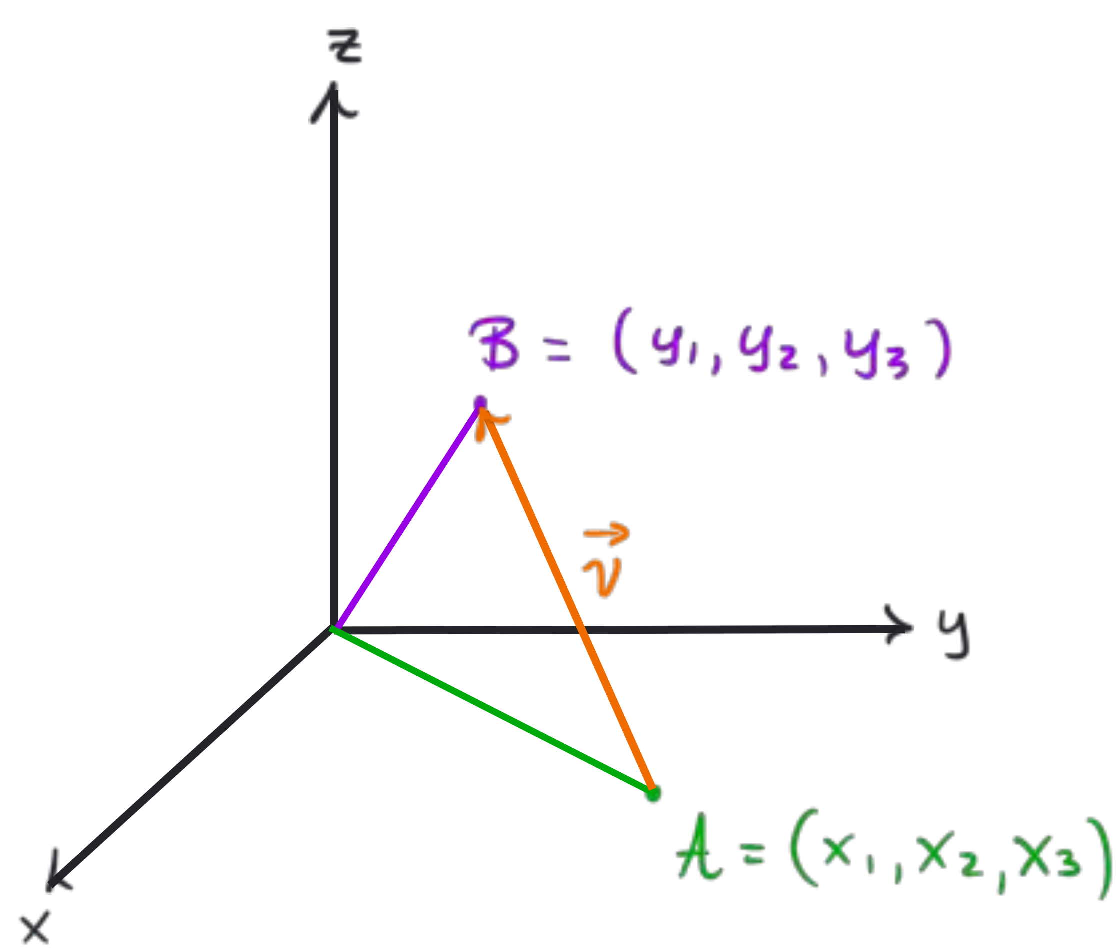
$$\vec{v} = (a, b, c) .$$

- We can clearly see that if we instead base $\vec{v}$ at a point $A = (x_1, x_2, x_3)$, that it should terminate at the point $B = (x_1 + a, x_2 + b, x_3 + c)$.

- Let's say we have two points $A = (x_1, x_2, x_3)$, and $B = (y_1, y_2, y_3)$.

What should the components of $\vec{v} = \overrightarrow{AB}$ be?

Fig. 8:



Geometrically we see that: $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

So we should have $\vec{v} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

which, in components would be:

$$\boxed{\vec{v} = (y_1, y_2, y_3) - (x_1, x_2, x_3) = (y_1 - x_1, y_2 - x_2, y_3 - x_3)}$$

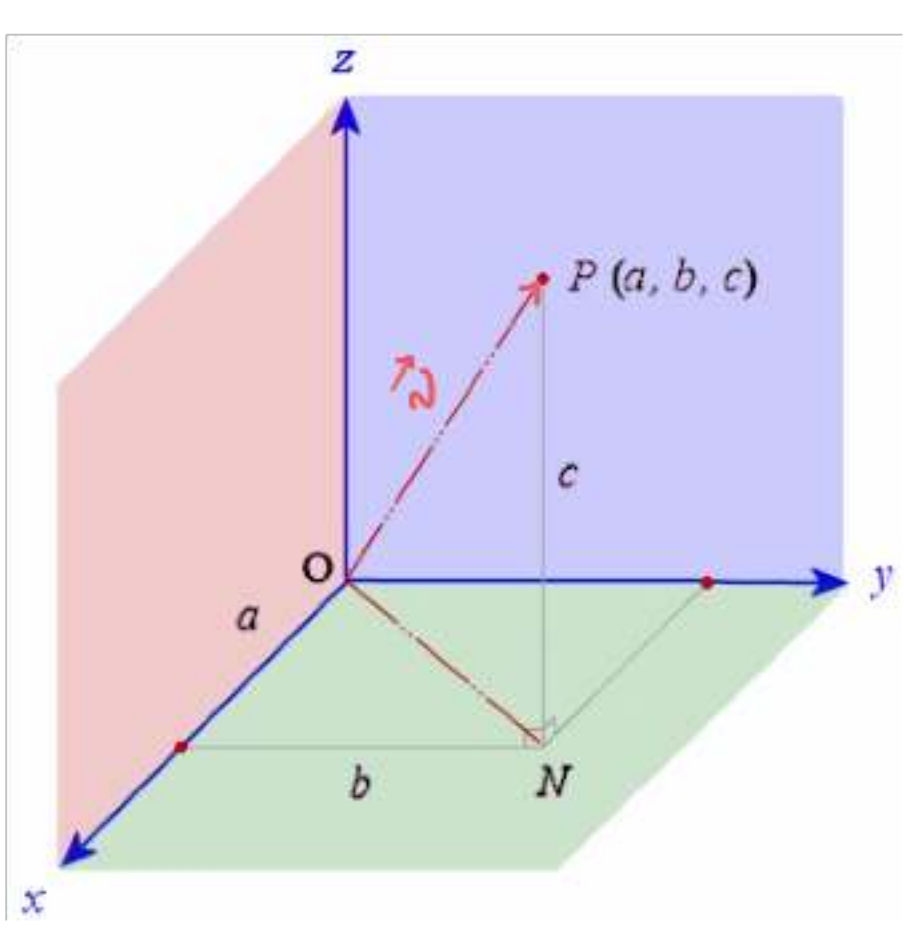
Main Point: $\vec{v}$ represented in components is:

$$\vec{v} = \begin{pmatrix} \text{displacement } \vec{v} \\ \text{causes in} \\ x\text{-direction} \end{pmatrix., \begin{matrix} \text{displacement } \vec{v} \\ \text{causes in} \\ y\text{-direction} \end{matrix}, \begin{pmatrix}. \text{displacement } \vec{v} \\ \text{causes in} \\ z\text{-direction} \end{pmatrix}$$

In particular, this representation doesn't care about where $\vec{v}$ is based.

Remark: Clearly the length of $\vec{v}$ is doesn't care about where $\vec{v}$ is based either.

Hence, we compute the length of $\vec{v}$, which we denote by $\|\vec{v}\|$, by using its representation in components:



Using pythagoras, we can see:

$$\boxed{\|\vec{v}\| = \sqrt{a^2+b^2+c^2}}$$

Questions: How do we add /subtract /scale vectors algebraically?

Answers: If $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$,

then:

(i) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

(iii) For $c \in \mathbb{R}$: $c\vec{a} = (ca_1, ca_2, ca_3)$

Remark: We could just have easily done all this in the 2-dimensional case, $\mathbb{R}^2$: $\vec{v} = (x_1, x_2)$, or the 4-dimensional case, $\mathbb{R}^4$: $\vec{v} = (x_1, x_2, x_3, x_4)$,

$\vdots$

or the n-dimensional case, $\mathbb{R}^n$: $\vec{v} = (x_1, \dots, x_n)$.

General Properties of vectors:

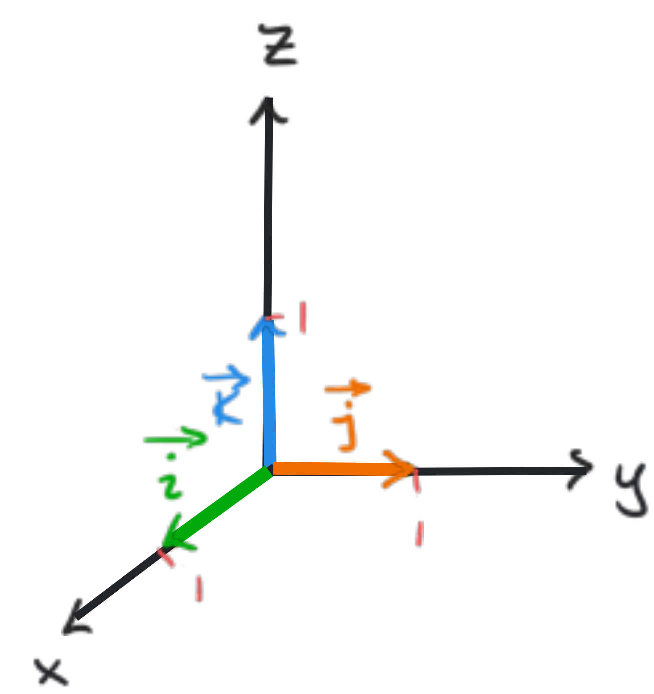
1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Remark: Every one of these properties can be justified geometrically.

Standard Basis Vectors:

Three vectors in $\mathbb{R}^3$ play a special role:

$$\left.\begin{aligned} \vec{i} &= (1,0,0) \\ \vec{j} &= (0,1,0) \\ \vec{k} &= (0,0,1) \end{aligned}\right\} \text{ Standard Basis Vectors for } \mathbb{R}^3$$



Why?

Because any vector $\vec{v}$ can be represented, algebraically as a linear combination of $\vec{i}$, $\vec{j}$, $\vec{k}$:

$$\vec{v} = (a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$$

$$= a\vec{i} + b\vec{j} + c\vec{k}$$

Remark: It is useful to think of $\vec{i}, \vec{j}, \vec{k}$ as the "building blocks" for $\mathbb{R}^3$.

Exercise: If $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{j} + 2\vec{k}$, represent $\vec{u}+\vec{v}$ in as a linear combination of $\vec{i}, \vec{j}, \vec{k}$ and compute $\|\vec{u} + \vec{v}\|$.

Definition: A unit vector is a vector with length 1.

Example: $\vec{i}$, $\vec{j}$ and $\vec{k}$ are all unit vectors.

Remark: For any non-zero vector $\vec{u}$, there is a unit vector pointing in the same direction as $\vec{u}$. This vector is usually denoted as $\hat{u}$, and is given algebraically by:

$$\boxed{\hat{u} = \frac{1}{\|\vec{u}\|} \cdot \vec{u}}$$

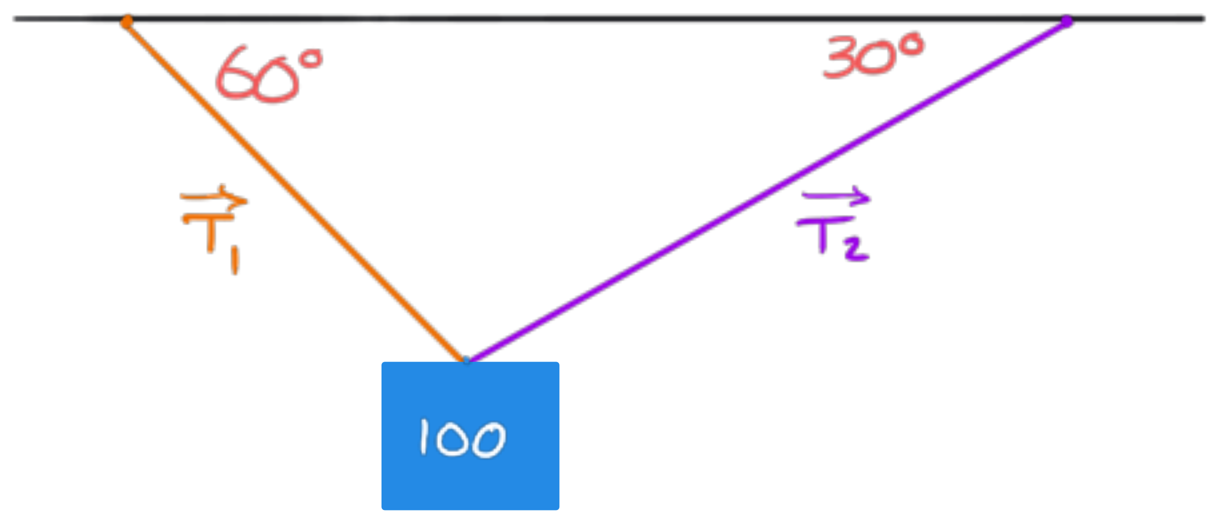
This process $(\vec{u} \rightsquigarrow \hat{u})$ is called normalizing $\vec{u}$.

Remark: Unit vectors are sometimes referred to as directions.

Exercise: Normalize $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$.

Applications:

1. A 100 lb weight hangs from two wires as shown:



Find the tension forces T_1 and T_2 in both wires and the magnitudes of the tensions, assuming the system is in equilibrium.

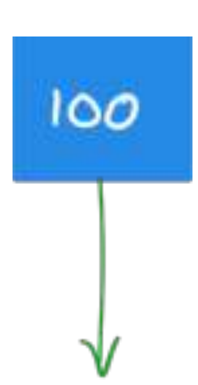
Sol^n: As our system is in equilibrium, we have:

$$\vec{T_1} + \vec{T_2} + \vec{w} = \vec{0} \quad , \text{ or :}$$

$$\vec{T_1} + \vec{T_2} = -\vec{w} = 100\,\vec{j} \qquad (+)$$

Force Diagrams:

$\vec{w}$:



$\Rightarrow \quad \vec{w} = -100\,\vec{j}$

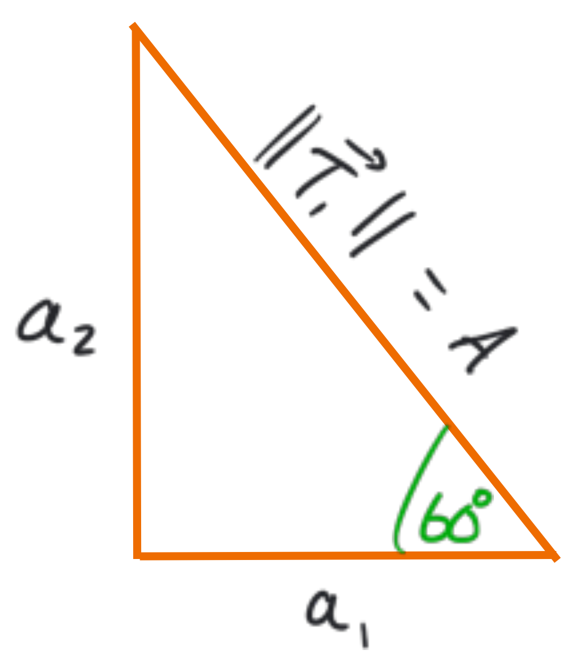
Let us denote the magnitude of the tension forces :

$$\|\vec{T}_1\| =: A \quad \text{\&} \quad \|\vec{T}_2\| =: B.$$

So, breaking our forces up into components :

$$\vec{T}_1 = a_1 \vec{i} + a_2 \vec{j} \quad \text{and} \quad \vec{T}_2 = b_1 \vec{i} + b_2 \vec{j}$$

$\vec{T}_1$:

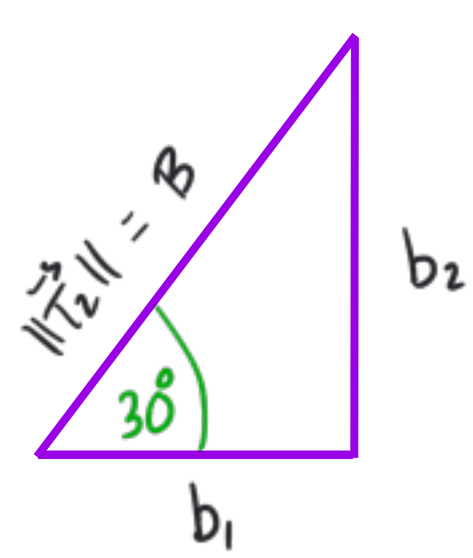


$$\Rightarrow a_1 = A\cos 60° = \frac{A}{2}$$

$$a_2 = A\cos 30° = \frac{\sqrt{3}}{2}A$$

Hence : $\boxed{\vec{T}_1 = -\frac{A}{2}\vec{i} + \frac{\sqrt{3}}{2}A\vec{j}}$

$\vec{T}_2$:



$$\Rightarrow b_1 = B\cos 30° = \frac{\sqrt{3}}{2}B$$

$$b_2 = B\sin 30° = \frac{B}{2}$$

Hence : $\boxed{\vec{T}_2 = \frac{\sqrt{3}}{2}B\vec{i} + \frac{B}{2}\vec{j}}$

So, rewriting (+):

$$\left(-\frac{A}{2}\vec{i} + \frac{\sqrt{3}}{2}A\vec{j}\right) + \left(\frac{\sqrt{3}}{2}B\vec{i} + \frac{B}{2}\vec{j}\right) = 100\vec{j} + 0\vec{i}$$

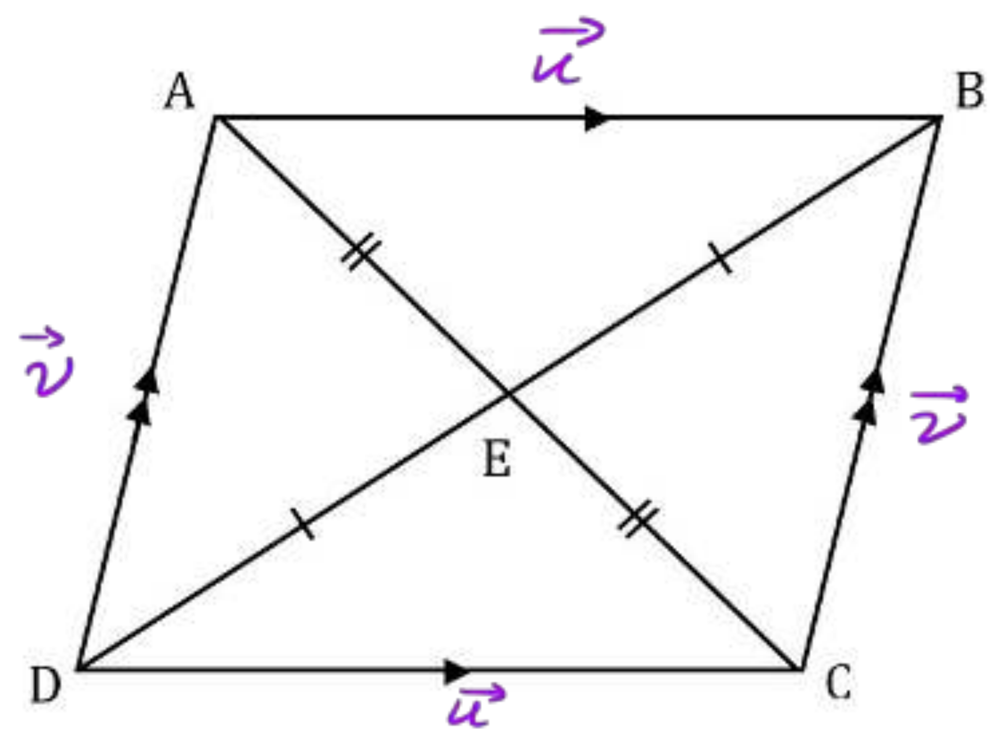
Equating components:

$$-\frac{A}{2} + \frac{\sqrt{3}}{2}B = 0 \quad \& \quad \frac{\sqrt{3}}{2}A + \frac{B}{2} = 100$$

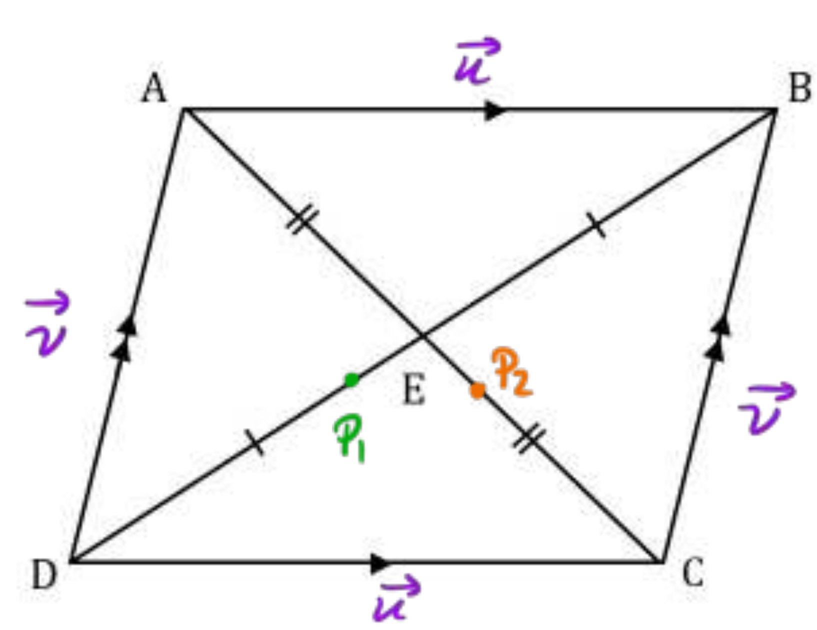
Solving this system yields: $A = 50\sqrt{3}$ & $B = 50$.

So
$$\boxed{\begin{aligned}\vec{T}_1 &= -25\sqrt{3}\,\vec{i} + 75\,\vec{j}\\ \vec{T}_2 &= 25\sqrt{3}\,\vec{i} + 25\,\vec{j}\end{aligned}}$$

2. Show that the diagonals of a parallelogram bisect eachother :



Solⁿ: Let's denote the halfway point of the cord DB by P_1 and the halfway point of the cord CA by P_2.



Our claim is that $P_1 = P_2$.

We can see $\vec{DB} = \vec{u} + \vec{v}$, so to get to P_1 we start at D and follow $\frac{1}{2}(\vec{u} + \vec{v})$.

Let's represent the vector $\overrightarrow{CA}$ by $\vec{w}$.

We can see that to get to P_2, starting from D, we follow $\vec{u}$ and then $\frac{1}{2}\vec{w}$.

So, in components:

$$P_1 = D + \frac{1}{2}(\vec{u} + \vec{v})$$

and

$$P_2 = D + \vec{u} + 1/2\,\vec{w}.$$

But we can see from the diagram that

$\vec{w} = -\vec{u} + \vec{v}$.

So $$P_2 = D + \vec{u} + 1/2(-\vec{u} + \vec{v})$$
$$= D + 1/2(\vec{u} + \vec{v})$$
$$= P_1$$

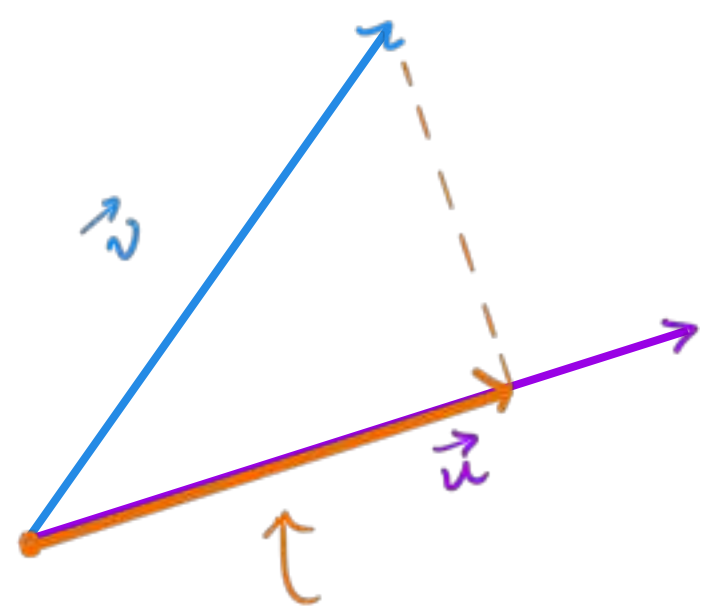
— o —

3. Dot Product & Cross Product:

Motivation: Say I have two vectors $\vec{u}$ and $\vec{v}$,

and I would like to know:

How much of $\vec{v}$ points in the direction of $\vec{u}$?

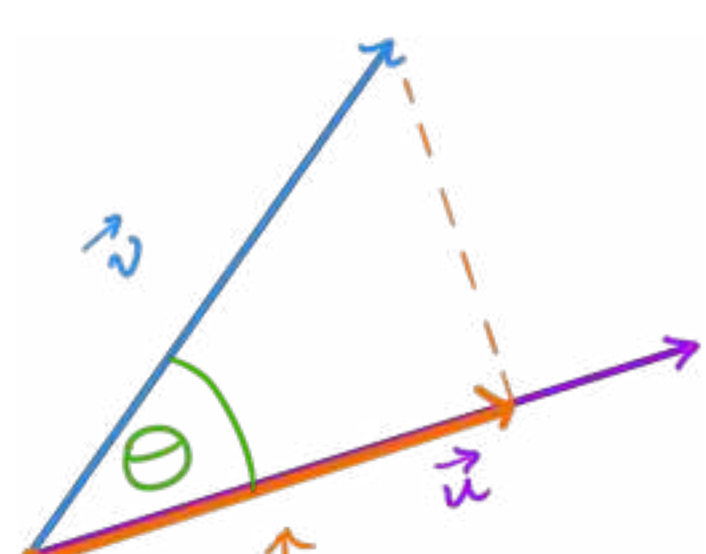


Let's call this: $\text{proj}_{\vec{u}}(\vec{v})$

But how can we find a formula for $\text{proj}_{\vec{u}}(\vec{v})$?

Well, if we knew θ, we could find it's length:

(Assume θ is acute - if not, use $-\vec{v}$)



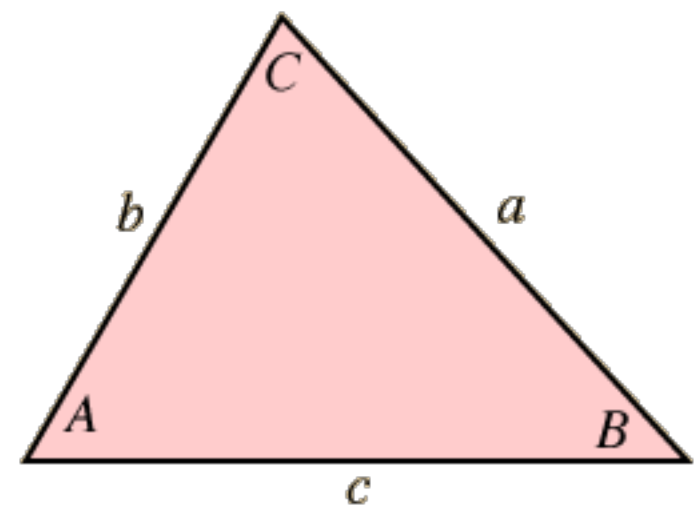
$\|\text{proj}_{\vec{u}}(\vec{v})\| = |\vec{v}| \cos\theta$

Let's call this length: $\boxed{\text{comp}_{\vec{u}}(\vec{v}) := \|\text{proj}_{\vec{u}}(\vec{v})\|}$

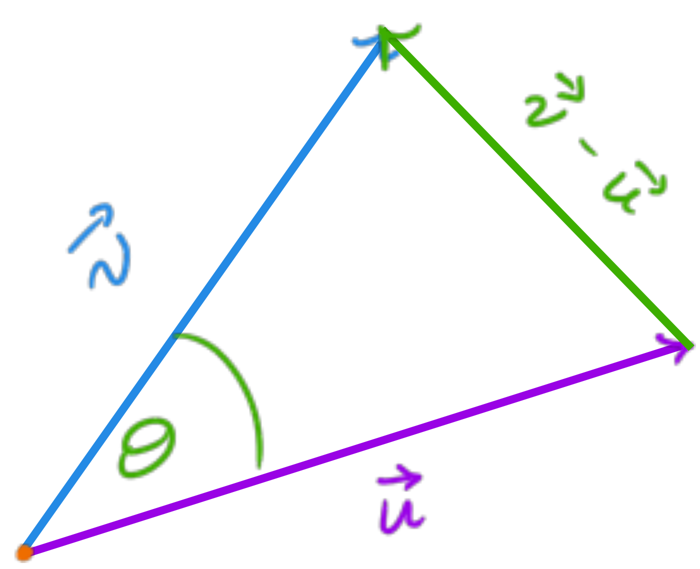
So, how can we find θ?

Recall : Cosine Rule :

$$c^2 = a^2 + b^2 - 2ab\cos C$$



So :



$$\Rightarrow \quad \|\vec{v}-\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2 = v_1^2 + v_2^2 + v_3^2 + u_1^2 + u_2^2 + u_3^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad -2v_1u_1 - 2v_2u_2 - 2v_3u_3 = -2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad v_1u_1 + v_2u_2 + v_3u_3 = \|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad \boxed{\cos\theta = \frac{v_1u_1 + v_2u_2 + v_3u_3}{\|\vec{v}\|\|\vec{u}\|}}$$

← Assuming $\vec{v}$ and $\vec{u}$ are non-zero.

This motivates :

Definition: The Dot Product of two vectors $\vec{v}$ and $\vec{u}$ is :

$$\boxed{\vec{v}\cdot\vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3}$$

Remark: Hence, we see :

$$\boxed{\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\,\|\vec{u}\|}}$$

OR :

$$\boxed{\vec{v}\cdot\vec{u} = \|\vec{v}\|\,\|\vec{u}\|\cos\theta}$$

Finally:

$$\boxed{\operatorname{comp}_{\vec{u}}(\vec{v}) = \|\vec{v}\|\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}}$$

Recall that by definition, $\operatorname{proj}_{\vec{u}}(\vec{v})$ points in the same direction as $\vec{u}$. Hence :

$$\boxed{\operatorname{proj}_{\vec{u}}(\vec{v}) = \operatorname{comp}_{\vec{u}}(\vec{v})\cdot\hat{u} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}\right)\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u}}$$

Properties of the Dot Product:

1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5) $\mathbf{0} \cdot \mathbf{a} = 0$

NB: $\vec{v}$ and $\vec{u}$ are orthogonal if and only if:

$$\boxed{\vec{v} \cdot \vec{u} = 0}$$

Exercise: Why?

Motivation: Given two vectors $\vec{u}$ and $\vec{v}$, imagine you wanted a vector which is orthogonal to both $\vec{u}$ and $\vec{v}$. Say we find such a vector: $\vec{w}$.

Then: $\vec{u} \cdot \vec{w} = 0$ & $\vec{v} \cdot \vec{w} = 0$.

↳ If you solve this algebraically, the easiest solution is:

$$w_1 = u_2 v_3 - v_2 u_3 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}$$

$$w_2 = -(u_1 v_3 - v_1 u_3) = -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$$

$$w_3 = u_1 v_2 - v_1 u_2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

So
$$\vec{w} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

This motivates the following definition:

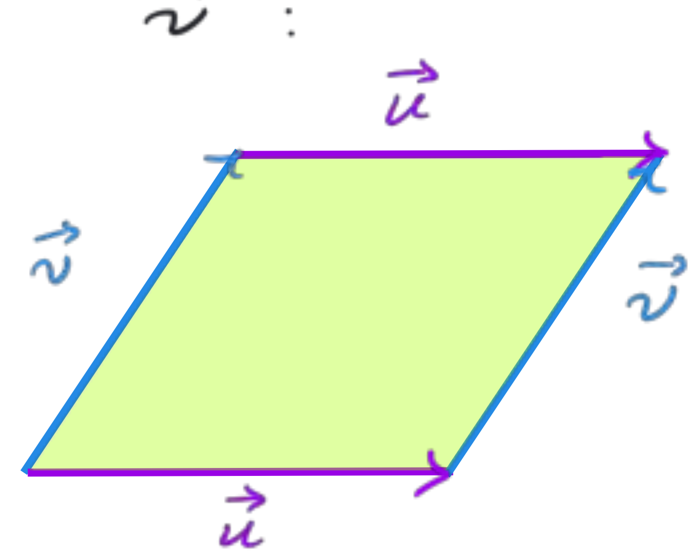
Definition: The Cross Product of two vectors $\vec{u}$ and $\vec{v}$ is:

$$\boxed{\vec{u} \times \vec{v} := \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

Key Properties of $\vec{u} \times \vec{v}$:

1) $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{v}$

2) For $\vec{u}$ and $\vec{v}$:



$\|\vec{u} \times \vec{v}\|$ = Area of shaded parallelogram

3) The direction of $\vec{u} \times \vec{v}$ follows the Right Hand Rule: If $\vec{u}$ is the direction of your index finger, and $\vec{v}$ is the direction of your middle finger, then $\vec{u} \times \vec{v}$ points in the direction of your thumb.

Other properties: If $\vec{a}$ and $\vec{b}$ are vectors & c is a scalar:

i) $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

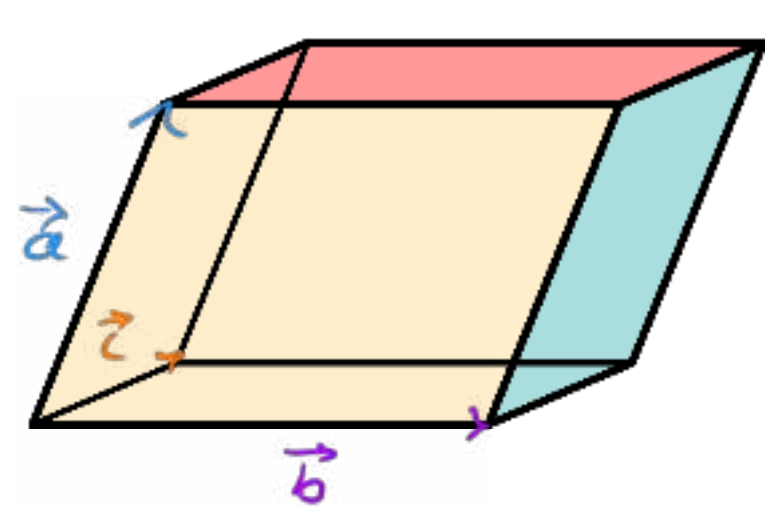
ii) $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

iii)
1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Definition: The vector triple product is defined as: $\vec{a} \cdot (\vec{b} \times \vec{c})$

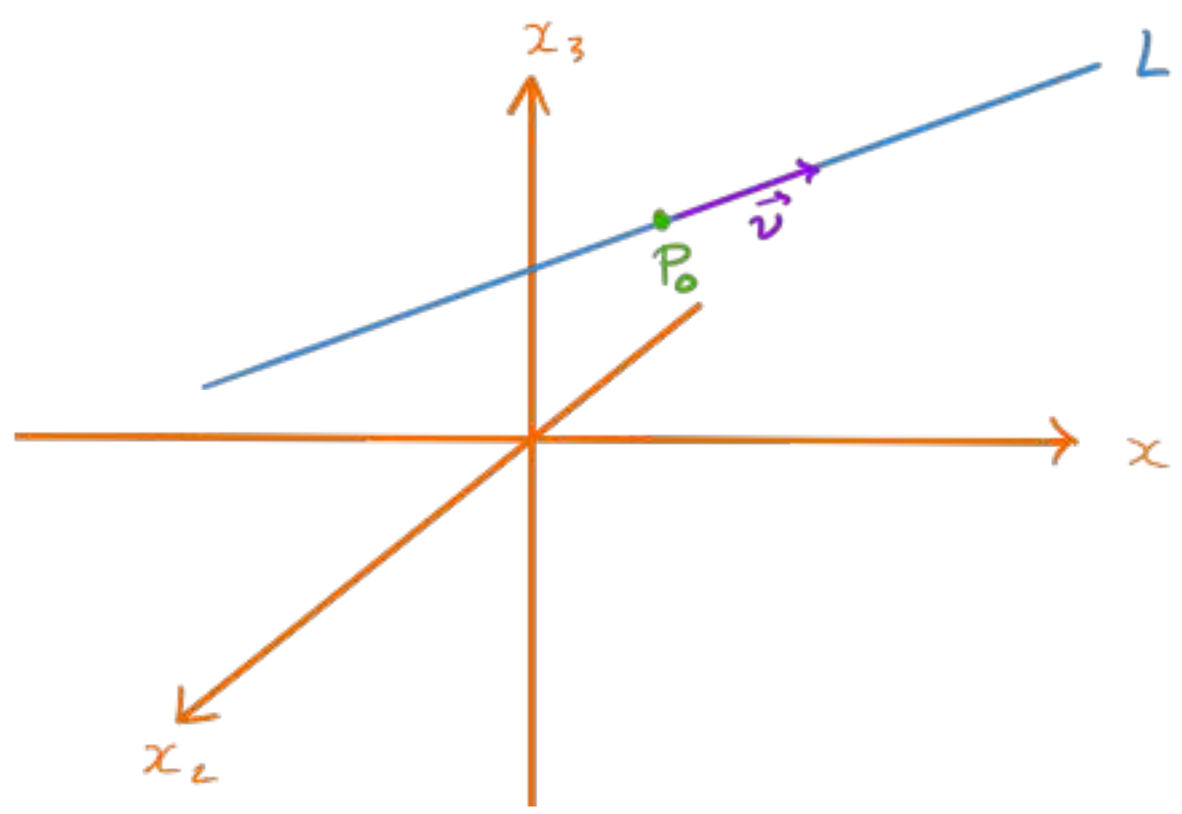
Remark: We can compute the volume of a parallelopiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$ using the vector triple product:

$$\text{Volume}(\vec{a}, \vec{b}, \vec{c}) = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$



4. Lines and Planes:

- We can see geometrically that a line is uniquely defined by a point on the line and the direction of the line (or, alternatively by two points on the line).



$(\vec{v} \neq \vec{0})$

Equation of L: $\boxed{L(t) = \vec{P_0} + t\,\vec{v}}$ (vector form)

Remarks:

1) We can think of this as a criteria for a point to be on the line. i.e., a point $P = (x, y, z)$ is on the line L $\iff$ there is a t_* such that $\vec{P} = \vec{P_0} + t_* \vec{v}$.

2) We can also think of this as a machine:

You give it a value of t and it gives you a point on the line L.

e.g. $L(1) = \vec{P_0} + (1)\vec{v} = \vec{P_0} + \vec{v}$ is on L

$L(\pi) = \vec{P_0} + \pi\vec{v}$ is on L

$L(-3) = \vec{P_0} - 3\vec{v}$ is on L

$\vdots$

- If $\vec{P_0} = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$, then:

$$L(t) = (x(t), y(t), z(t)) = (x_0 + at, y_0 + bt, z_0 + ct)$$

Where we now think of $x(t)$, $y(t)$ and $z(t)$ as "component machines".

- Say a point (x, y, z) is on L. Then there must be some time, say t_*, where

$$\left.\begin{aligned} x &= x_0 + at_* \\ y &= y_0 + bt_* \\ z &= z_0 + ct_* \end{aligned}\right\} \text{Parametric equations of } L$$

Isolating t_* :

$$t_* = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

We can also see if an x, y, z satisfy the last two equalities, they will be the components of $L(t)$ for some time. i.e.

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

is an equivalent description of a line through $P_0 = (x_0, y_0, z_0)$ with direction $\vec{v} = (a, b, c)$.

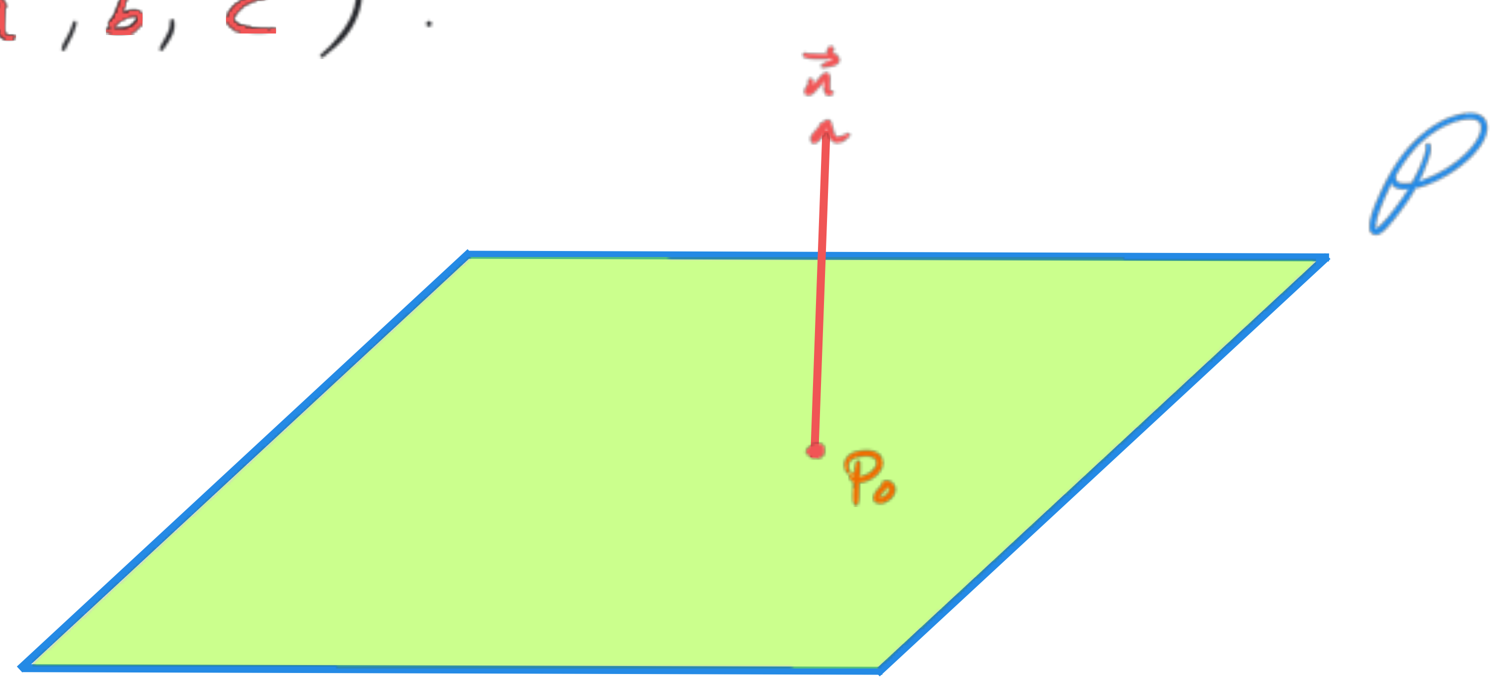
These are called the symmetric equations of L.

- We can geometrically see that a Plane $\mathcal{P}$ is described uniquely by a point it contains, and two vectors that lie in $\mathcal{P}$.

  Alternatively, and more usefully, $\mathcal{P}$ can be described by a point it contains and a vector that is orthogonal to the plane.

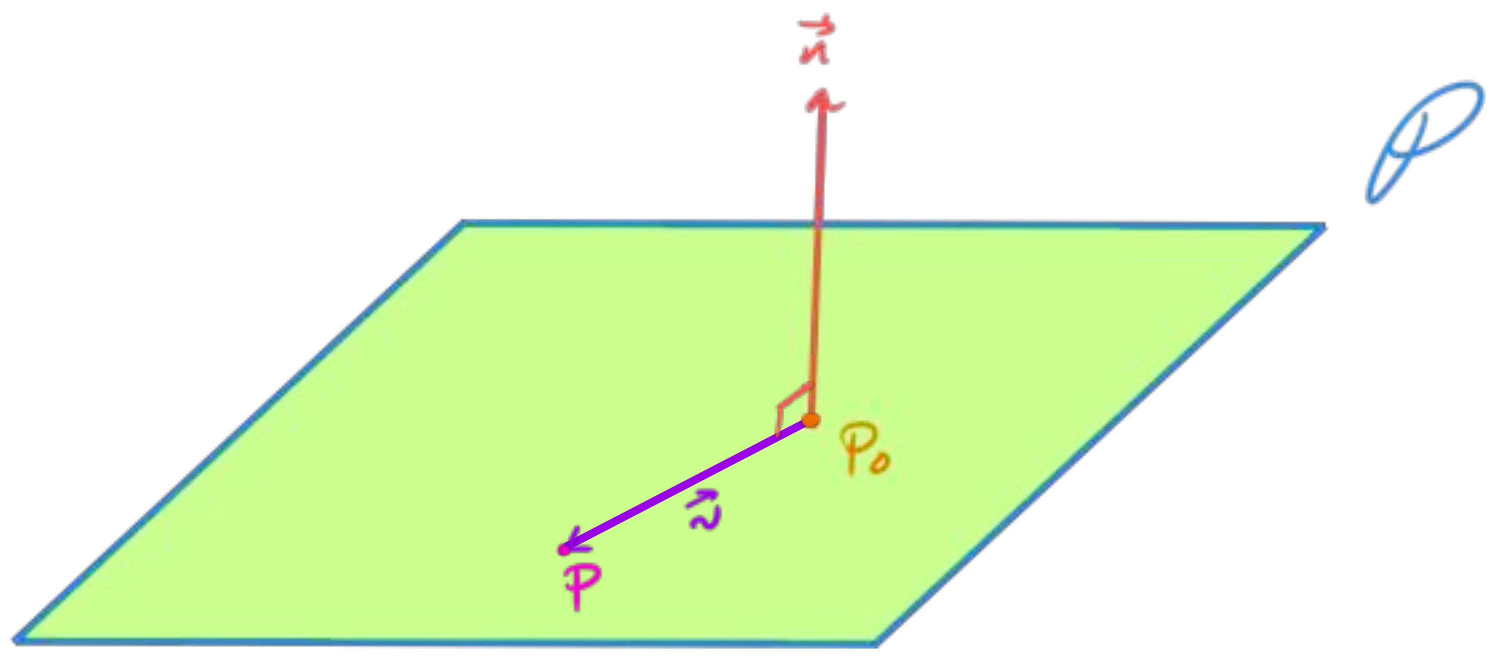
- This orthogonal vector is referred to as being <u>normal to $\mathcal{P}$</u>, and is usually denoted by $\vec{n}$.

- Consider the plane $\mathcal{P}$ below, which contains the point $P_0 = (x_0, y_0, z_0)$ and has normal vector $\vec{n} = (a, b, c)$:



Say $P = (x, y, z)$ is on $\mathcal{P}$.

Then : $\vec{v} = \overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ lies in $\mathcal{P}$:



So we must have $\vec{v} \perp \vec{n}$.

$\iff \quad \vec{v} . \vec{n} = 0$

$\iff \quad (x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$

$\iff \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\iff \quad \boxed{ax + by + cz = ax_0 + by_0 + cz_0}$

The above uniquely describes $\mathcal{P}$ (criteria) .

<u>Remark</u> : We are usually given planes by equations of the form : $\quad ax + by + cz = d$.

Here d is just $ax_0 + by_0 + cz_0$ (simplified).

<u>NB</u>: We can read off $n = (a, b, c)$ from this equation .

Example: A normal vector to the plane given by

$P: \ 2x + y - z = 10$ is $\vec{n} = (2, 1, -1)$.

Another is $\vec{N} = (4, 2, -2)$. Why?

Exercise: Given 3 points, how would you find the plane containing them?

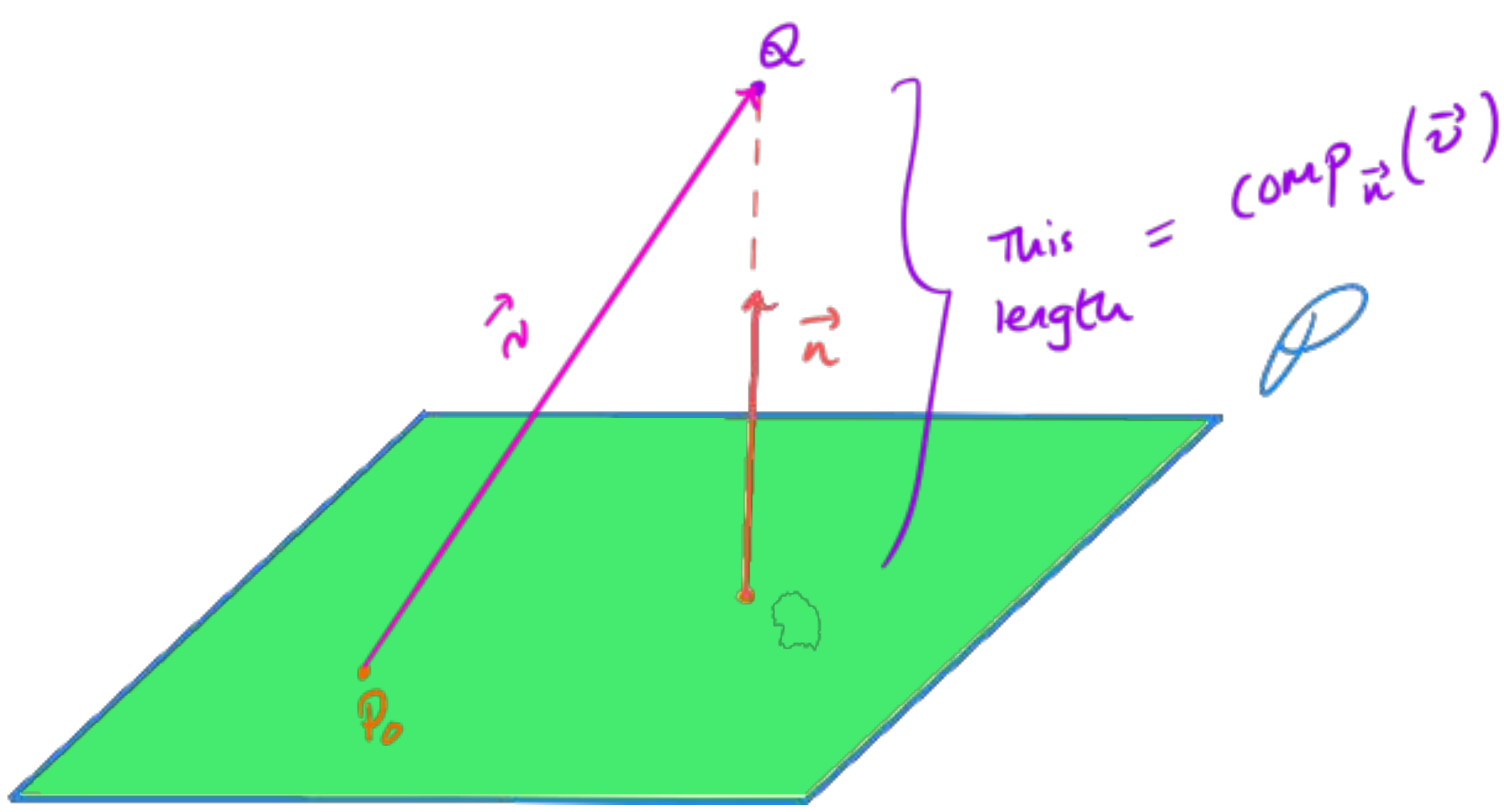
Remark: The angle between two planes can be found by finding the angle between their respective normal vectors.

- If two planes intersect to give a line, you can find the line's direction vector by taking the cross product of the normal vectors of the planes:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2$$

- Alternatively, you can try find the symmetric eq$^{\underline{n}}$s of the line by manipulating the equations of both planes (tutorial problem).

- The formula for the distance of a point to a plane is:



$$\boxed{\text{Dist}(Q, \mathcal{P}) = \frac{|\vec{v} \cdot \vec{n}|}{|\vec{n}|}}$$

5. Vector Functions and Space Curves:

For us, these are two different ways of thinking about the same thing (a function versus it's graph):

$$r(t) = (x(t), y(t), z(t))$$

You can think about $r(t)$ as being the position of a particle at time t (allow "negative time").

Example: $r(t) = (\cos(t), \sin(t), -t)$.

Draw this space curve / vector valued function / particle trajectory.

Sol^n:

6. Derivatives of Space Curves:

- If $r(t) = (x(t), y(t), z(t))$, then

(i) $r'(t) = (x'(t), y'(t), z'(t))$

(ii) $r''(t) = (x''(t), y''(t), z''(t))$

(iii) $\int r(t)\,dt = \left(\int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt\right)$

Remark: If we consider $r(t)$ to be the position of a particle at time t, then $r'(t) = v(t)$ is its velocity and $r''(t) = a(t)$ is its acceleration.

- If $\vec{u}(t)$ and $\vec{v}(t)$ are vector valued functions and $f : \mathbb{R} \xrightarrow{C^1} \mathbb{R}$, then:

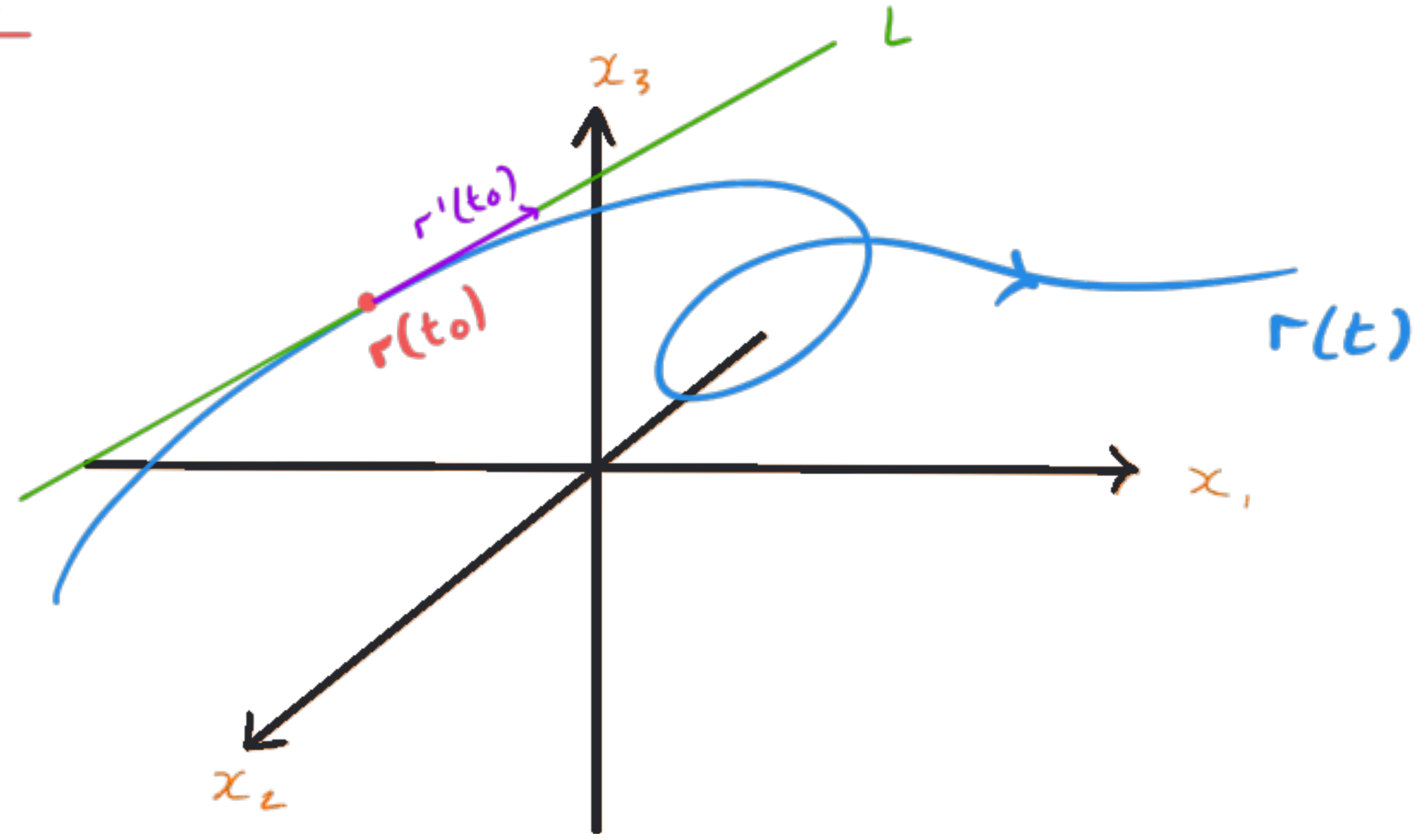
(i) $(\vec{u}(t) \cdot \vec{v}(t))' = u'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

(ii) $(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

(iii) $(f(t)\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)u'(t)$

(iv) $(\vec{u}(f(t)))' = f'(t)\,\vec{u}(f(t))$

Definition:

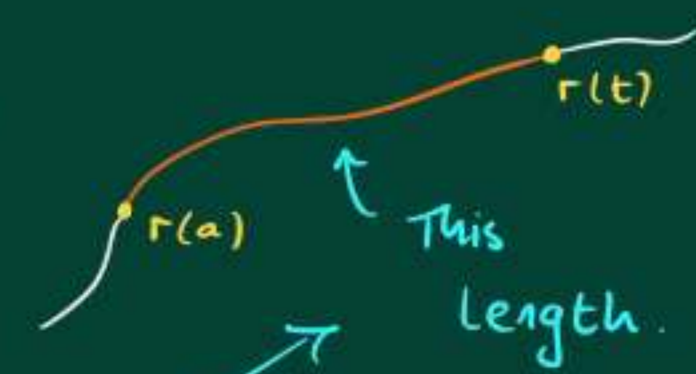


The tangent line to r at $P_0 = r(t_0)$ is the line containing P_0 with direction vector $r'(t_0)$:

$$\boxed{L_{P_0}(s) = P_0 + s\,r'(t_0)}\ , \quad s \in \mathbb{R}$$

7. Arc Length

·) Arc Length : $\boxed{s(t) = \int_a^t |r'(\tau)| \, d\tau}$ ←



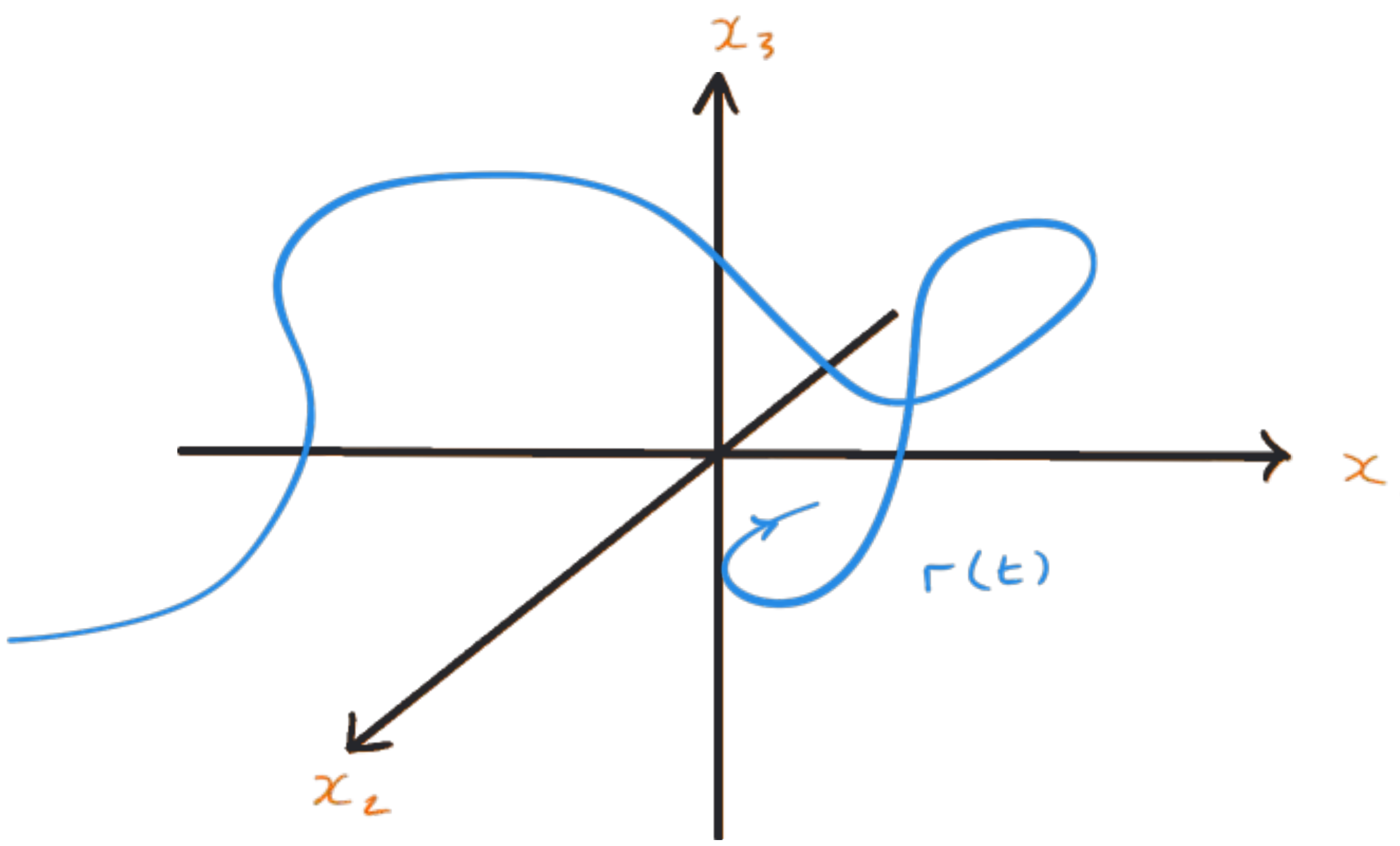
=

Example: Find the length of the arc of the circular helix $r(t) = (\cos(t), \sin(t), t)$ from the point $(1,0,0)$ to $(1,0,2\pi)$ and from $(1,0,0)$ to $(1,0,4\pi)$.

Sol$\underline{n}$:

8. TNB Frame :

Consider the space curve $r(t)$:



Goal : To describe the "shape of the curve".

Remark : The curve is shaped differently at each point $r(t)$, so whatever description we come up with, if it's "good" will vary from point to point i.e. vary with time.

Idea : One piece of information that tells us something about the shape of the curve is where it's "pointing" / "the direction it's heading" at at each point.

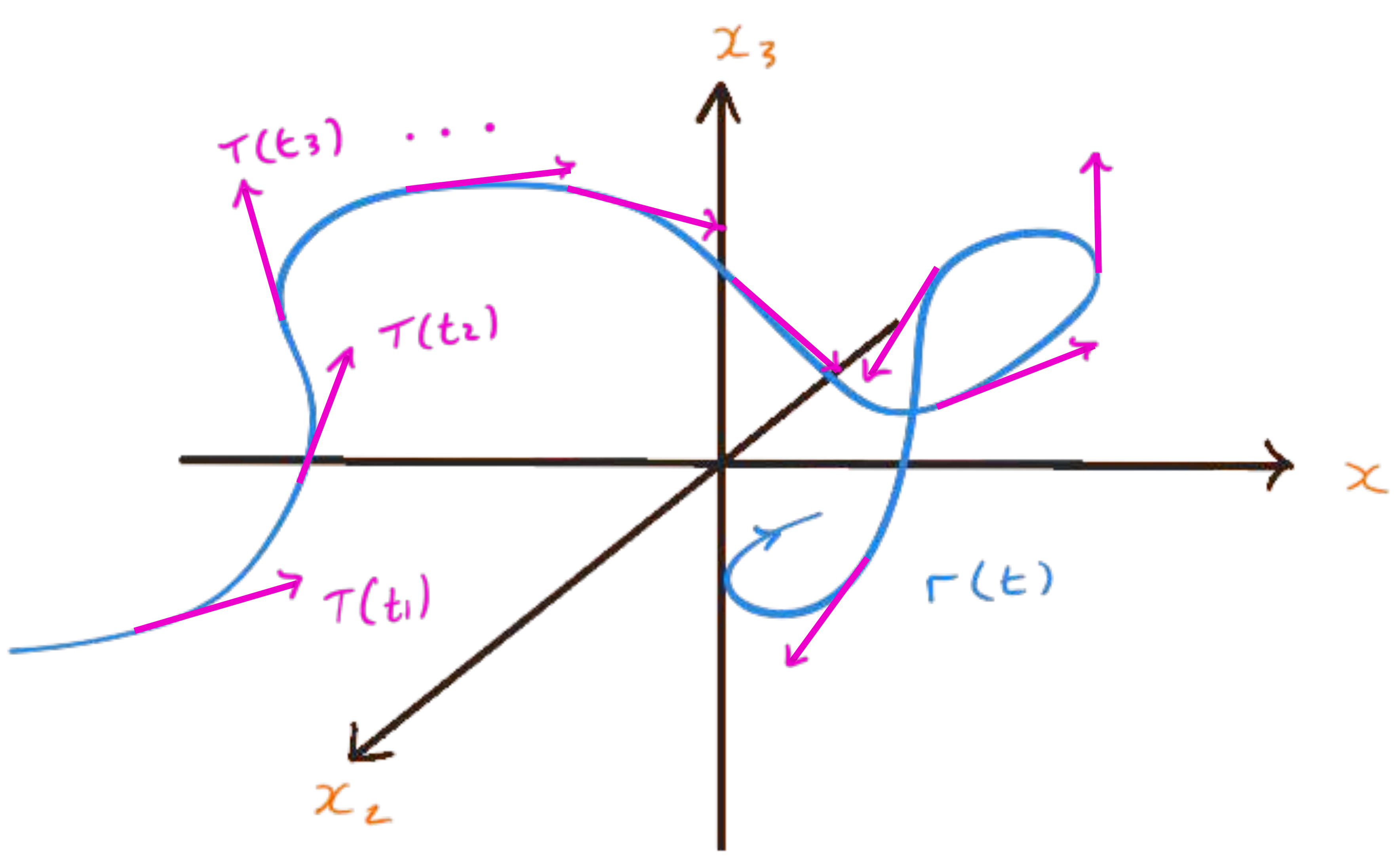
i.e. we should consider $r'(t)$.

But $r'(t)$ has a magnitude, which we don't care about. This is why we concern ourselves with the Unit Tangent Vector:

$$\boxed{T(t) = \frac{r'(t)}{|r'(t)|}}$$

Remark: Here we are assuming our particle never "stops moving": i.e. $r(t) \neq 0$ for any t.

- Let's consider how $T(t)$ varies with time:



Remark: We can see, at the places where the curve is most ... curved, that $T(t)$ changes direction quite rapidly.

This gives us an idea of how the curve is "bending" at a point. This motivates the following definition:

Definition: The unit Normal to r at a point $r(t)$ is given by:

$$\boxed{N(t) := \frac{T'(t)}{|T'(t)|}}$$

Remark: $N(t) \perp T(t)$ for all t.

Definition: We define the Unit Binormal to r at $r(t)$ by:

$$\boxed{B(t) = T(t) \times N(t)}$$

Remark: $B(t) \perp T(t)$ and $N(t)$ for all t.

- To summarize | visualize :

For a parametrised curve $r(t)$:

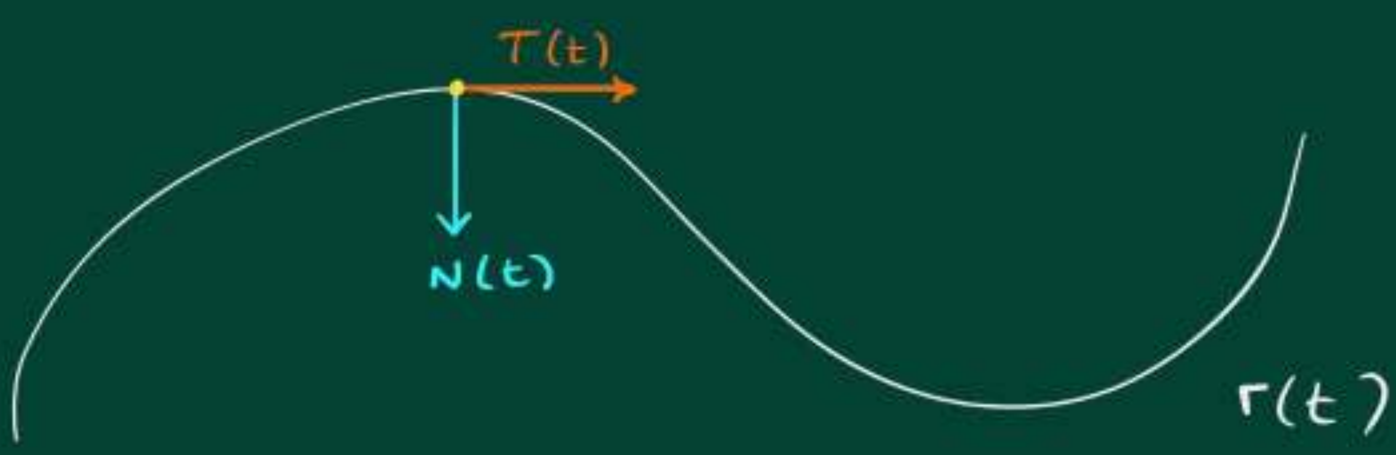
Unit Tangent : $\boxed{T(t) := \frac{r'(t)}{|r'(t)|}}$

Unit Normal : $\boxed{N(t) := \frac{T'(t)}{|T'(t)|}}$

Unit Binormal : $\boxed{B(t) := T(t) \times N(t)}$

Calc. III is a

Big Terrible Nightmare



$B(t) = T(t) \times N(t)$

Can show : 1) $\boxed{r''(t) =: \vec{a}(t) = \underbrace{a_T(t)}_{\in \mathbb{R}} \vec{T}(t) + \underbrace{a_N(t)}_{\geq 0} \vec{N}(t)}$

2) $\boxed{B(t) = \frac{r'(t) \times r''(t)}{|r'(t) \times r''(t)|}}$

3) $\boxed{N(t) = B(t) \times T(t)}$

... but it's

Not Bad Today

NB: • When you are asked to calculate these quantities, the above formulas make things a lot easier.

• If you are asked to evaluate these at a given point, always evaluate $r'(t)$ and $r''(t)$ at the point first and then compute!

•) $$\boxed{a_T(t) = \frac{r'(t) \cdot r''(t)}{|r'(t)|}}$$ $$\boxed{a_N(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|}}$$

•) Trick : $$\boxed{a_N(t) = \sqrt{|\vec{a}(t)|^2 - a_T(t)^2}}$$

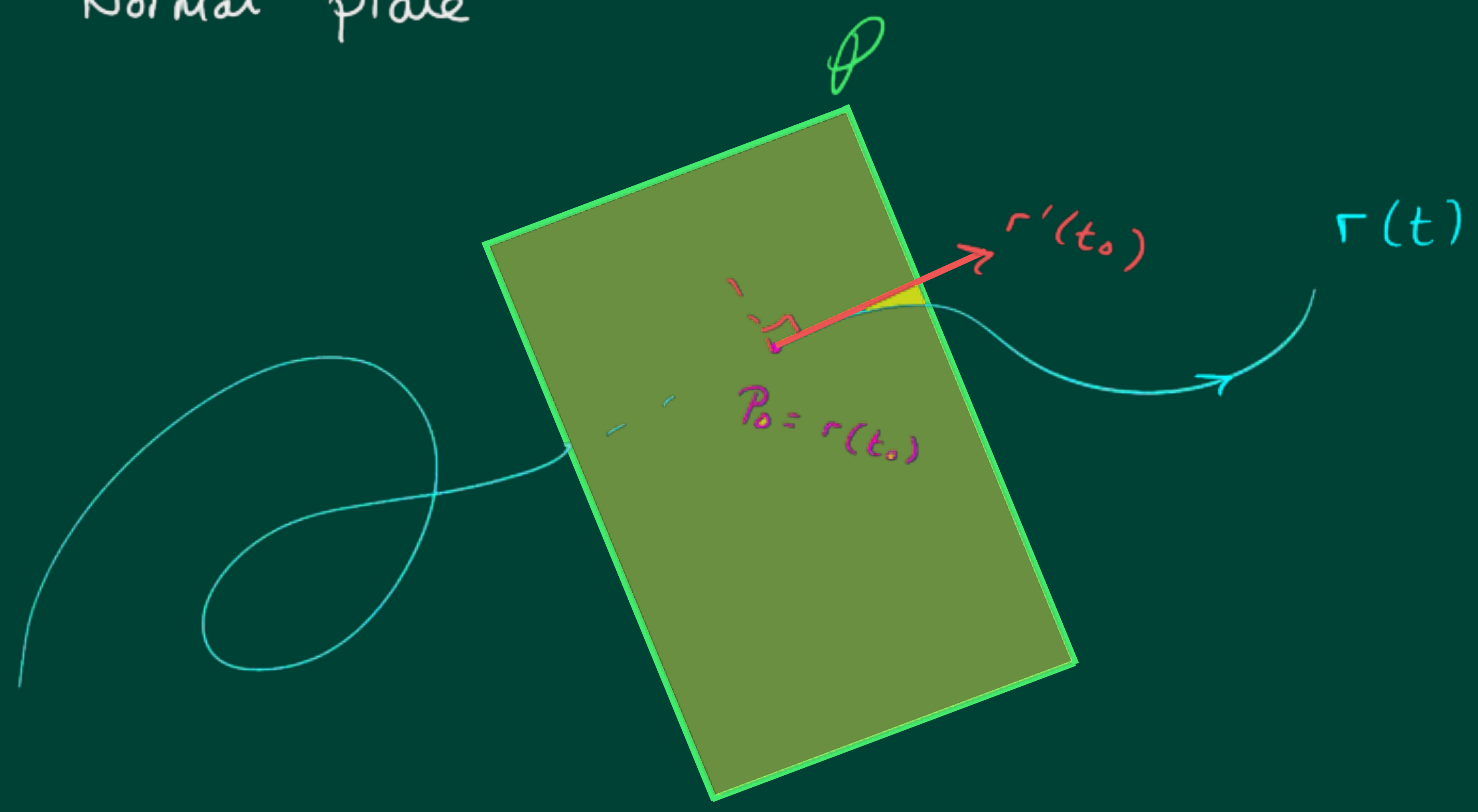
Definition: The plane at a point $r(t_0)$ on a space curve r determined by $N(t_0)$ and $B(t_0)$ is called the Normal Plane of r at $r(t_0)$.

Shortcut: $r'(t_0)$ is normal to this plane.

NB

Remark: It consists of all lines $\perp$ to the tangent line.

Fig 1 : Normal plane



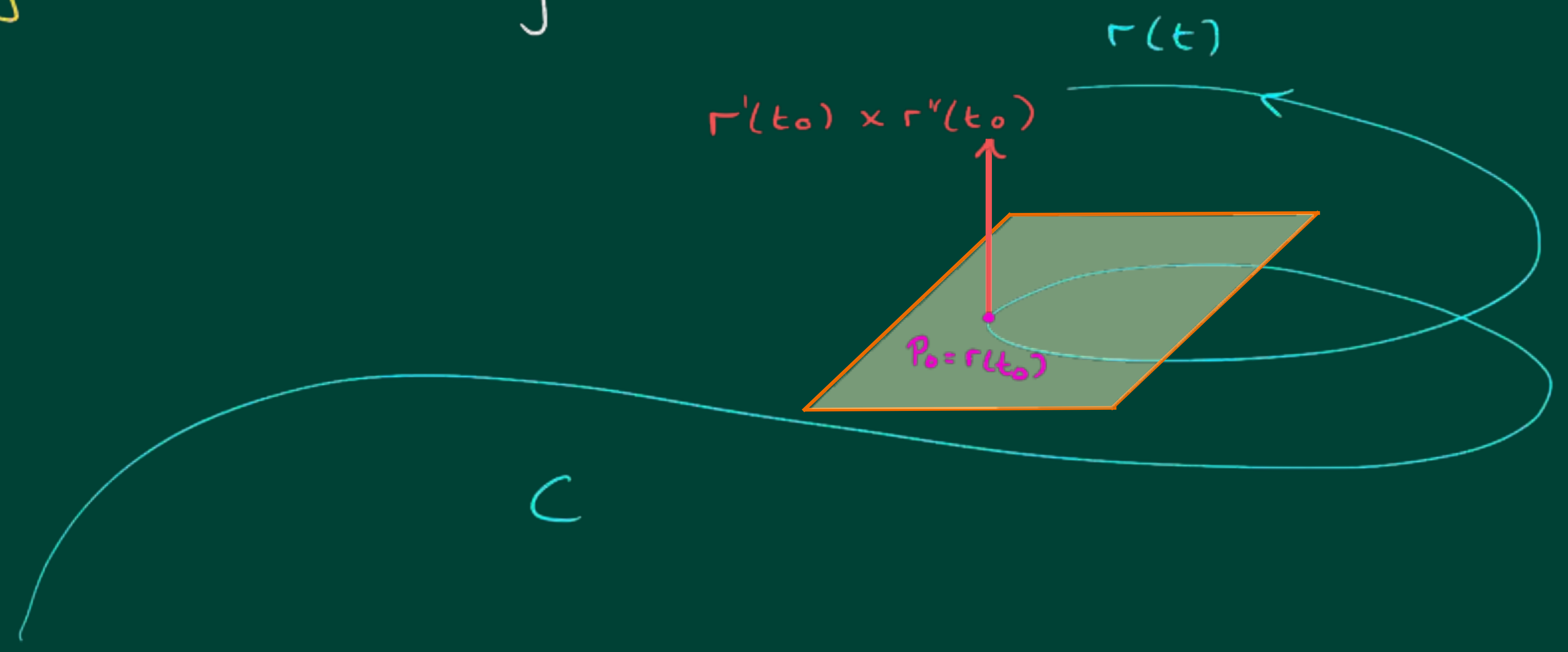
Definition: The Osculating Plane of a curve C (parametrized by $r(t)$) at a point $P_0 = r(t_0)$ is the plane determined by $T(t_0)$ and $N(t_0)$.

Shortcut: $r'(t_0) \times r''(t_0)$ is normal to this plane.

NB

Remark: Intuitively, the osculating plane is the plane that comes closest to containing part of the curve near P_0.

Fig 2 : Osculating Plane



$\int$ Recap :

What we've seen so far :

↳ Functions of several variables :

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$: (x,y) \longmapsto f(x,y)$$

$$g : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$: (x,y,z) \longmapsto g(x,y,z)$$

Remark : Graphs of functions :

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$: (x,y) \longmapsto f(x,y)$$

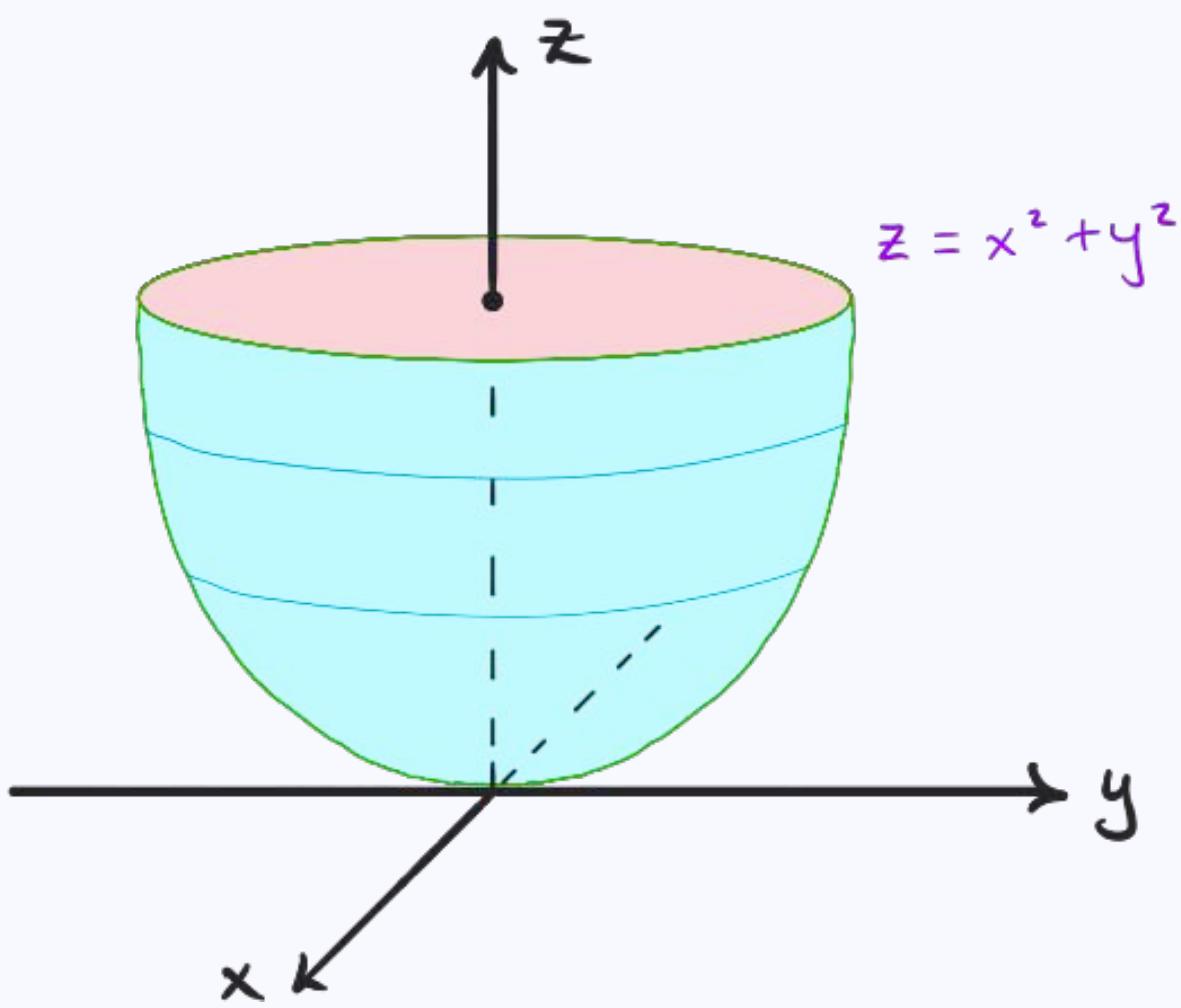
give rise to "canopies" or "surfaces".

↑ technically not always true.

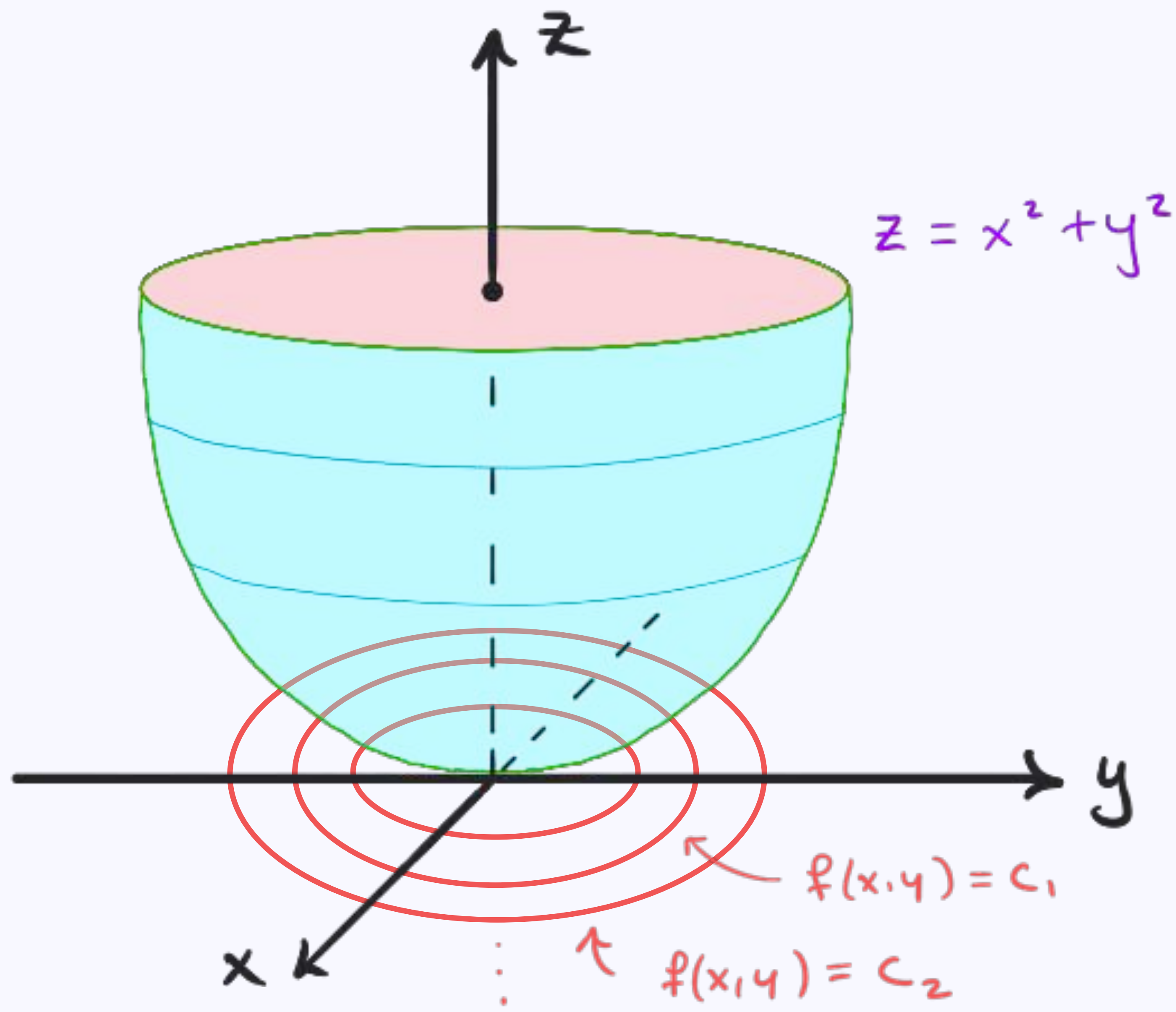
Fig 1 :

Graph of

$f(x,y) = x^2+y^2$



↳ Looked at level sets/curves:



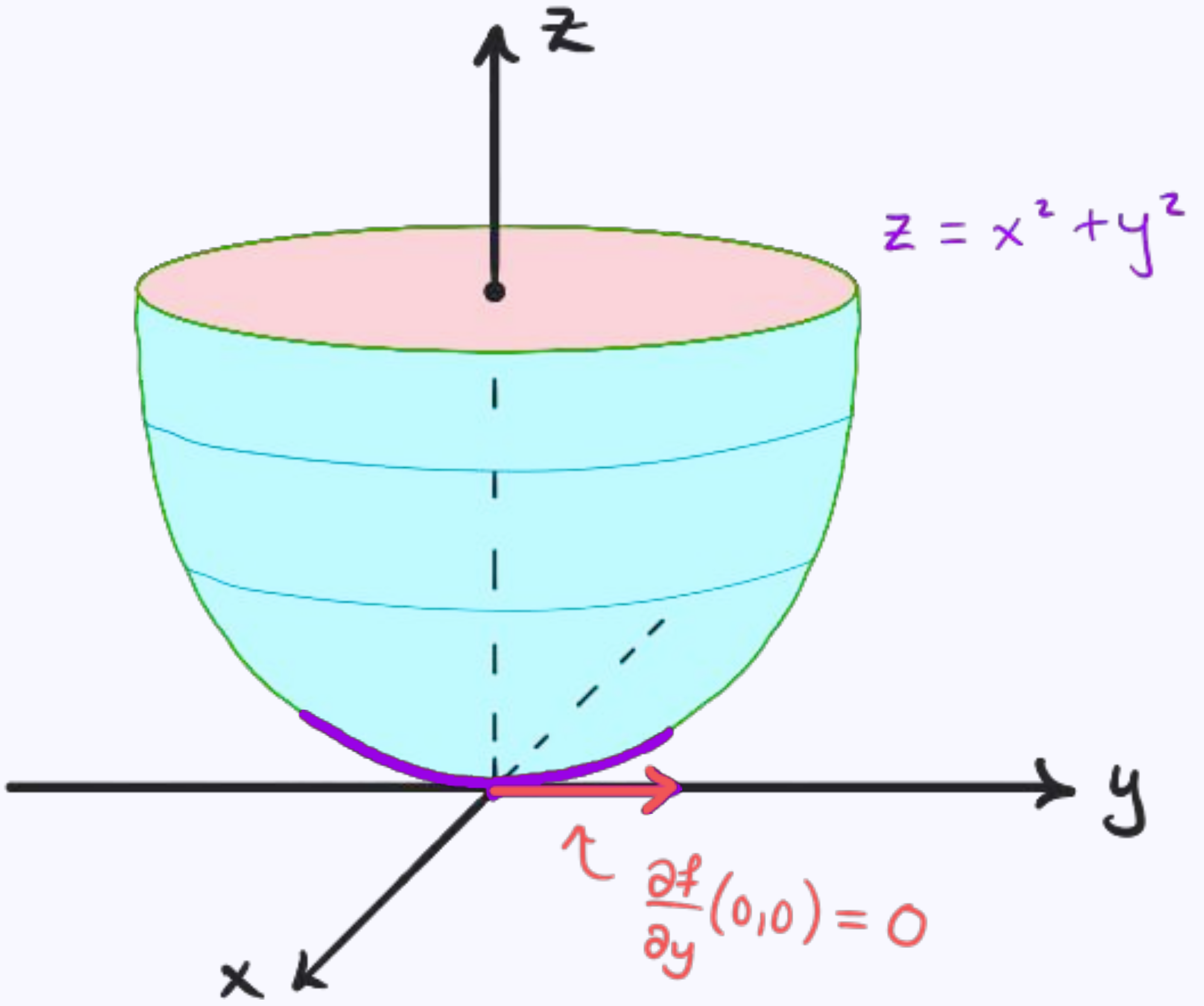
↳ Limits and continuity for functions of several variables.

↳ Partial Derivatives:

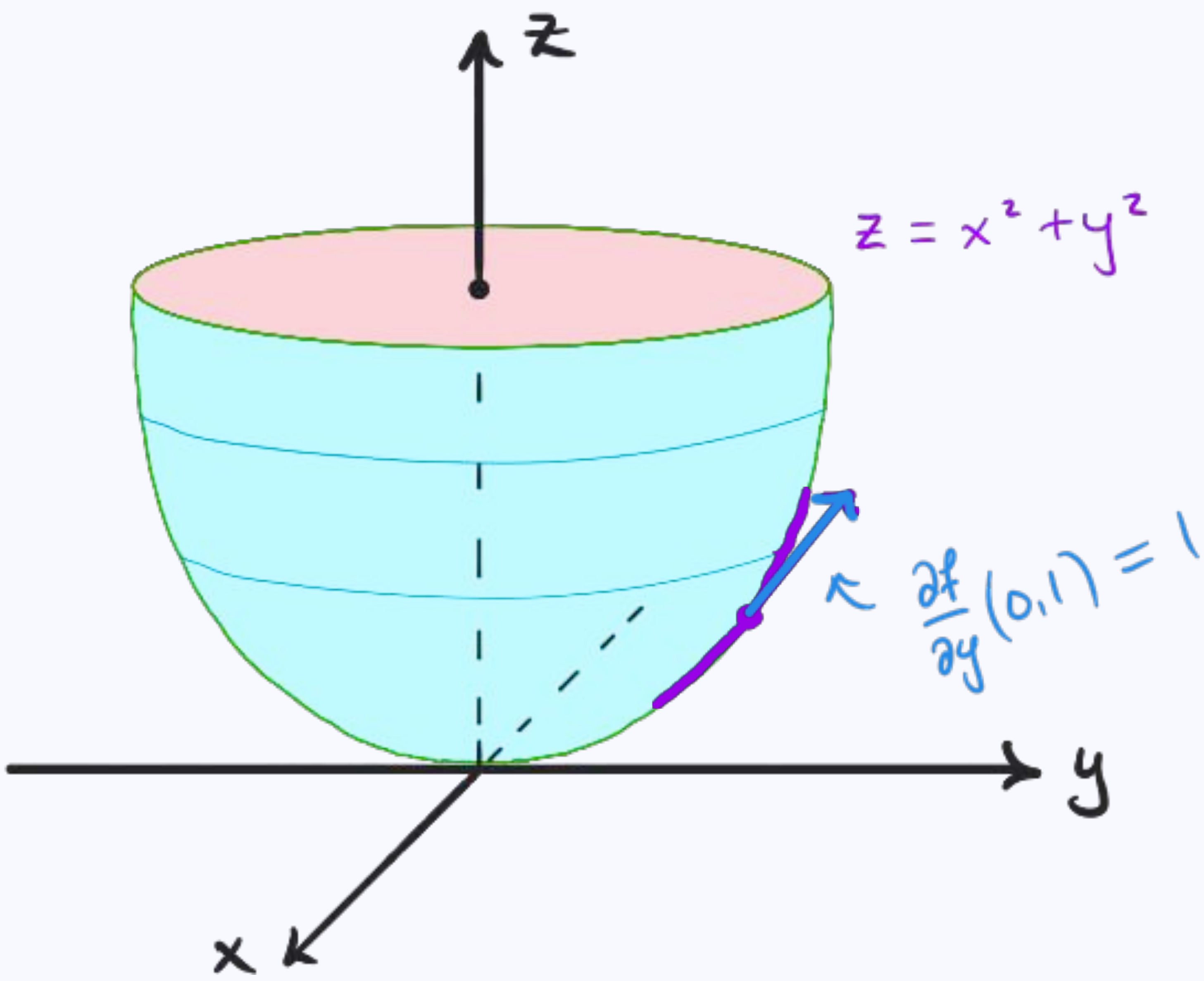
Remark: For intuition purposes, you can think of $\frac{\partial f}{\partial y}$ as how the value of $f(x,y)$ changes if we vary y and keep x fixed.

Example: $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial y}(x,y) = 2y \Rightarrow \frac{\partial f}{\partial y}(0,0) = 0 \;\; \& \;\; \frac{\partial f}{\partial y}(0,1) = 1$$



If we restrict to the purple curve (fixed $x=0$) we can see the tangent to this curve is horizontal.



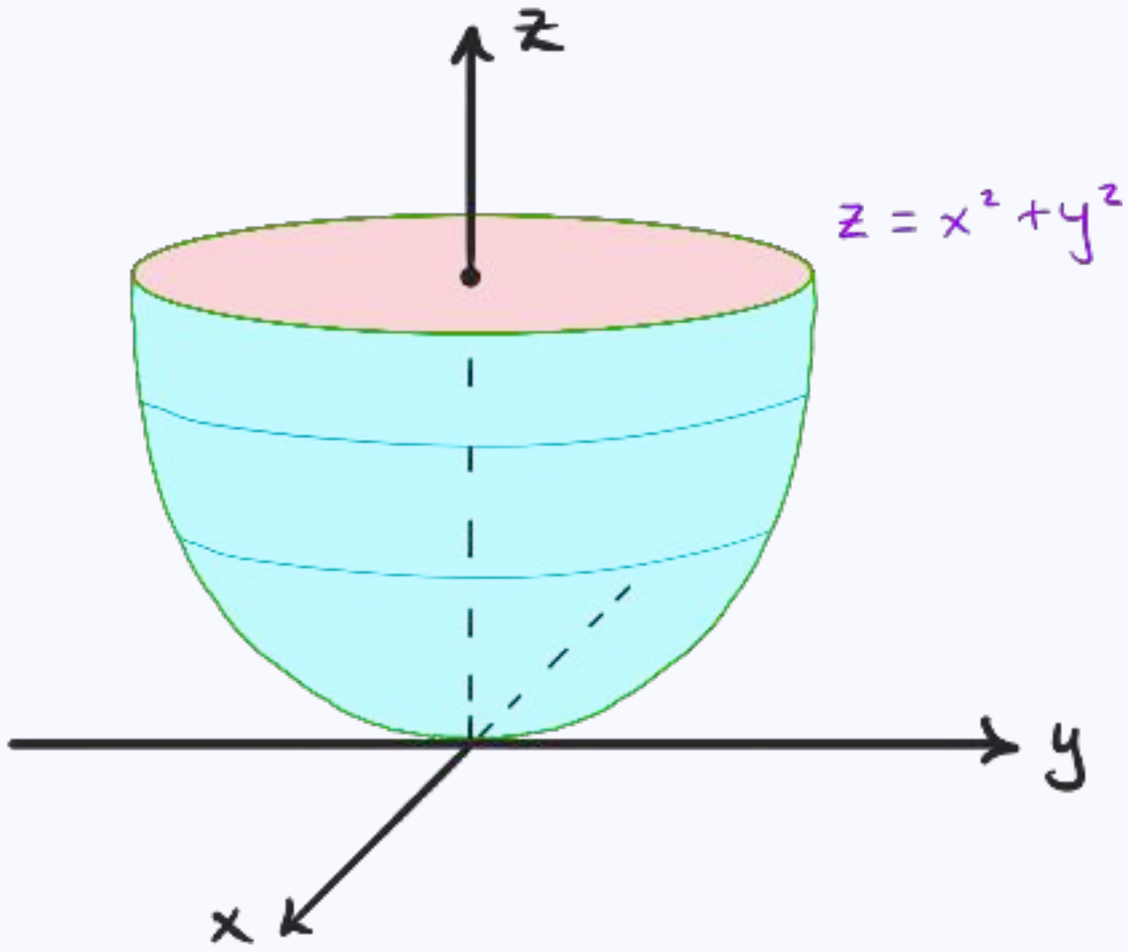
If we restrict to this purple curve (fixed $x=0$) we can see the tangent to this curve has an upward slope of 1 (i.e the height, z, is changing at a rate of 1 at this point on this curve).

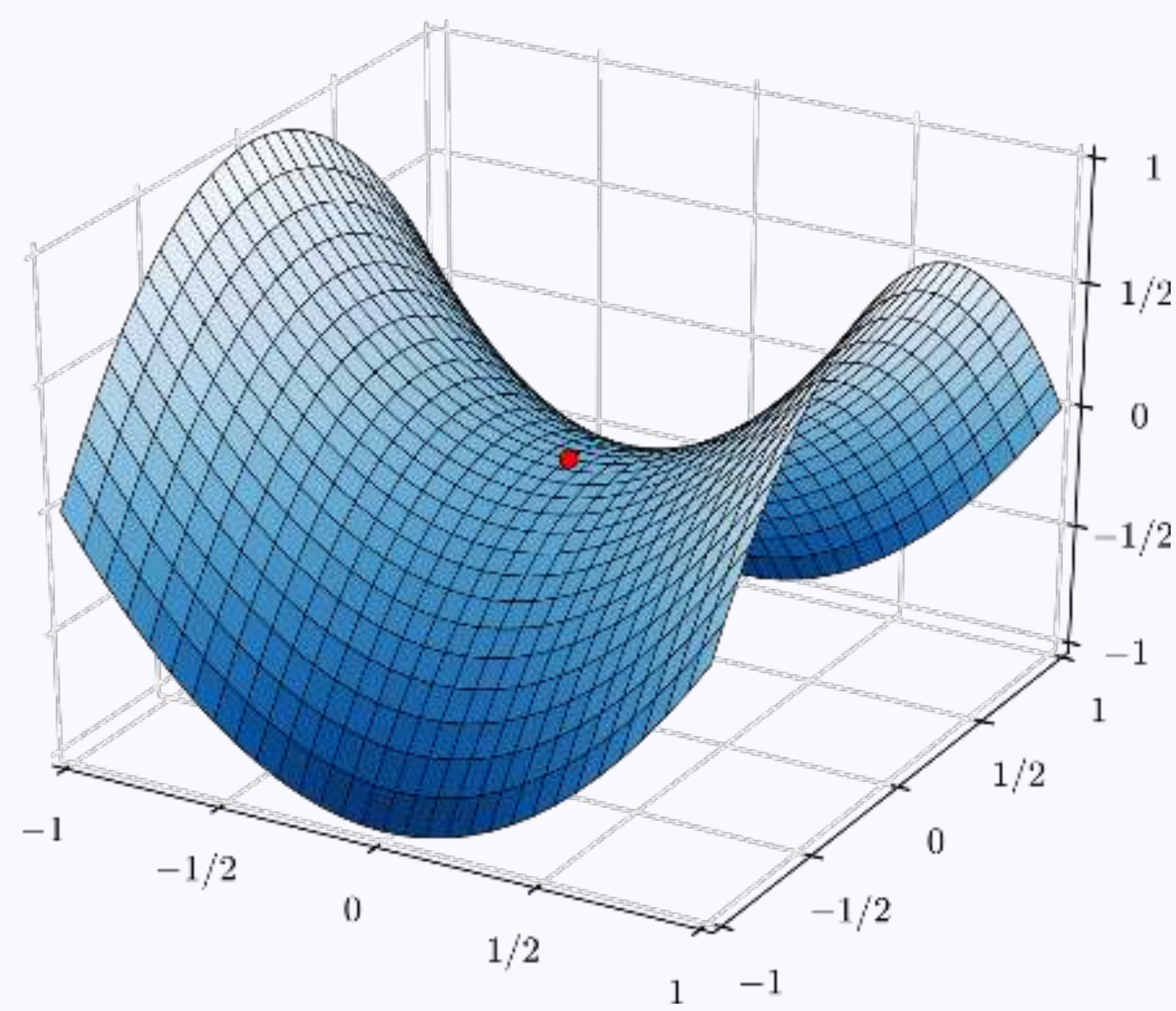
Question : What is $\frac{\partial f}{\partial x}(0,1)$?

Can you see why from the picture ?

Remark: Functions of several variables are sensitive to multiple inputs (& even linear combinations of them).

Examples:





§14. Chain Rule:

Goals for Today:

1) Introduce the Chain Rule for a function of several variables.

2) Use this to expand our previous knowledge of Implicit Differentiation.

Recall: In Calc. I we see how to break complicated relationships between a single input x and a single output $F(x)$:

$$x \longrightarrow \boxed{\quad F \quad} \longrightarrow F(x)$$

By breaking it into a chain of simpler processes:

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

such that $F(x) = f(g(x))$

i.e. $F = f \circ g$

We arrived at the one dimensional chain rule:

$$\boxed{(f \circ g)'(x) = f'(g(x))\, g'(x)}$$

So, let's proceed with our objectives for this lecture:

<u>Example:</u> Say we want to design a new car.

What will determine / influence it's top speed?

1) Weight (w)

2) Aero dynamic efficiency (s)

3) Engine Power (p)

4) Tire grip (g)

5) Transmission efficiency (t)

Say F is our top speed function.

Algebraically it might look something like:

$$F(w,s,p,g,t) = \frac{(s+p+t)g}{w}$$

So $\frac{\partial F}{\partial w}$ would be ____________________.

$\frac{\partial F}{\partial s}$ would be ____________________.

$\vdots$

But if we think a layer deeper, each variable w,s,p,g,t could depend on other variables:

1) Weight :

↳ Density of material (ρ)

↳ Surface Area of Car (A)

2) Aero-dynamic efficiency :

↳ Surface Area of Car (A)

↳ "Narrowness" of Car (N)

3) Engine power :

↳ Number of cylinders (c)

↳ Size of engine (z)

4) Tire grip :

↳ Thread depth (d)

↳ Tire size (τ)

↳ Material density (m)

5) Trasmission :

↳ Number of cogs (σ)

↳ Friction between cogs (μ).

We can represent these dependencies by writing

$$w(\rho, A), \ s(A, N), \ p(c, z), \ g(d, \tau, m), \ t(\sigma, \mu)$$

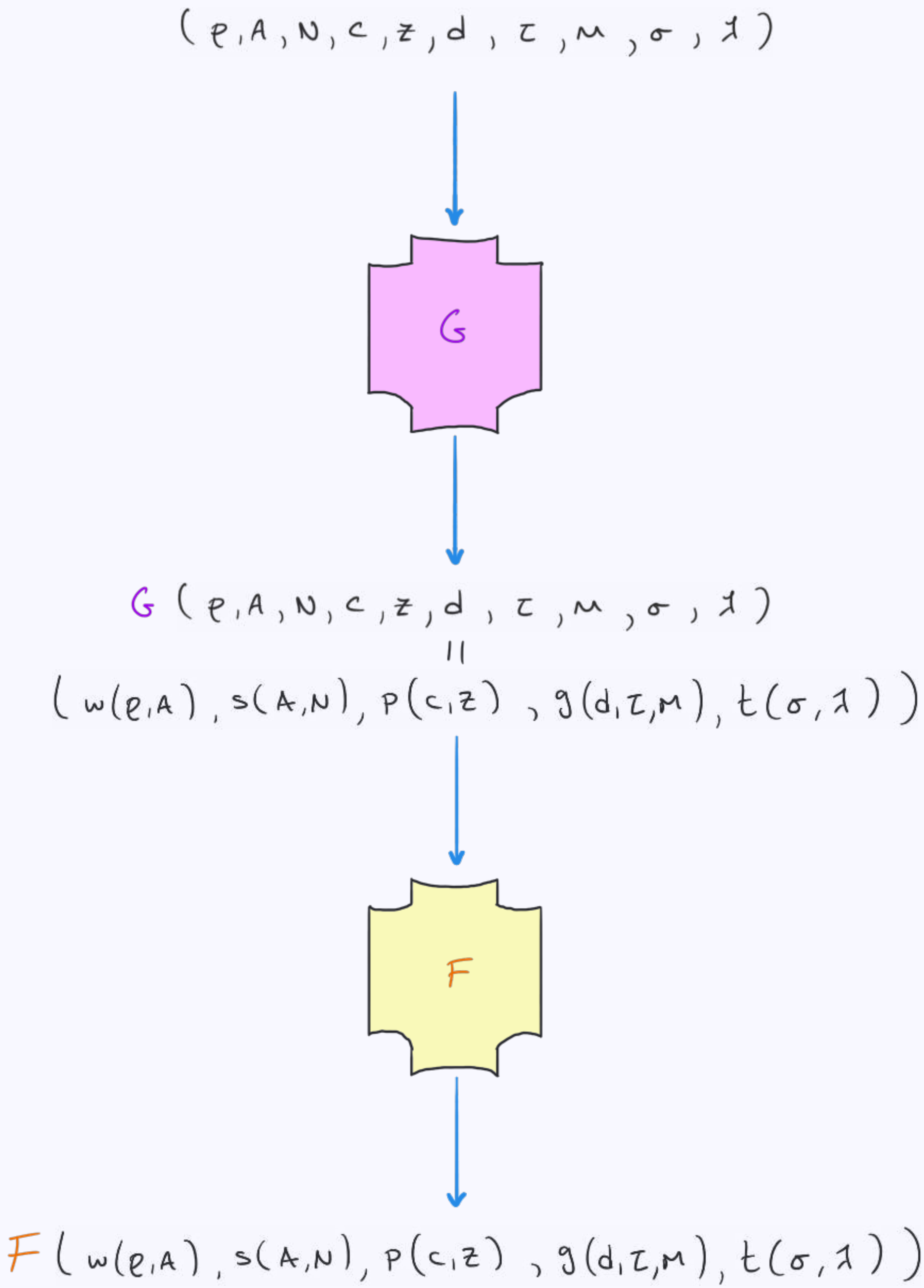
So now we can think of the machine that takes in values for $(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$ and spits out the corresponding top speed:

$$(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$$

$$\mathcal{H}(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$$

This machine would have very complicated relationships with all of the variables it depends on.

So we try to break it into a chain of simpler machines:


$(\rho, A, N, c, z, d, \tau, M, \sigma, \lambda)$
G
$G(\rho, A, N, c, z, d, \tau, M, \sigma, \lambda)$
$=$
$(w(\rho, A), s(A, N), p(c, z), g(d, \tau, M), t(\sigma, \lambda))$
F
$F(w(\rho, A), s(A, N), p(c, z), g(d, \tau, M), t(\sigma, \lambda))$

Such that $H = F \circ G$.

Then the multivariable chain rule says:

$$\frac{\partial H}{\partial e} = \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial e} + \frac{\partial F}{\partial s} \cdot \frac{\partial s}{\partial e} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial e} + \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial e} + \frac{\partial F}{\partial t} \cdot \frac{\partial t}{\partial e}$$

$$\frac{\partial H}{\partial A} = \dots$$

$$\vdots$$

In general:

The Chain Rule

If u is a differentiable function of n variables: $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of m variables: $t_1, t_2, \ldots, t_m$, then u is a function of $t_1, t_2, \ldots, t_m$ and:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

for any $i = 1, 2, \ldots, m$.

Remark: Don't worry too much about this formula. It is very general, and we are mainly concerned with the following two versions, which correspond to the cases:

↳ $m = 1$, $n = 2$

↳ $m = 2$, $n = 2$

"Just t":

If we have a function $f : \mathbb{R}^2 \to \mathbb{R} \; : \; f(x,y)$, and we are given $x = g(t)$, $y = h(t)$, i.e. x and y are now considered to be functions of t, we can write $z(t) = f(g(t), h(t))$. i.e. we have the process:

$$t \longrightarrow \boxed{(g,h)} \longrightarrow (g(t), h(t)) \longrightarrow \boxed{f} \longrightarrow f(g(t), h(t))$$

Combined to:

$$t \longrightarrow \boxed{z} \longrightarrow z(t) \quad \left(= f(g(t), h(t))\right)$$

And we want to see how sensitive $z(t)$ is to a change in t:

$$\boxed{\frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(g(t), h(t)) \cdot \frac{dg}{dt}(t) + \frac{\partial f}{\partial y}(g(t), h(t)) \cdot \frac{dh}{dt}(t)}$$

More compactly:

$$\boxed{\frac{dz}{dt}(t) = f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)}$$

Examples:

1) **3.**(6 pts) If $z = f(x, y)$, where f is differentiable, and $x = g(t), y = h(t)$, $g(1) = 3$, $h(1) = 4$, $g'(1) = -2$, $h'(1) = 5$, $f_x(3, 4) = 7$ and $f_y(3, 4) = 6$. Find dz/dt when $t = 1$.

(a) 13 (b) 44 (c) 32 (d) 23 (e) 16

Sol$\underline{n}$: ("Just t")

So $x = g(t)$, $y = h(t)$ and hence $z(t) = f(g(t), h(t))$.

By the formula:

$$\frac{dz}{dt}(1) = \frac{\partial f}{\partial x}(g(1), h(1)) \cdot g'(1) + \frac{\partial f}{\partial y}(g(1), h(1))\, h'(1)$$

$$= f_x(3, 4)(-2) + f_y(3, 4)(5)$$

$$= 7(-2) + 6(5)$$

$$= 16$$

"s and t":

If we have a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y)$ and we are given $x = g(s,t)$, $y = h(s,t)$, i.e x & y are considered to be functions of s and t, we can write $z(s,t) = f(g(s,t), h(s,t))$ i.e. we have the process:

$$(s,t) \longrightarrow \boxed{(g,h)} \longrightarrow (g(s,t), h(s,t)) \longrightarrow \boxed{f} \longrightarrow f(g(s,t), h(s,t))$$

Combined to:

$$(s,t) \longrightarrow \boxed{z} \longrightarrow z(s,t) \quad \big(= f(g(s,t), h(s,t))\big)$$

And we want to see how sensitive $z(s,t)$ is to a change in s or t:

$$\frac{\partial z}{\partial t}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t)) \frac{\partial h}{\partial t}(s,t)$$

&

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t)) \frac{\partial h}{\partial s}(s,t)$$

2) **5.**(6 pts) Let $f(x,y)$ be a function of $x(s,t)=st$ and $y(s,t)=2s+t$. If you know that $f_x(1,3)=2$ and $f_y(1,3)=-3$ then what is $\partial f/\partial s$ at when $s=1$ and $t=1$?

(a) -1

(b) not enough information to determine the value

(c) 3

(d) -4

(e) 0

Sol$^{\underline{n}}$: ("s and t")

Here we are given $x = x(s,t) = st$ and $y = y(s,t) = 2s+t$.

Hence $z(s,t) = f\left(x(s,t), y(s,t)\right)$.

We are asked for $\frac{\partial z}{\partial s}(1,1)$.

By the formula:

$$\frac{\partial z}{\partial s}(1,1) = f_x\left(x(1,1), y(1,1)\right)\frac{\partial x}{\partial s}(1,1) + f_y\left(x(1,1), y(1,1)\right)\frac{\partial y}{\partial s}(1,1)$$

- $x(s,t) = st \Rightarrow \frac{\partial x}{\partial s}(s,t) = t$

  Hence $x(1,1) = 1$ & $\frac{\partial x}{\partial s}(1,1) = 1$

- $y(s,t) = 2s + t \Rightarrow \frac{\partial y}{\partial s}(s,t) = 2$

  Hence $y(1,1) = 3$ & $\frac{\partial y}{\partial s}(1,1) = 2$

$$\Rightarrow \frac{\partial z}{\partial s}(1,1) = f_x(1,3)(1) + f_y(1,3)(2) = (2)(1) + (-3)(2)$$

$$= -4$$

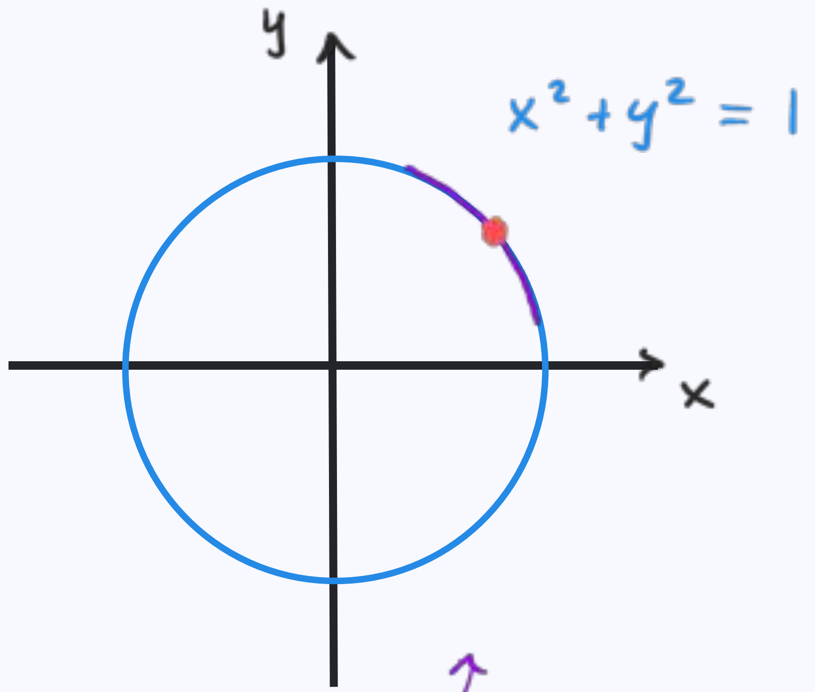
Implicit Differentiation: Recall in Calc. I we learned how to find tangents to curves defined by equations. e.g. $x^2 + xy = 2y^2$

By treating y (locally) as a function of x we were able to find $\frac{dy}{dx}$:

e.g.

This is not the graph of a function $y = f(x)$, but we can write y locally as a function of x:

$y = \sqrt{1-x^2}$ around red point



$x^2+y^2=1$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

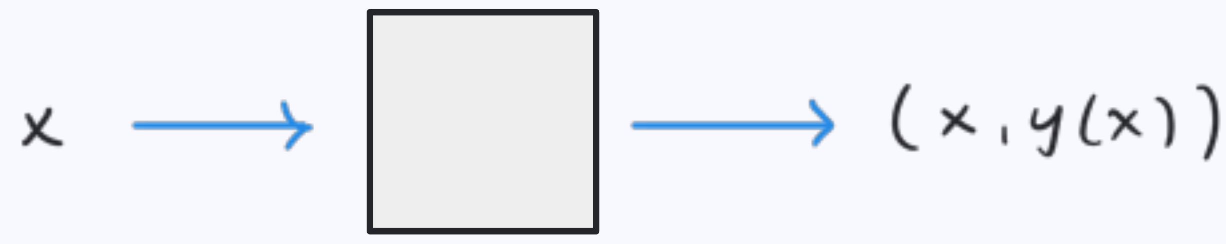
$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$, for $y \neq 0$.

depends on y. Why?

- Let's apply what we've learned about chain rule.

Let's think of y as a function of x (locally).

So we can think of the machine:



Let's go back to our example curve:

Example: Find the slope of the tangent line to the curve $x^2 + xy = 2y^2$ at the point $(1,1)$.

Sol^{n}: We will try to outline a general method here.

Step 1: Bring everything to the LHS.

$$x^2 + xy - 2y^2 = 0 \qquad (*)$$

Step 2: Let $F(x,y) = \text{LHS}$.

$$F(x,y) = x^2 + xy - 2y^2$$

Idea: If y is a function of x, we can think of the LHS as $F(x, y(x))$. Writing this as a chain:

$$x \longrightarrow \boxed{} \longrightarrow (x, y(x)) \longrightarrow \boxed{} \longrightarrow F(x, y(x))$$

From the "just t" chain rule (here "t = x"), if we take the derivative of $F(x, y(x))$ w.r.t. x we get:

$$F_x(x, y(x)) \underbrace{\frac{dx}{dx}}_{1}(x) + F_y(x, y(x)) \frac{dy}{dx}(x)$$

$$= F_x(x, y(x)) + F_y(x, y(x)) \frac{dy}{dx}(x) \qquad (*)$$

But $(*)$ says $F(x, y(x)) = 0$ on this curve.

So that derivative we got, (*), should be zero.

i.e. $F_x(x, y(x)) + F_y(x, y(x)) \frac{dy}{dx}(x) = 0$

Solving for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx}(x) = \frac{-F_x(x, y(x))}{F_y(x, y(x))}}$$

More compactly:

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}} \qquad (*)$$

If you are not concerned with theory, you can just memorize the formula in the purple box and proceed:

Step 3 : $F_x(x, y) = 2x + y$

$F_y(x, y) = x - 4y$

$(*) \Rightarrow \frac{dy}{dx} = -\frac{(2x + y)}{x - 4y}$

$$\Rightarrow \frac{dy}{dx}(1,1) = -\frac{3}{-3} = \boxed{1}$$

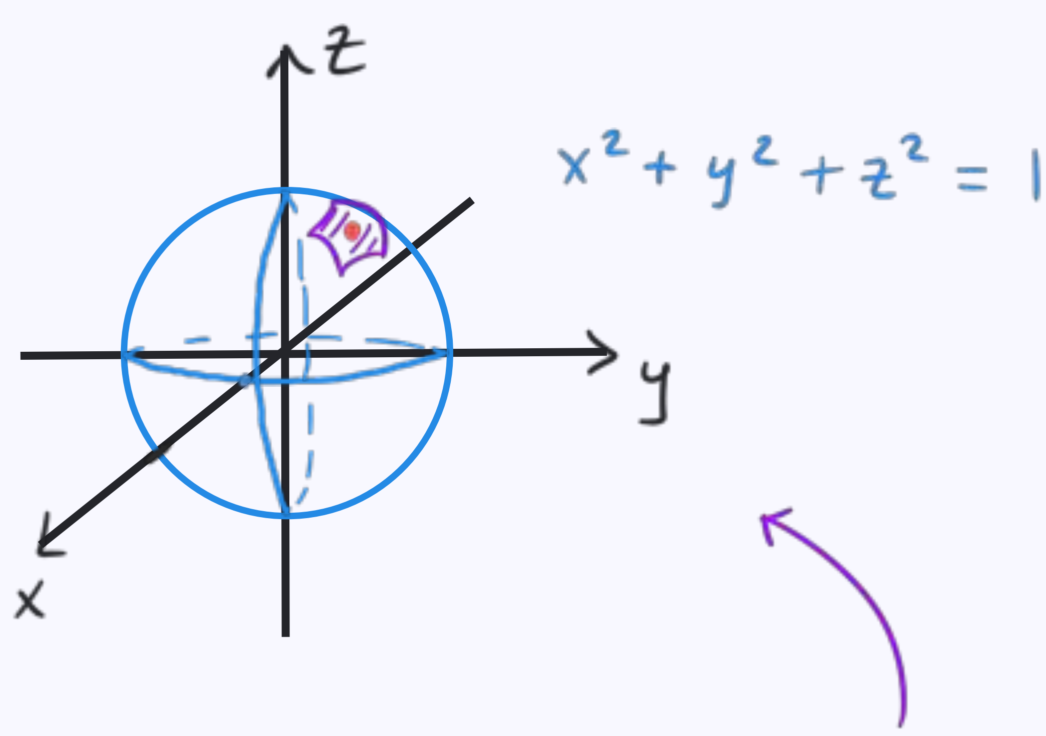
- Again, when answering a question, go straight from step 2 to step 3.

Even more variables: If instead of having a function define a "surface" $z = f(x,y)$, what if we were just given an equation:

e.g. $$z^2 + 2xz + 3yz = 2xy$$

Under certain circumstances (which will always be satisfied in examples we see in this course) we can think of z as being locally a function of x & y.

e.g.



This is not the graph of a function $z = f(x,y)$, but we can write z locally as a function of x & y: $z = \sqrt{1-x^2-y^2}$ "around" the red point.

- Let's apply what we've learned about chain rule.
  Let's think of z as a function of x and y (locally).

So we can think of the machine:

$$(x,y) \longrightarrow \boxed{} \longrightarrow (x, y, z(x,y))$$

Back to our example:

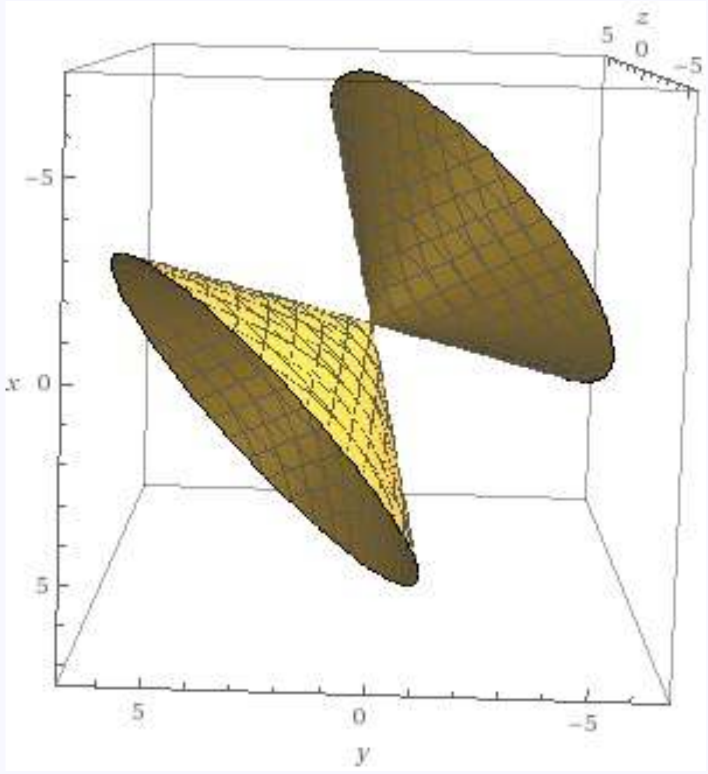
Example: Find $\frac{\partial z}{\partial x}$ at $(1,-3,1)$ for the surface

described by: $z^2 + 2xz + 3yz = 2xy$

Aside:

This is a graph of the "surface":

What does $\frac{\partial z}{\partial x}$ "mean" at $(1,-3,1)$?



Sol$\underline{n}$: We will again try to give a general method here, with some motivation/explanation between step 2 and step 3 which you can ignore if you're not interested.

Step 1: Gather everything to the LHS.

$$z^2 + 2xz + 3yz - 2xy = 0 \qquad (*)$$

Step 2: Let $F(x,y,z) = \text{LHS}$.

$$F(x,y,z) = z^2 + 2xz + 3yz - 2xy$$

Idea: If z is a function of x and y, we can think of the LHS as $F(x,y,z(x,y))$. Writing this as a chain:

$$(x,y) \longrightarrow \boxed{} \longrightarrow (x,y,z(x,y)) \longrightarrow \boxed{} \longrightarrow F(x,y,z(x,y))$$

We can see how sensitive this end output is to a change in x by using chain rule (very similar to "s and t" case):

$$F_x(x,y,z(x,y)) \cdot \underbrace{\frac{\partial x}{\partial x}(x,y)}_{=1} + F_y(x,y,z(x,y)) \underbrace{\frac{\partial y}{\partial x}(x,y)}_{\substack{=0 \\ \text{as } y \text{ is} \\ \text{indep. of} \\ x \text{ here}}} + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y)$$

$$= F_x(x,y,z(x,y)) + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y) \qquad (*)$$

But $(*)$ says $F(x,y,z(x,y)) = 0$ on this surface, so it's partial derivative w.r.t. x should be zero.

i.e. by $(*)$ we have:

$$F_x(x,y,z(x,y)) + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y) = 0$$

Isolating $\frac{\partial z}{\partial x}(x,y)$ we arrive at:

$$\boxed{\frac{\partial z}{\partial x}(x,y) = \frac{-F_x(x,y,z(x,y))}{F_z(x,y,z(x,y))}}$$

or, more compactly:

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}} \qquad (*)$$

similarly with y instead of x:

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}} \qquad (*)$$

Step 3: Apply relevant formula in purple box.

$$F_x(x,y,z) = 2z - 2y$$

$$F_z(x,y,z) = 2z + 2x + 3y$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-(2z-2y)}{2z+2x+3y}$$

So, at $(1,-3,1)$:

$$\frac{\partial z}{\partial x} = -\frac{8}{-5} = \boxed{\frac{8}{5}}$$

Example :

2.(6 pts) Use implicit differentiation to find $\partial z/\partial x$ when $xz + z^2 = y$.

(a) $\dfrac{\partial z}{\partial x} = \dfrac{-z}{x+2z}$

(b) $\dfrac{\partial z}{\partial x} = \dfrac{y}{x+z}$

(c) $\dfrac{\partial z}{\partial x} = \dfrac{-x}{2z}$

(d) $\dfrac{\partial z}{\partial x} = \dfrac{y-z}{x+2z}$

(e) $\dfrac{\partial z}{\partial x} = \dfrac{y-x}{2z}$

Exam Question Method :

Step 1 : $xz + z^2 - y = 0$

Step 2 : $F(x,y,z) = xz + z^2 - y$

Step 3 : $F_x(x,y,z) = z$

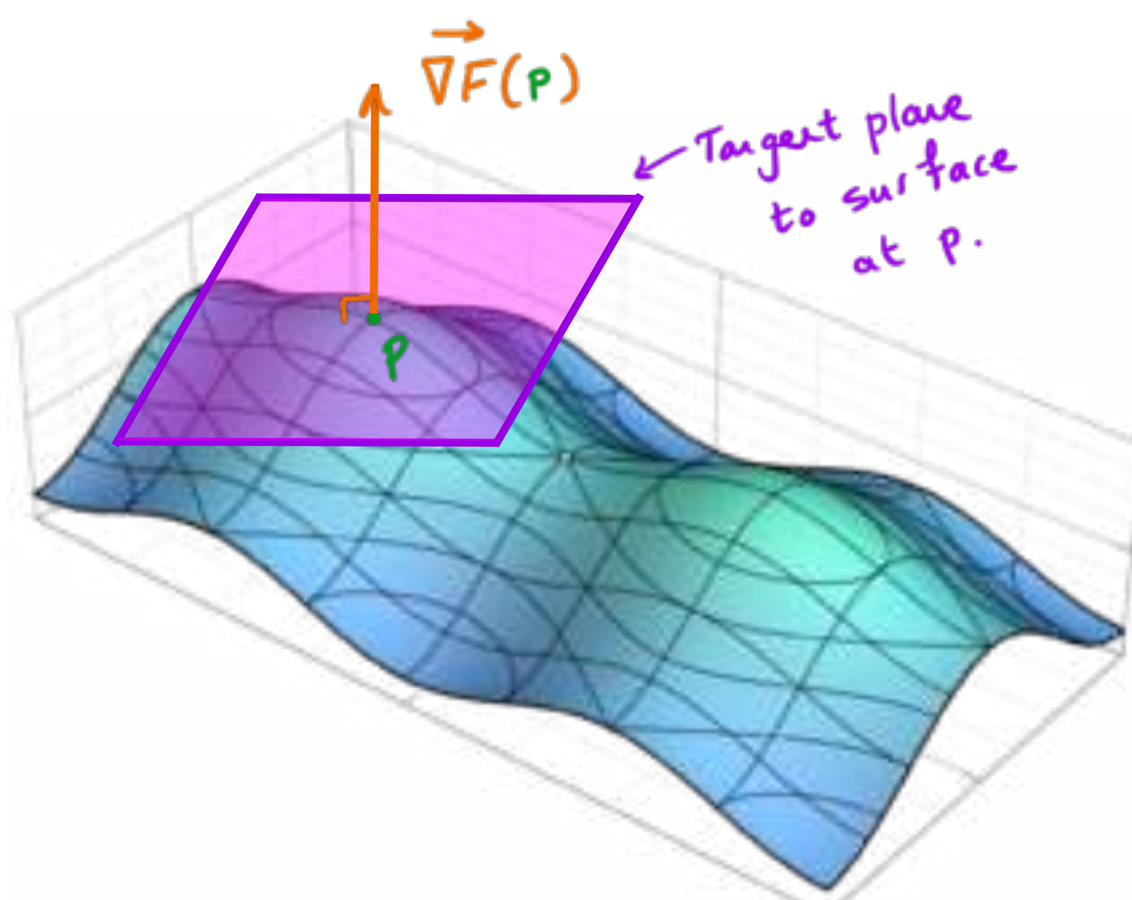
$F_z(x,y,z) = x + 2z$

Step 4 : $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_y} = \dfrac{-z}{x+2z}$

Problem Session 1 :

Definition : $\vec{\nabla} f(x,y,z) := \left(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \right)$

Theory : If a surface S is given by $F(x,y,z) = C$, and p is a point on the surface, then $\vec{\nabla} F(p)$ is normal to the tangent plane to the surface at p.



Special Case : If S is the graph of a function : $z = g(x,y)$, define $F(x,y,z) = z - g(x,y)$. So S is now given by $F(x,y,z) = 0$, and hence for any point p on S $\nabla \vec{F}(p) = (-g_x, -g_y, 1)$ is normal to the tangent plane to the surface at p.

Theory : $\vec{\nabla} F(p)$ points in the direction which will cause the greatest rate of change in the outputs of F "near" p.

Formula: If $x = g(t)$, $y = h(t)$ and $z(t) = f(g(t), h(t))$:

$$\boxed{\frac{dz}{dt}(t) = f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)}$$

Formula: If $x = g(s,t)$, $y = h(s,t)$ and $z(s,t) = f(g(s,t), h(s,t))$:

$$\frac{\partial z}{\partial t}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t))\, \frac{\partial h}{\partial t}(s,t)$$

&

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t))\, \frac{\partial h}{\partial s}(s,t)$$

Formula:

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla_{\vec{u}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \vec{\nabla} f(x,y,z) \cdot \hat{u}}$$

Method: If two surfaces $S_1 : F(x,y,z) = C_1$ & $S_2 : G(x,y,z) = C_2$ intersect at a point P, a direction for the tangent line to the curve of intersection at P can be given by $\boxed{\vec{v} = \vec{\nabla} F(P) \times \vec{\nabla} G(P)}$.

Formulas: If a surface is given by $F(x,y,z) = C$:

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}} \quad \& \quad \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Problem Session 2:

Theory: If a function has a local maximum or a local minimum at a point p, then $\vec{\nabla} f(p) = 0$.

Definition: A point p is called a critical point of f if $\vec{\nabla} f(p) = \vec{0}$.

NB: Critical points need not be local max. or mins.

Method: Suppose that f is a "nice" function, and that p is a critical point of f: $\vec{\nabla} f(p) = \vec{0}$.

Define $\boxed{D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2}$, or equivalently:

$$\boxed{D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{vmatrix}}$$

Then:

(i) If $D(p) > 0$ and $f_{xx}(p) > 0$, f has a local min. at p.

(ii) If $D(p) > 0$ and $f_{xx}(p) < 0$, f has a local max. at p.

(iii) If $D(p) < 0$, f has a saddle point at p.

Method: To find the absolute max./min. of a function f on a closed set whose boundary is made up of straight lines (e.g. a triangle or square):

Step 1: Find all critical points of f.

Step 2: Evaluate f at these points, ignoring ones that are outside of the region.

Step 3: Evaluate f at each "corner".

Step 4: Pick a side of the boundary and write it as $y = mx + c$, for x values in some interval (if the side is vertical, write it as $x = k$).

Step 5: Define $h(x) = f(x, mx + c)$ (or $g(y) = f(k, y)$).

Step 6: Find all critical points of h: $h'(x) = 0$ in the interval for x (or $\frac{dg}{dy} = 0$).

Step 7: Evaluate h at each critical point (or g).

Step 8: Do this for each side.

Step 9: Pick out the absolute max. and absolute min.

Method: To find the maximum and minimum values of a function f, subject to a constraint $g(x,y,z)=k$:

Step 1: Find all (x,y,z) such that there is a $\lambda \in \mathbb{R}$;

$$\boxed{\begin{aligned} \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \\ \text{and} \quad g(x,y,z) &= k \end{aligned}}$$

Step 2: Evaluate f at these points and pick out the maximum and minimum values.

Method: To find the maximum and minimum values of a function f, subject to two constraints:

$g(x,y,z)=k$ and $h(x,y,z)=l$:

Step 1: Find all (x,y,z) such that there is a λ & $\mu \in \mathbb{R}$;

$$\boxed{\begin{aligned} &\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) + \mu \vec{\nabla} h(x,y,z) \quad \& \\ &g(x,y,z)=k \quad \text{and} \quad h(x,y,z)=l \end{aligned}}$$

Step 2: Evaluate f at these points and pick out the max/min values.

Method: To find the absolute max./min. of a function f on a closed set whose boundary is given by a curve $g(x,y,z) = k$ (e.g. an ellipse or a disc):

Step 1: Find all critical points of f.

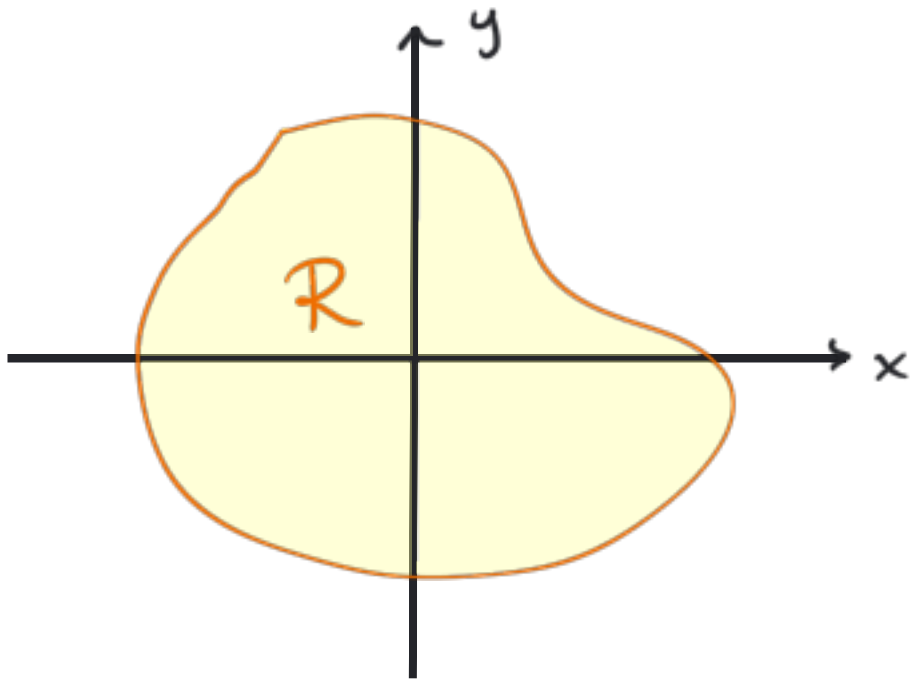
Step 2: Evaluate f at these points, ignoring ones that are outside of the region.

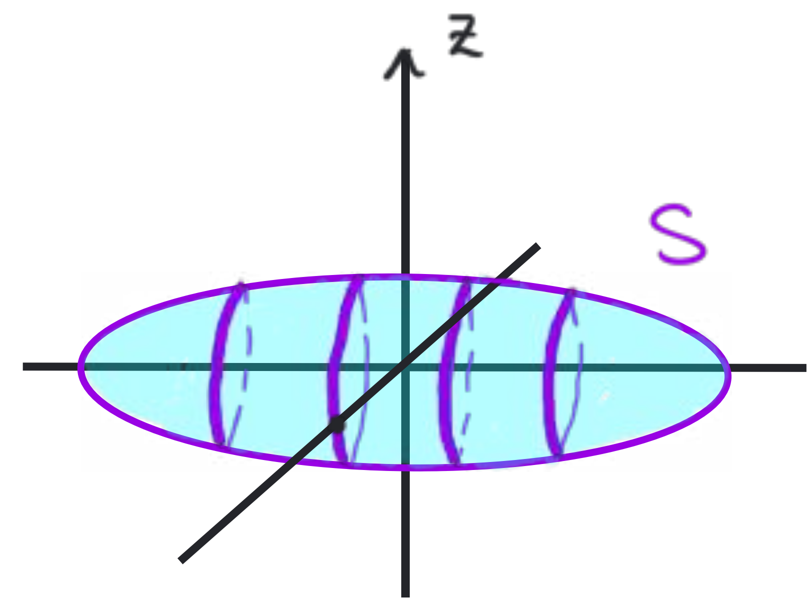
Step 3: Apply the method of Lagrange Multipliers to find the maximum/minimum of f on the boundary $g(x,y,z) = k$ i.e. subject to the constraint $g(x,y,z) = k$.

Step 4: Pick out the maximum and minimum values.

Problem Session 3:

Theory: If R is a region in $\mathbb{R}^2$ (the "xy-plane"), and S is some solid sitting in $\mathbb{R}^3$:





$$\text{Area}(R) = \iint_R dA \qquad\qquad \text{Volume}(S) = \iiint_S dV$$

- In Cartesian / Retangular coordinates:

$$\boxed{dA = dx\,dy} \qquad\qquad \boxed{dV = dx\,dy\,dz}$$

- In Polar / Cylindrical Coordinates:

$$\boxed{dA = r\,dr\,d\theta} \qquad\qquad \boxed{dV = r\,dz\,dr\,d\theta}$$

Method: It is almost always useful to sketch the region of integration.

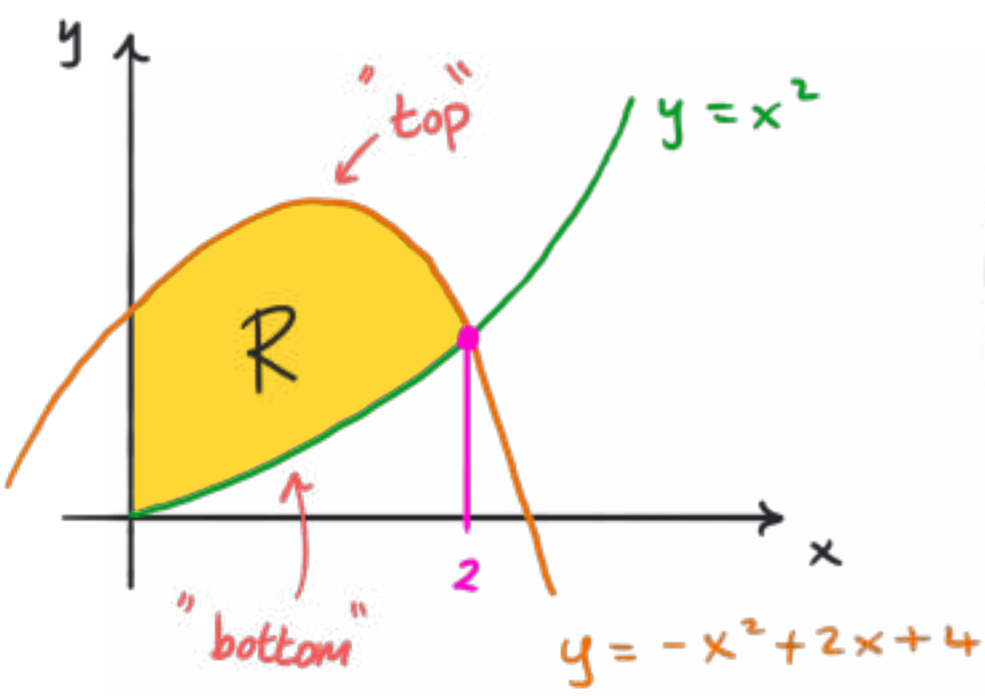
Types of problems:

1) Compute the area of a region $R \subset \mathbb{R}^2$:

Example: Find the area of the region given by:

$$R = \{(x,y) \; ; \; 0 \le x \le 2 \; , \; x^2 \le y \le x^2 - 2x + 4\}.$$

Solution:



(Aside: $x^2 = -x^2 + 2x + 4$
$\Rightarrow 2x^2 - 2x - 4 = 0$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow x = 2$ or $x = -1$)

$$\text{Area}(R) = \iint_R dA = \int_0^2 \int_{\underset{\text{"bottom"}}{x^2}}^{\overset{\text{"top"}}{-x^2+2x+4}} dy\,dx = \int_0^2 y \Big|_{x^2}^{-x^2+2x+4} dx$$

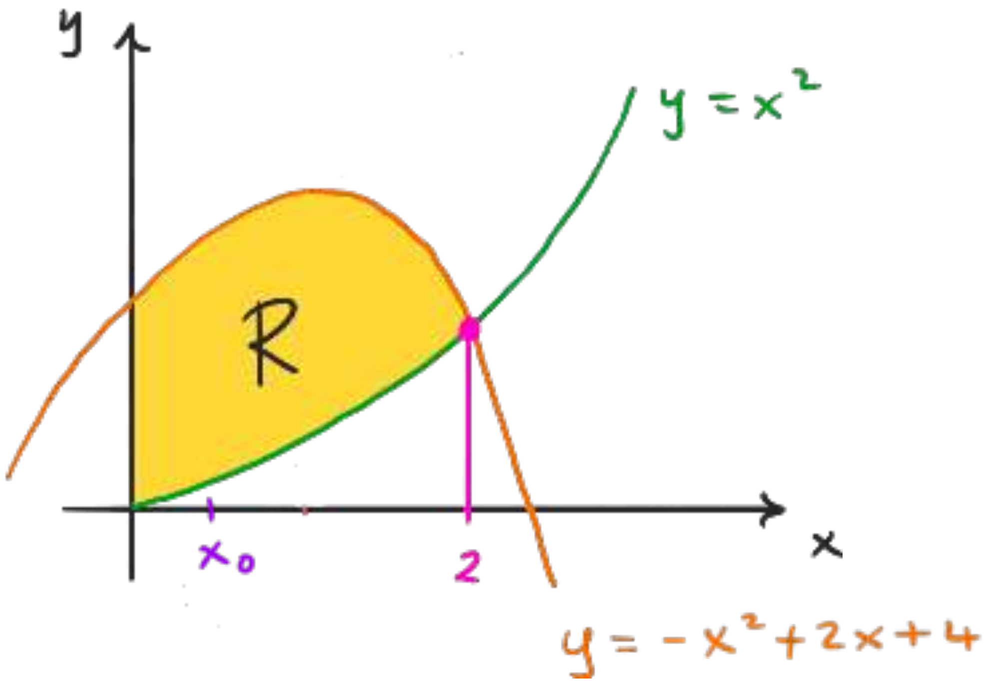
$$= \int_0^2 \left(-x^2 + 2x + 4 - x^2\right) dx = \int_0^2 -2x^2 + 2x + 4 \, dx$$

$$= -\frac{2}{3}x^3 + x^2 + 4x \Big|_0^2 = -\frac{16}{3} + 4 + 8 = \frac{20}{3}$$

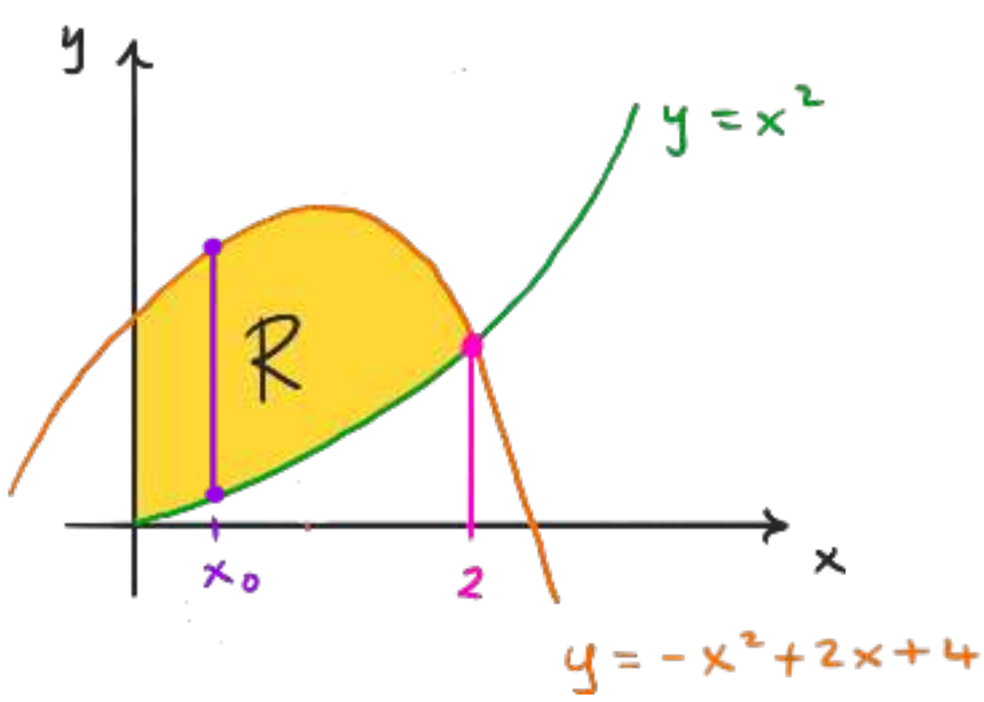
Remark: At this stage: $\int_0^2 \int_{x^2}^{-x^2+2x+4} dy\,dx$, you should read your variables of integration "from the outside in".

i.e. you are handed an $x_0 \in [0,2]$: $\int_0^2 \int_{x^2}^{-x^2+2x+4} dy\,dx$

outside



and you compute the length of this cord:



which is: $\displaystyle\int_{x_0^2}^{-x_0^2+2x_0+4} dy = y\Big|_{x_0^2}^{-x_0^2+2x_0+4}$

$$= (-x_0^2+2x_0+4) - x_0^2$$

$$= -2x_0^2 + 2x_0 + 4$$

You do this for all $x \in [0,2]$: Lenght of "x-cord" $= -2x^2+2x+4$, and integrate x over $[0,2]$ to "shade in" the yellow area:

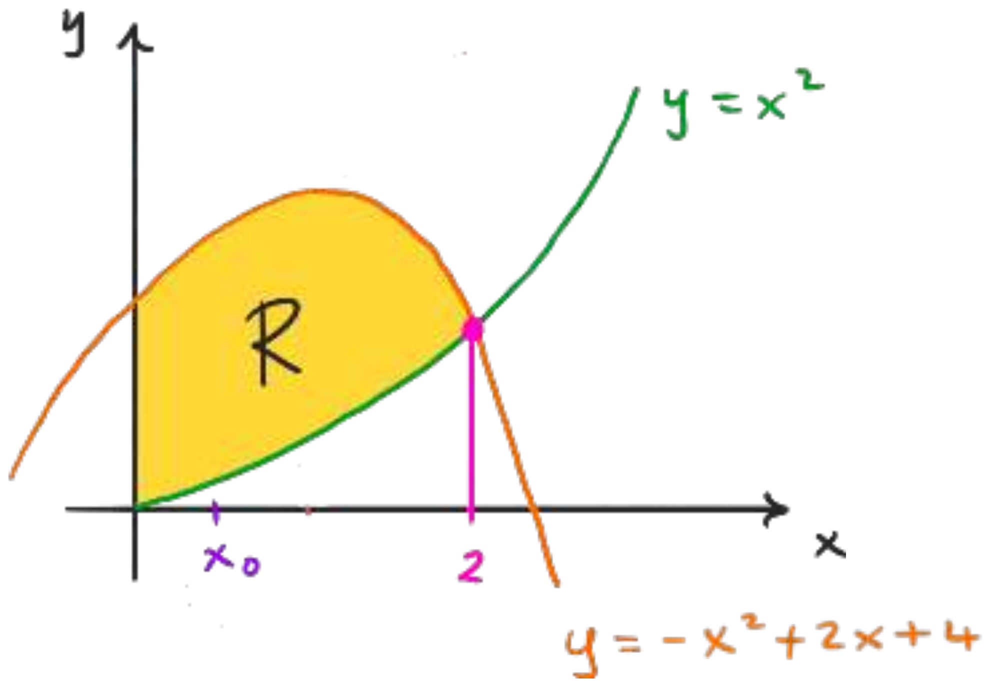
$$\text{Area}(R) = \int_0^2 (-2x^2+2x+4)\,dx = \frac{20}{3}.$$

This way of thinking about it can help you decide your bounds for each of your variables.

2) Integrate a function over a region $R \subset \mathbb{R}^2$:

Example : Let's use the same region as before

for simplicity : $R = \{ (x,y) \; ; \; 0 \leq x \leq 2 \; , \; x^2 \leq y \leq -x^2+2x+4 \}$



Compute $\iint_R f(x,y)\,dA$

where $f(x,y) = x$

Sol$\underline{n}$: Write bounds as before :

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{-x^2+2x+4} f(x,y)\,dy\,dx = \int_0^2 \int_{x^2}^{-x^2+2x+4} (x)\,dy\,dx$$

$$= \int_0^2 xy \Big|_{x^2}^{-x^2+2x+4} dx$$

$$= \int_0^2 x(-x^2+2x+4) - x(x^2)\,dx$$

$$= \int_0^2 -2x^3 + 2x^2 + 4x \;dx$$

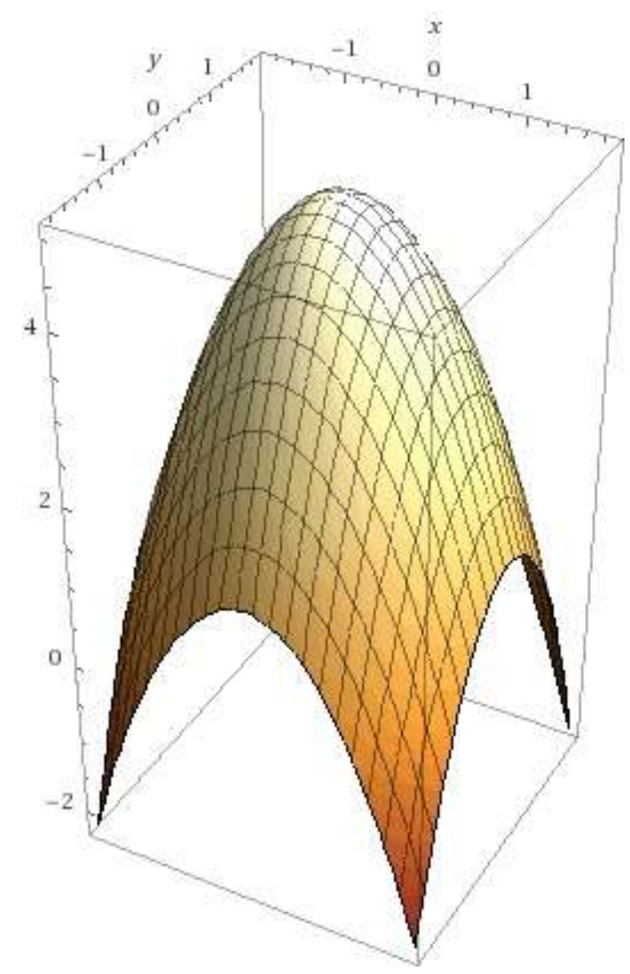
$$= -\frac{1}{2}x^4 + \frac{2}{3}x^3 + 2x^2 \Big|_0^2 = \frac{16}{3}$$

3) Find the volume of a solid:

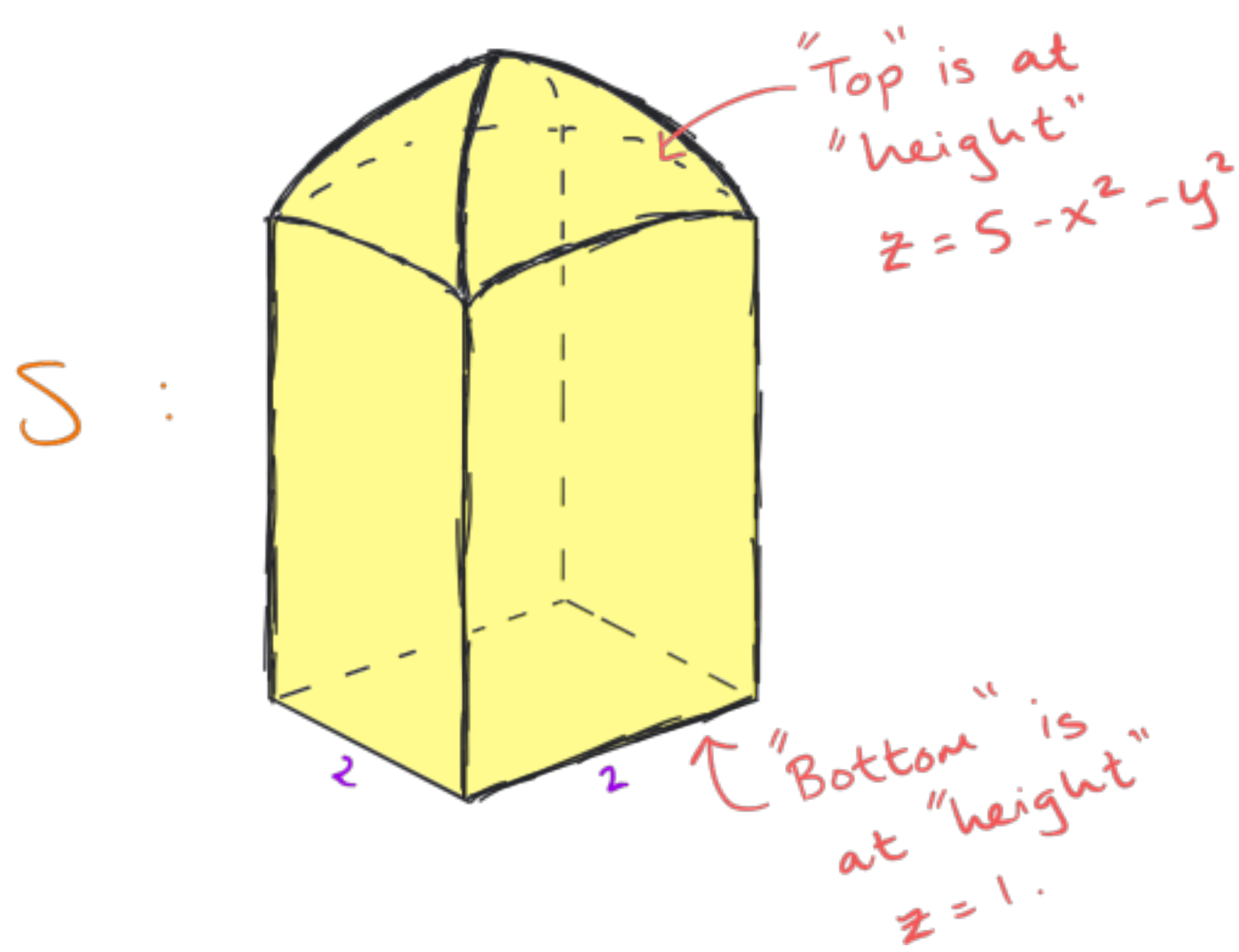
Find the volume of the solid bounded by the planes:

$x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1$

and the graph of $z = 5 - x^2 - y^2$:



So, the shape looks like:



So $S = \{ (x,y,z) ; -1 \le x \le 1, -1 \le y \le 1, 1 \le z \le 5 - x^2 - y^2 \}$

So : $\text{Vol}(S) = \iiint\limits_S dV = \int_{-1}^{1} \int_{-1}^{1} \int_{1}^{5-x^2-y^2} dz\,dy\,dx$

$$= \int_{-1}^{1}\int_{-1}^{1} z \Big|_{1}^{5-x^2-y^2} dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} (5-x^2-y^2) - 1 \; dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 4 - x^2 - y^2 \; dy\,dx$$

$$= \int_{-1}^{1} \left(4y - x^2 y - \frac{y^3}{3}\right)\Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left\{4 - x^2 - 1/3\right\} - \left\{-4 + x^2 + 1/3\right\} dx$$

$$= \int_{-1}^{1} \left(\frac{22}{3} - 2x^2\right) dx$$

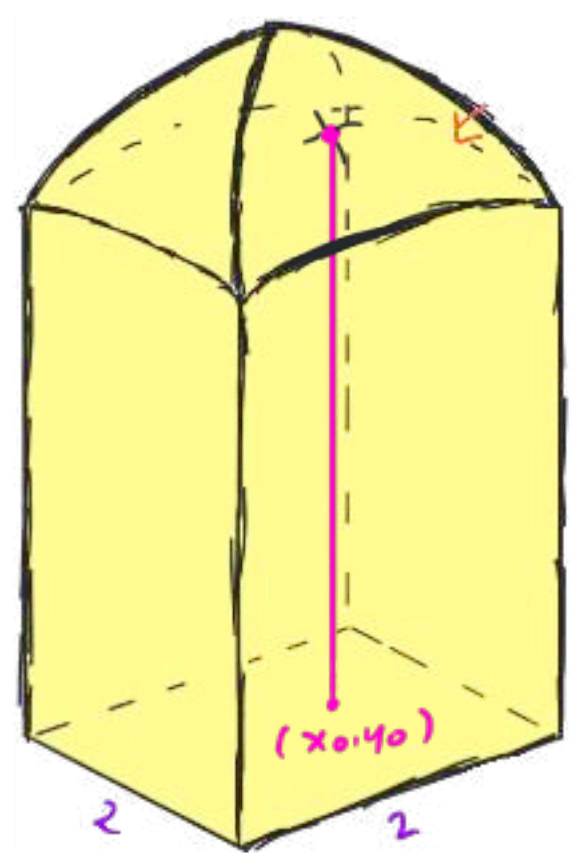
$$= \frac{22}{3}x - \frac{2x^3}{3} \Big|_{-1}^{1}$$

$$= \left(\frac{22}{3} - \frac{2}{3}\right) - \left(-\frac{22}{3} + \frac{2}{3}\right)$$

$$= \frac{40}{3}$$

Remark: Similar to the area problem, we can think of this volume being constructed by: Picking (x_0, y_0) and computing the length of the "cord" above (x_0, y_0):

S :



Then integrate over all $(x,y) \in [-1,1] \times [-1,1]$ to "fill in" S with cords.

Bonus Questions: 1) $\frac{40}{3} = 8 + \frac{16}{3}$

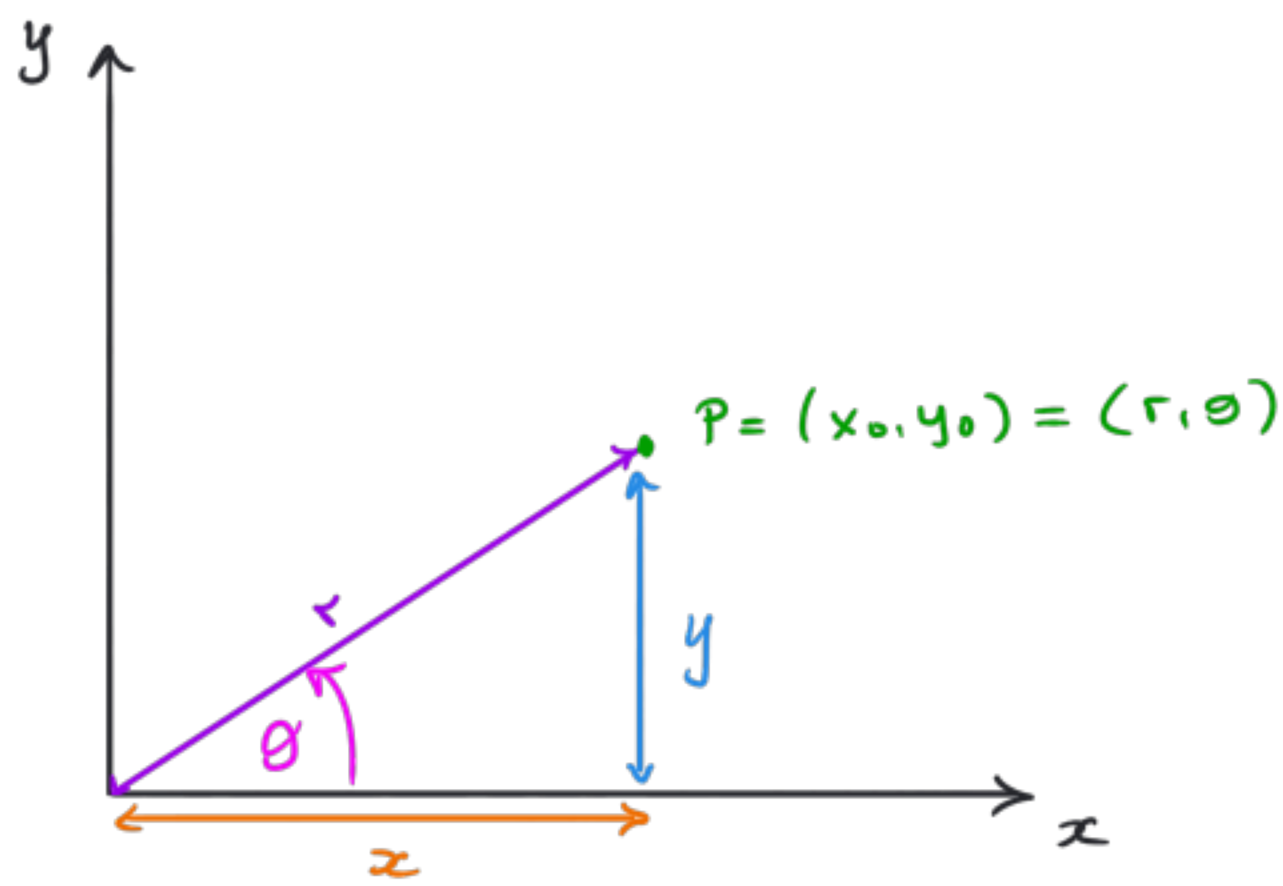
Explain from the picture why $\int_{-1}^{1}\int_{-1}^{1} (2 - x^2 - y^2)\,dy\,dx = \frac{16}{3}$.

Hint: Think of S as "a cap sitting on a box".

2) Assume $f(x,y) \geq 0$ on a region R. Why does $\iint_R f(x,y)\,dA$ = the Volume "trapped" under the graph of f over R?

$\int$ Calculus III : Exam 3 Notes :

Polar Coordinates :



We can represent the point p in either cartesian : (x, y), or polar : (r, θ) coordinates.

We can see from the picture that we should have :

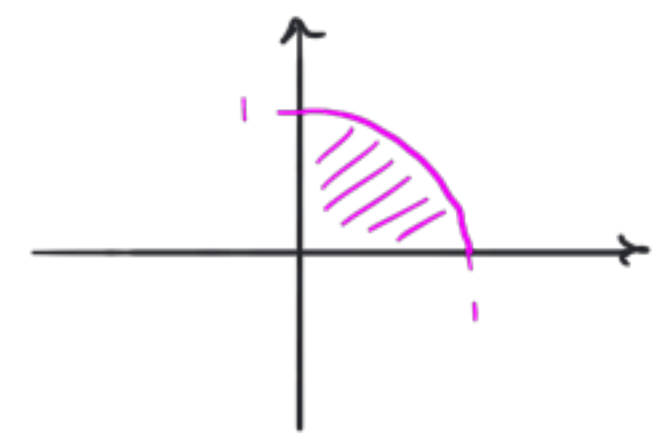
$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2}$$

We also have :

$$\boxed{dA = r\,dr\,d\theta}$$

Remark: Some regions in $\mathbb{R}^2$ are easier to describe in polar coordinates.

Example: Describe this region:



Remark: Hence it is sometimes useful to integrate over regions in $\mathbb{R}^2$ using polar coordinates.

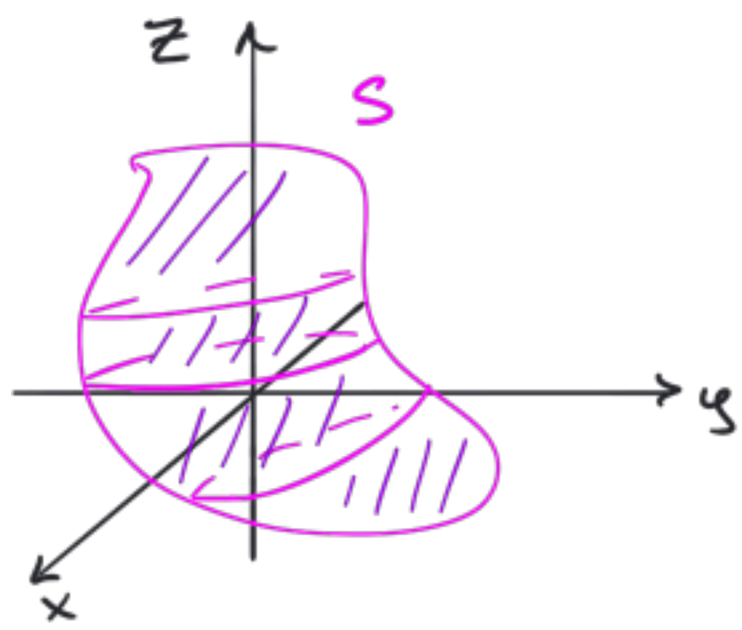
Here's how to switch from cartesian to polar:

$$\boxed{\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = \iint_{R_{(r,\theta)}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Where $R_{(x,y)}$ means the region expressed in cartersian coordinates, and $R_{(r,\theta)}$ means the region expressed in polar coordinates.

Triple Integrals :

Consider a solid $S \subset \mathbb{R}^3$:



$$\boxed{\text{Volume}(S) = \iiint_S dV}$$

To integrate a function f over a solid S :

$$\boxed{\iiint_S f \, dV}$$

In Cartesian coordinates : $\boxed{dV = dx\,dy\,dz}$ and

$$\boxed{\text{Vol}(S) = \iiint_{S_{(x,y,z)}} dx\,dy\,dz}$$

$$\boxed{\iiint_S f \, dV = \iiint_{S_{(x,y,z)}} f(x,y,z)\, dx\,dy\,dz}$$

Applications of Double \ Triple Integrals :

2D: For a lamina $D \subset \mathbb{R}^2$ with density function δ :

- $$\boxed{\text{Mass}(D) = \iint_D \delta \, dA = \iint_{D_{(x,y)}} \delta(x,y)\, dx\, dy =: m}$$

- Moment of lamina around :

  x-axis : $$\boxed{M_x = \iint_{D_{(x,y)}} y\,\delta(x,y)\, dx\, dy}$$

  y-axis : $$\boxed{M_y = \iint_{D_{(x,y)}} x\,\delta(x,y)\, dx\, dy}$$

- Center of Mass of D : $(\bar{x}, \bar{y})$ where :

  $$\boxed{\bar{x} = \frac{M_y}{m}} \quad \text{and} \quad \boxed{\bar{y} = \frac{M_x}{m}}$$

- Moment of Inertia around :

  x-axis : $$\boxed{I_x = \iint_{D_{(x,y)}} y^2\,\delta(x,y)\, dx\, dy}$$ y-axis : $$\boxed{I_y = \iint_{D_{(x,y)}} x^2\,\delta(x,y)\, dx\, dy}$$

  origin : $$\boxed{I_0 = \iint_{D_{(x,y)}} (x^2+y^2)\,\delta(x,y)\, dx\, dy}$$

3D: For a solid $S \subset \mathbb{R}^3$ with density function δ:

- $$\boxed{\text{Mass}(S) = \iiint_S \delta \, dV = \iiint_{S_{(x,y,z)}} \delta(x,y,z) \, dV_{(x,y,z)} =: M}$$

- Moments around each coordinate plane:

$$\boxed{\begin{aligned} M_{yz} &= \iiint_{S_{(x,y,z)}} x\, \delta(x,y,z) \, dV_{(x,y,z)} \\ M_{xz} &= \iiint_{S_{(x,y,z)}} y\, \delta(x,y,z) \, dV_{(x,y,z)} \\ M_{xy} &= \iiint_{S_{(x,y,z)}} z\, \delta(x,y,z) \, dV_{(x,y,z)} \end{aligned}}$$

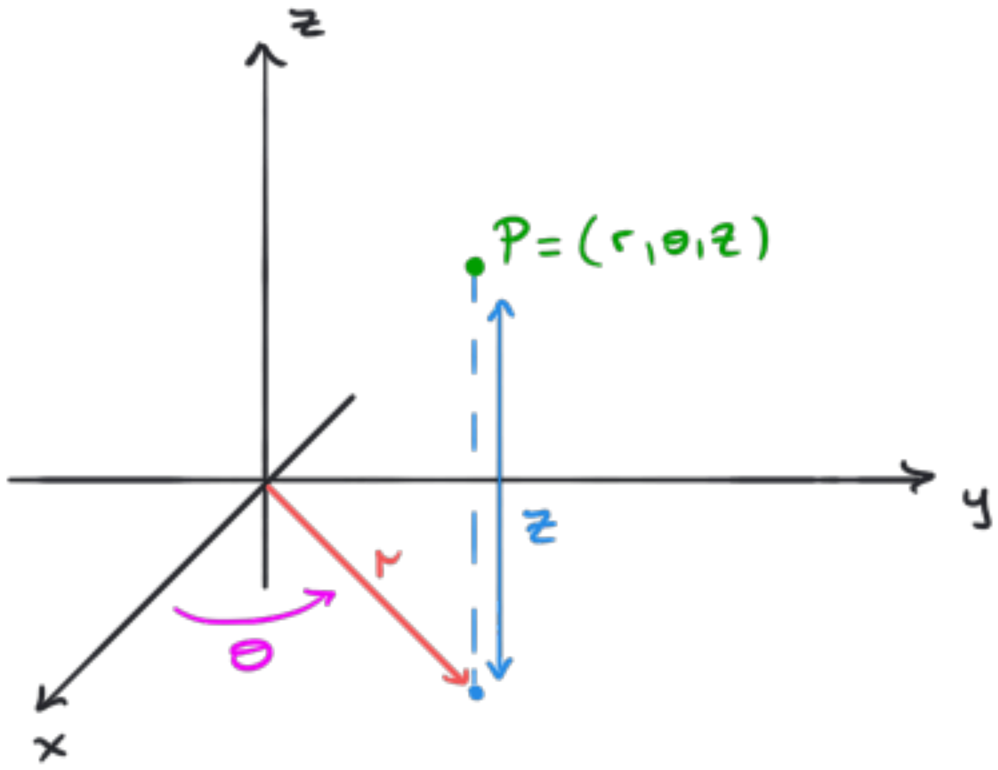
- Center of Mass of S: $(\bar{x}, \bar{y}, \bar{z})$ where:

$$\boxed{\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}}$$

Moments of Inertia around each coordinate axis:

$$\boxed{\begin{aligned} I_x &= \iiint_{S_{(x,y,z)}} (y^2+z^2)\, \delta(x,y,z) \, dV_{(x,y,z)} \\ I_y &= \iiint_{S_{(x,y,z)}} (x^2+z^2)\, \delta(x,y,z) \, dV_{(x,y,z)} \\ I_z &= \iiint_{S_{(x,y,z)}} (x^2+y^2)\, \delta(x,y,z) \, dV_{(x,y,z)} \end{aligned}}$$

Triple Integrals in Cylindrical Coordinates :



$0 \le r \quad , \quad 0 \le \theta \le 2\pi \quad , \quad -\infty < z < \infty$

$x = r\cos\theta$

$y = r\sin\theta$

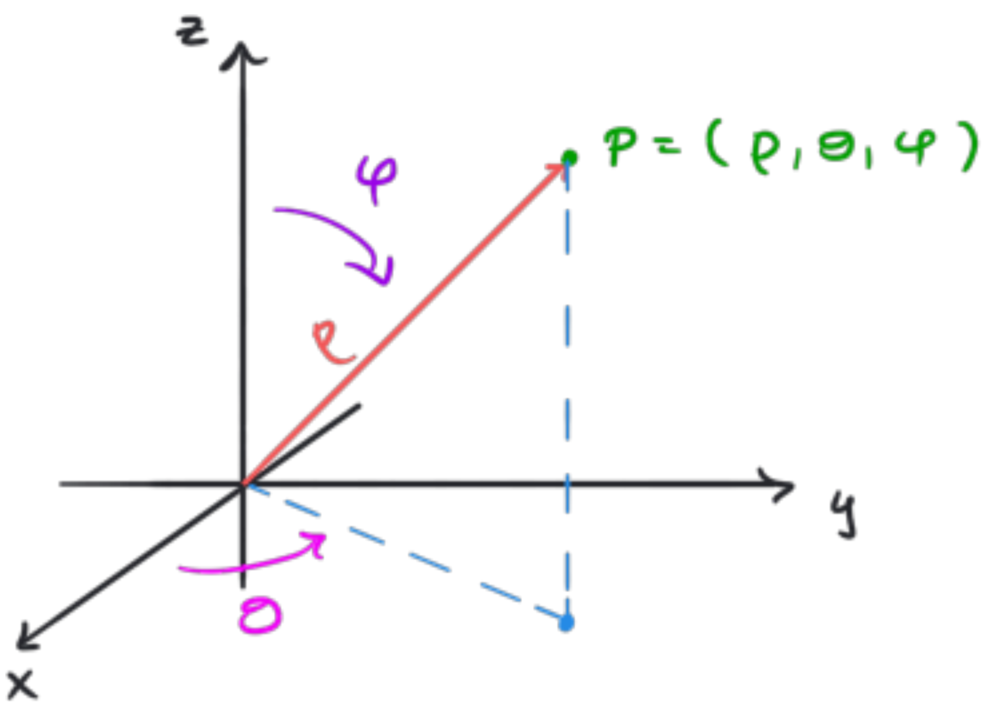
$z = z$

$dV_{(r,\theta,z)} = r\,dz\,dr\,d\theta$

$$\iiint_{S_{(x,y,z)}} f(x,y,z)\,dV_{(x,y,z)} = \iiint_{S_{(r,\theta,z)}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Example : (Old Exam Q1)

Triple Integrals in Spherical Coordinates:



$0 \le \rho$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$

$z = \rho \cos\varphi$

$dV_{(\rho,\theta,\varphi)} = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

$$\iiint_{S_{(x,y,z)}} f(x,y,z)\, dV_{(x,y,z)}$$

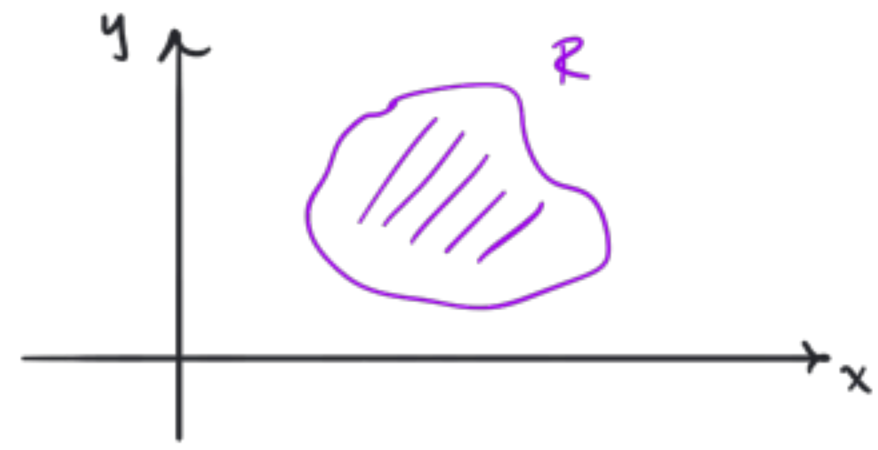
$$\|$$

$$\iiint_{S_{(\rho,\theta,\varphi)}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Example: (Old Exam Q4)

Change of Variables in Multivariate Integrals:

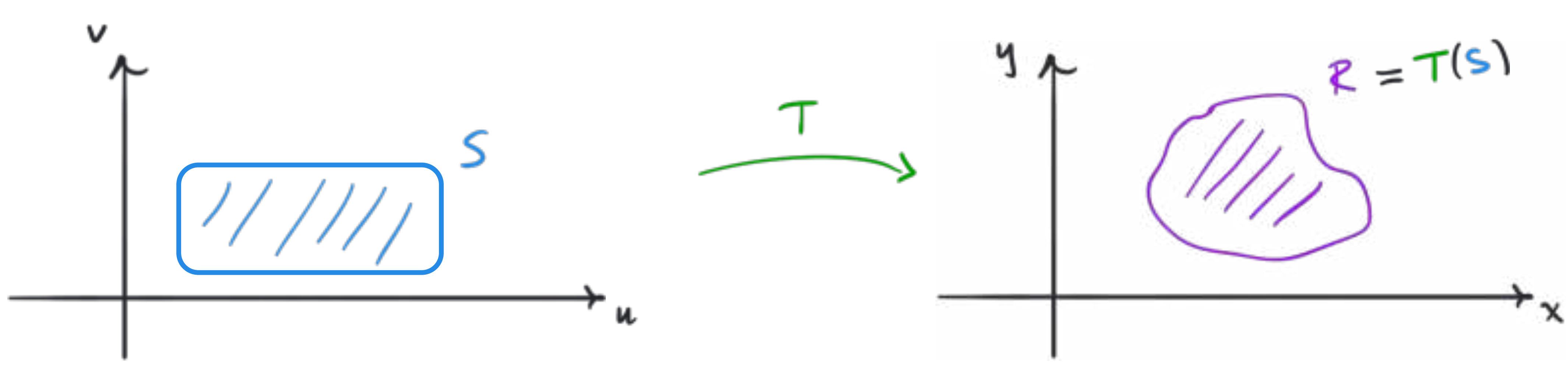
Let's say you're tasked with integrating a function f over a region $R \subset \mathbb{R}^2$:



$$\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = \,?$$

R may be very difficult to describe in Cartesian coordinates: $\left(R_{(x,y)} = \,?\right)$.

But imagine we have a simpler set $S \subset \mathbb{R}^2$, and a "nice" map T such that $T(S) = R$:



We can use a change of variables $T(u,v) = (x,y) = \big(x(u,v), y(u,v)\big)$ to integrate over the simpler set S:

$$\boxed{\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = \iint_{S_{(u,v)}} f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

where :

$$\boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right|}$$

This is referred to as the Jacobian of the transformation.

Example : $R_{(x,y)} = \{(x,y) \; ; \; 0 \leq x \leq 2 \; , \; 0 \leq y \leq 2\}$, $x = 2u$, $y = 2v$.

Find Area(R) .

Intuition :

3D: $T(u,v,w) = \big(x(u,v,w), y(u,v,w), z(u,v,w)\big).$

$$\boxed{\iiint_{R_{(x,y,z)}} f(x,y,z)\,dx\,dy\,dz = \iiint_{S_{(u,v,w)}} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

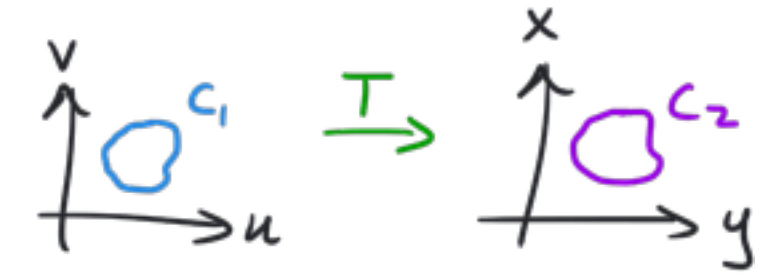
where

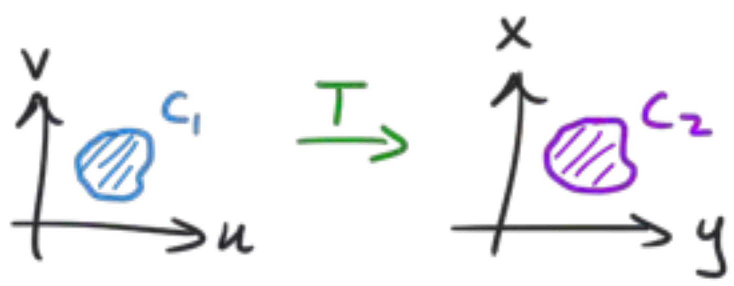
$$\boxed{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}}$$

Examples: (i) Cylindrical coordinates

(ii) Spherical coordinates

Remark: These transformations "map interiors to interiors".

i.e. If T sends a "loop" C_1 in the uv-plane to a "loop" C_2 in the xy plane: 

then it maps the region "inside" C_1 to the region "inside" C_2: 

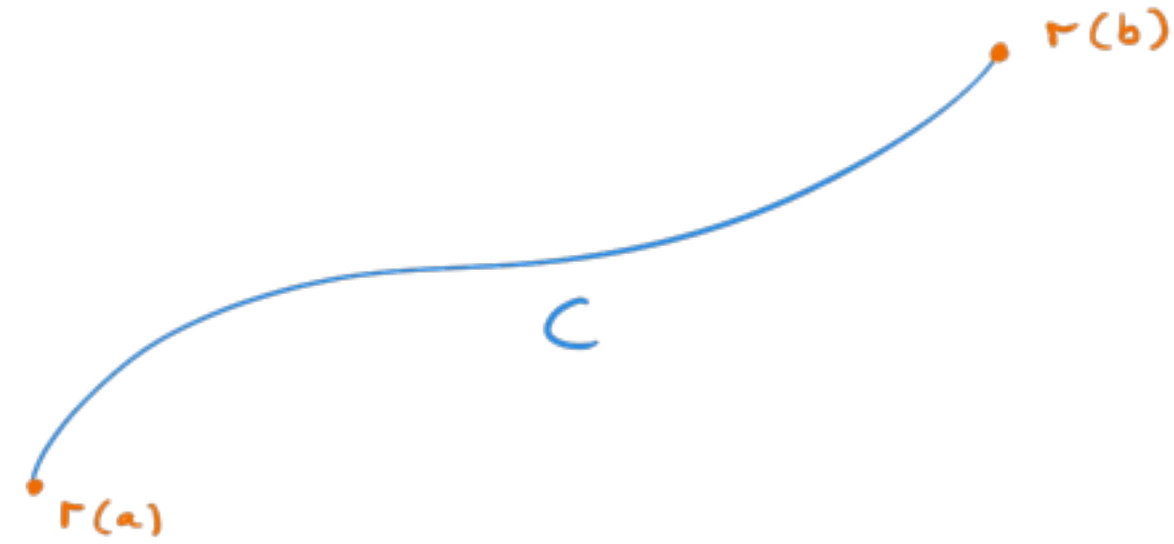
.

Example: Webassign parallelogram problems.

Line Integrals:

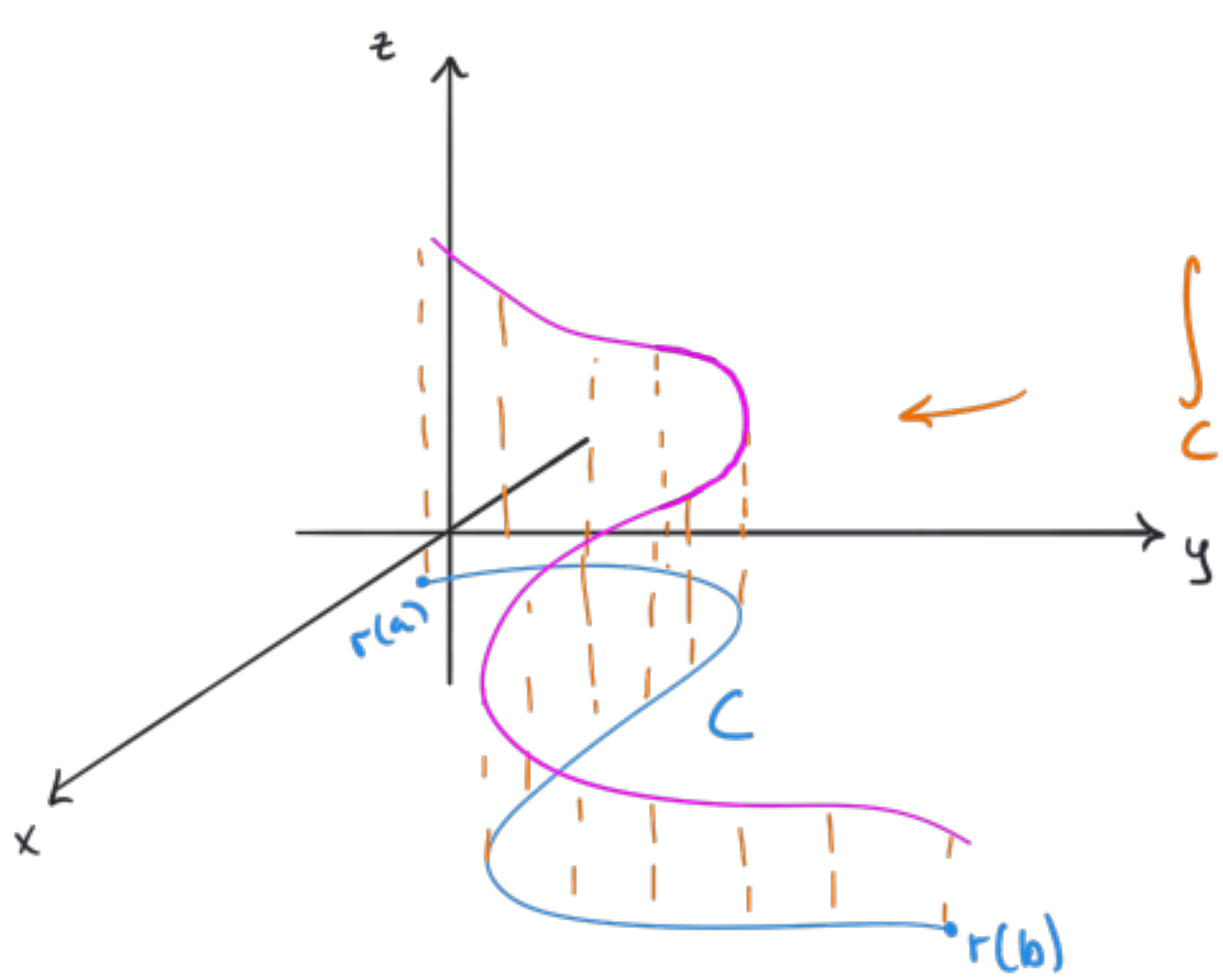
2D: Let C be a curve in $\mathbb{R}^2$ parametrized by $r(t) = (x(t), y(t))$

for $a \le t \le b$:



Consider a function $f: \mathbb{R}^2 \to \mathbb{R}$ and restrict the graph of

f to just those points "above C":



$\int_C f(x,y)\,ds$ = Area of this "curtain".

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

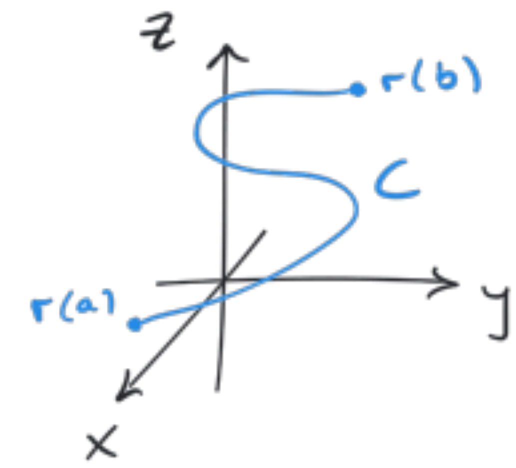
$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Example :

3D: If C is a curve in $\mathbb{R}^3$, parametrised by

$r(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$:



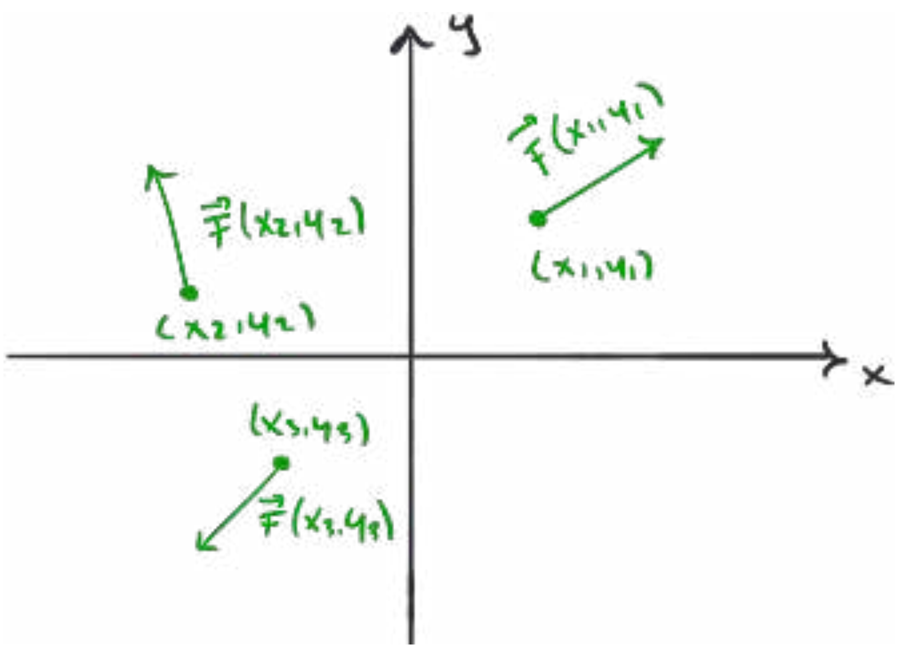
$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,|r'(t)|\,dt$$

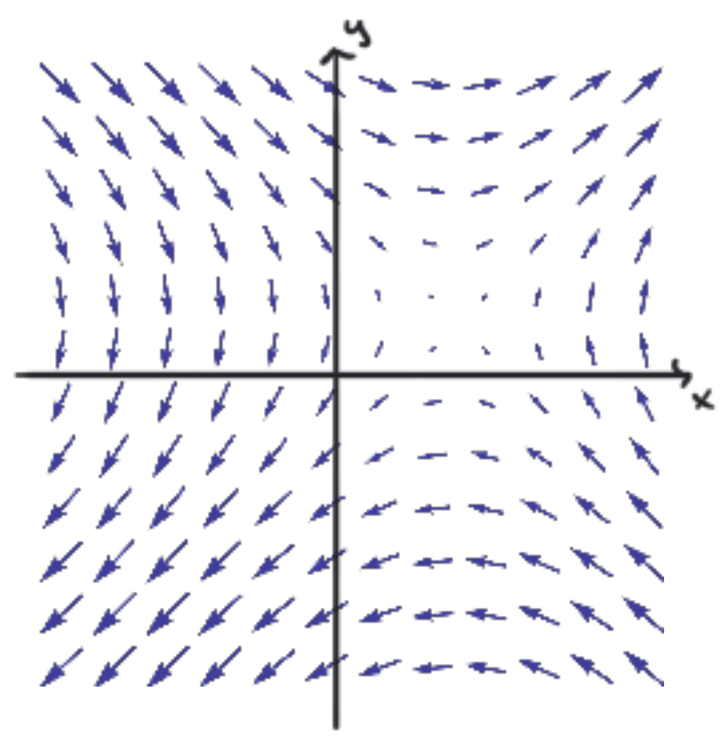
where $|r'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

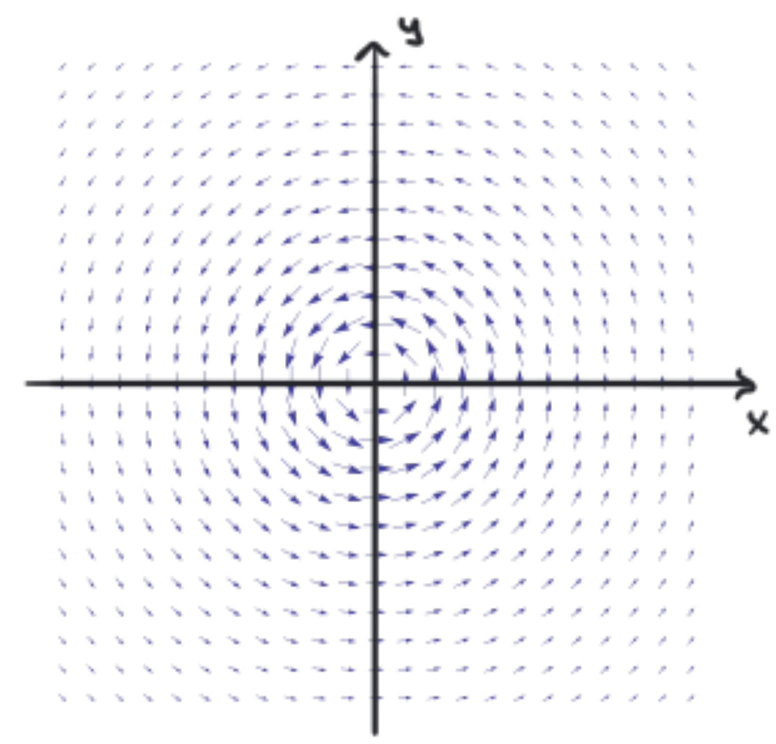
Example:

Vector Fields:

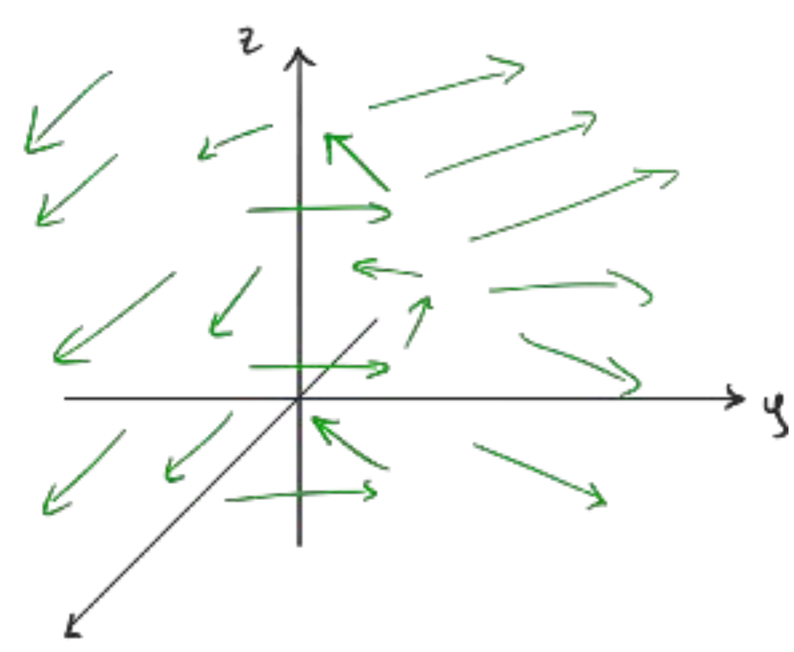
2D: A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that for every point (x,y) in $\mathbb{R}^2$ assigns a vector $\vec{F}(x,y)$ at that point:







3D: A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that for every point (x,y,z) in $\mathbb{R}^3$ assigns a vector $\vec{F}(x,y,z)$ at that point:



Remark The gradient of a function $f: \mathbb{R}^3 \to \mathbb{R}$ (or $\mathbb{R}^2$) : $\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f)$ is a vector field.

Definition: A vector field $\vec{F}$ is called conservative if there is a scalar function f such that $\boxed{\vec{F} = \vec{\nabla} f}$.

Line Integrals of Vector Fields:

Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ and let C be a curve in $\mathbb{R}^3$.

If we consider $\vec{F}$ to be a "force field", we can ask:

What is the work done by $\vec{F}$ in moving a particle along C?

Answer: If $r(t) = (x(t), y(t), z(t))$, $a \le t \le b$ parametrizes C, then:

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot r'(t)\, dt = \int_C \vec{F} . \vec{T}\, ds}$$

Why?

- If $\vec{F}$ is given in components by:

$$\vec{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$$

i.e. $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz}$$

The Fundamental Theorem for Line Integrals:

- Let C be a smooth curve parametrized by $r(t)$, $a \le t \le b$.

Let f be a differentiable function whose gradient $\vec{\nabla} f$ is continuous on C. Then:

$$\boxed{\int_C \vec{\nabla} f \cdot d\vec{r} = f(r(b)) - f(r(a))}$$

Remark: This implies that if C_1 and C_2 are two distinct smooth curves with the same start and endpoints, then:

$$\boxed{\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = \int_{C_2} \vec{\nabla} f \cdot d\vec{r}}$$

Definition: For an arbitrary vector field $\vec{F}$, continuous on a domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is <u>independent of path</u> if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1, C_2 in D with the same start and end points.

Remark: The work done by a conservative vector field along a path depends only on the start & end points.

Example :

Theorems :

- $\int_C \vec{F}.d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}.d\vec{r} = 0$ for every closed path C in D.

- Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F}.d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative. i.e. there exists a function f such that $\vec{F} = \vec{\nabla} f$.

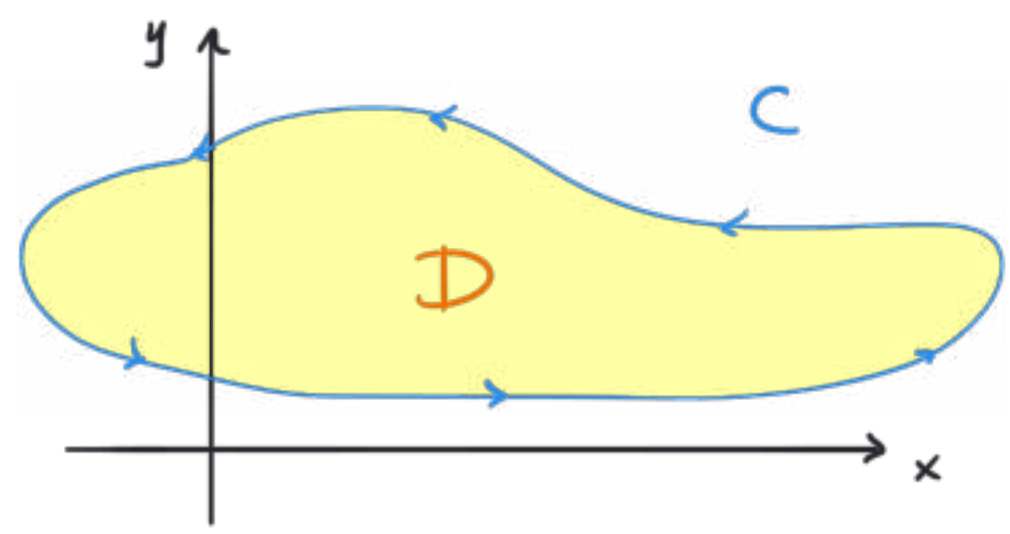
- If $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a conservative vector field, and P and Q have continuous first order partial derivatives on D, then we have: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

- Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D.

Then $\vec{F}$ is conservative.

Green's Theorem :

Let C be a positively oriented , piecewise smooth , simple closed curve in the plane and let D be the region enclosed by C :



If P and Q have continuous partial derivatives on an open region containing D , then :

$$\boxed{\int_C P\,dx + \int_C Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Example :

Curl and Divergence:

"Definition": $\boxed{\vec{\nabla} := \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}}$

Remark:

$$\vec{\nabla} f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

"Definition": For a "nice" vector field on $\mathbb{R}^3$, $\vec{F} = (P, Q, R)$:

$$\boxed{\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla} \times \vec{F}}$$

Theorem: If f is a scalar function of three variables that has continuous second order partial derivatives, then:

$$\boxed{\vec{\nabla} \times (\vec{\nabla} f) = 0}$$

Theorem: If $\vec{F}$ is a vector field on $\mathbb{R}^3$ whose component functions all have continuous first partials, then:

$$\boxed{\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} \text{ is conservative}}$$

Definition: $$\boxed{\operatorname{div}(\vec{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla} \cdot \vec{F}}$$

Theorem: $$\boxed{\operatorname{div}(\operatorname{curl}(\vec{F})) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0}$$

Theorem: If $\operatorname{div}(\vec{F}) = 0$, then there is a $\vec{G}$ such that $\vec{F} = \vec{\nabla} \times \vec{G}$.

Intuition:

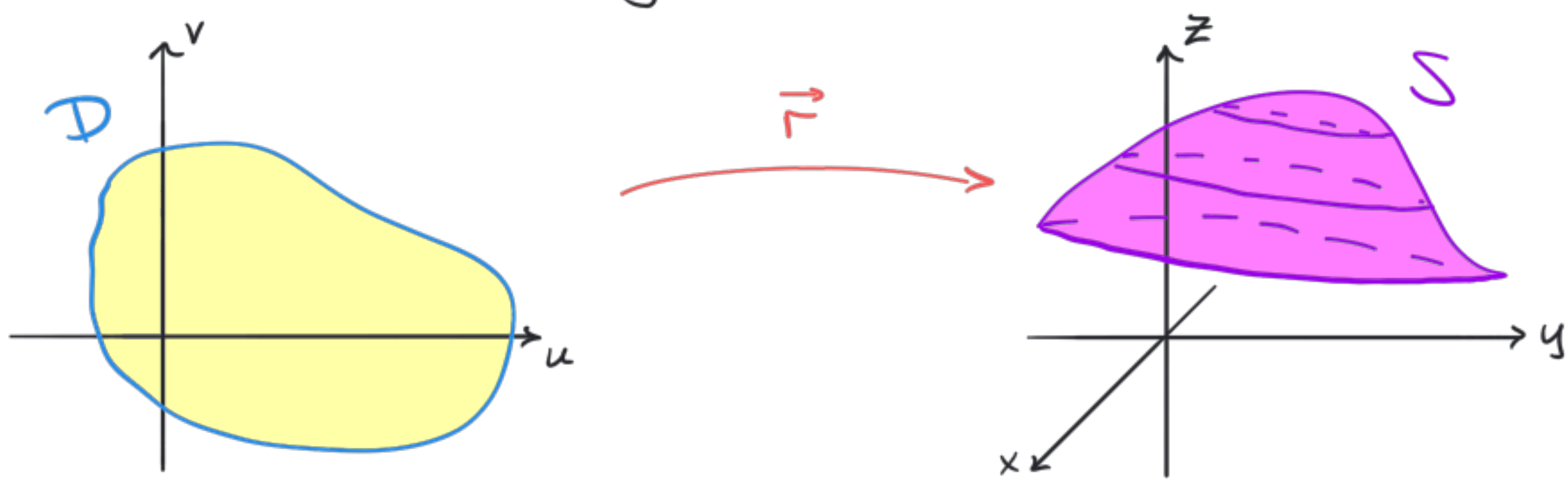
Example: Let $\vec{F}(x,y,z) = (e^y, zy, xy^2)$ and let

$\vec{G}(x,y,z) = (z^2, \frac{x}{3}, xy)$.

Compute $\operatorname{div} \vec{F}$ and $\vec{\nabla} \times \vec{G}$.

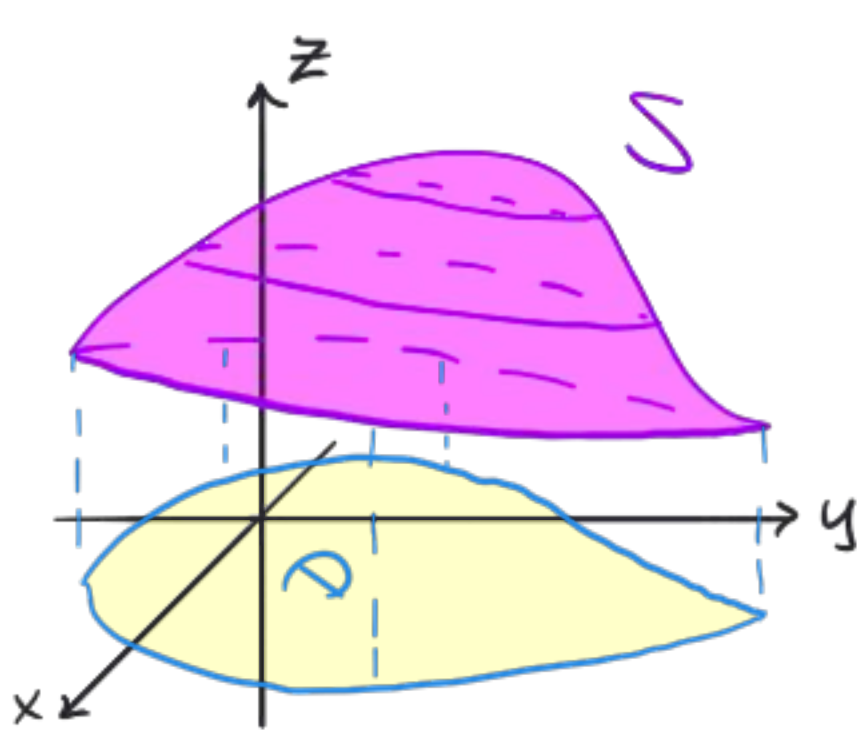
Parametric Surfaces and their Area:

- Let $S \subset \mathbb{R}^3$ be a surface described by a vector-valued function : $\vec{r}(u,v) = \left(x(u,v), y(u,v), z(u,v) \right)$ for (u,v) in a region $D \subset \mathbb{R}^2$:



- Then we say : $\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right\}$ are the parametric equations of S.

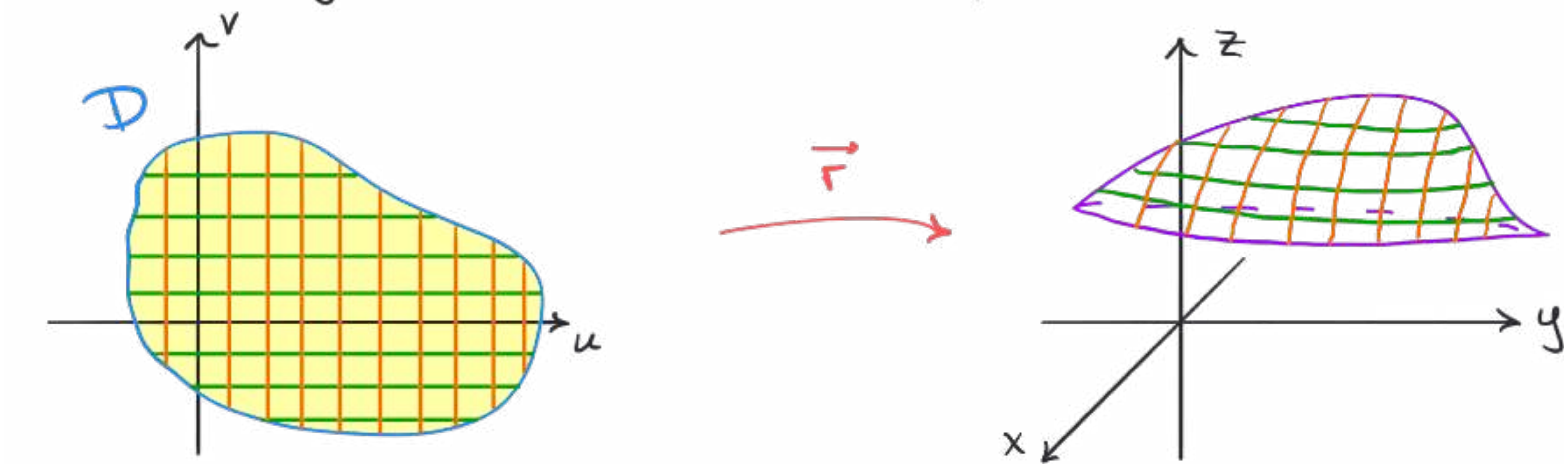
Special Case : If S is the graph of a function $z = g(x,y)$ over a region D in the xy plane :



we can parametrize S by :

$$\boxed{\vec{r}(x,y) = \left(x, y, g(x,y) \right)}$$

• The following picture illustrates grid curves:



The green lines/curves correspond to holding v constant.

The orange lines | curves correspond to holding u constant.

• We can think of restricting ourselves to a single grid curve: $\vec{r}(u, v_0)$ (Green) or $\vec{r}(u_0, v)$ (Orange).

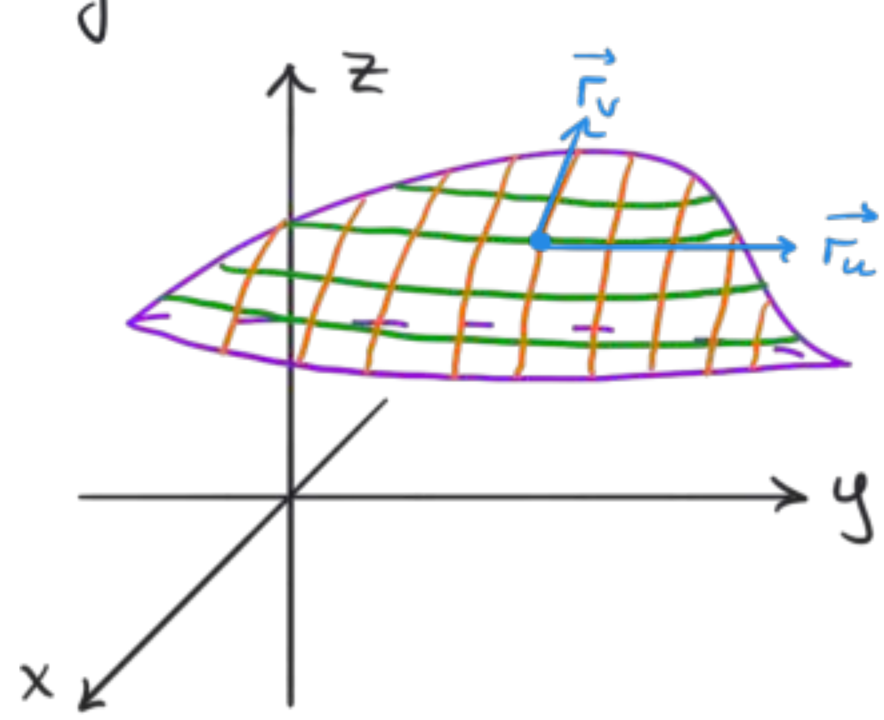
We can then think about limits like:

$$\lim_{h\to 0} \frac{r(u_0+h, v_0) - r(u_0, v_0)}{h} = \frac{\partial r}{\partial u}(u_0, v_0) = \vec{r}_u(u_0, v_0)$$

or

$$\lim_{h\to 0} \frac{r(u_0, v_0+h) - r(u_0, v_0)}{h} = \frac{\partial r}{\partial v}(u_0, v_0) = \vec{r}_v(u_0, v_0)$$

If we think about these geometrically, these correspond to tangent vectors to grid curves of S at $r(u_0, v_0)$:



Tangent Plane :

Hence, $r_u(u_0,v_0)$ and $r_v(u_0,v_0)$ span the tangent plane to S at $\vec{r}(u_0,v_0) = p = (x_0,y_0,z_0)$: T_pS.

So, if we wanted a normal vector to T_pS, we can compute $r_u(u_0,v_0) \times r_v(u_0,v_0)$.

This will allow us to find an equation for T_pS :

$$\boxed{(x-x_0,\ y-y_0,\ z-z_0)\cdot\left(r_u(u_0,v_0)\times r_v(u_0,v_0)\right) = 0}$$

Example :

1.(7 pts.) Compute the tangent plane to the surface parametrized by $\mathbf{r} = u\mathbf{i} + uv\mathbf{j} + (u+v)\mathbf{k}$ at the point $(1,2,3)$.

(a) $3x + 2y - z = 10$

(b) $\langle x,y,z\rangle = \langle 1+u, 2+uv, 3+u+v\rangle$

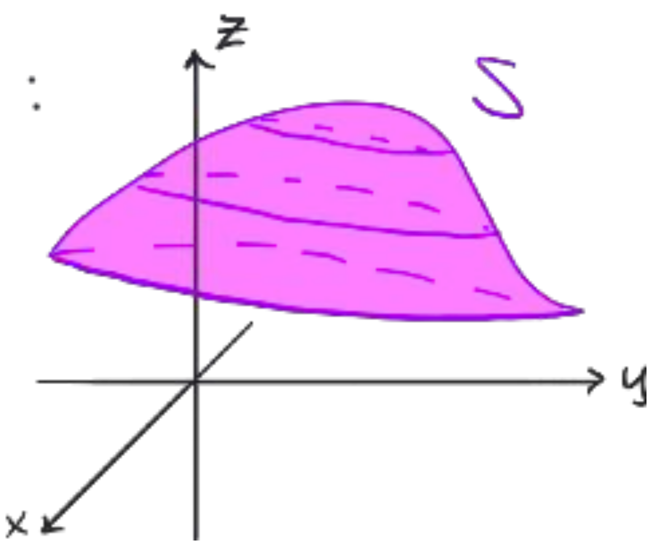
(c) $x - y + z = 2$

(d) $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$

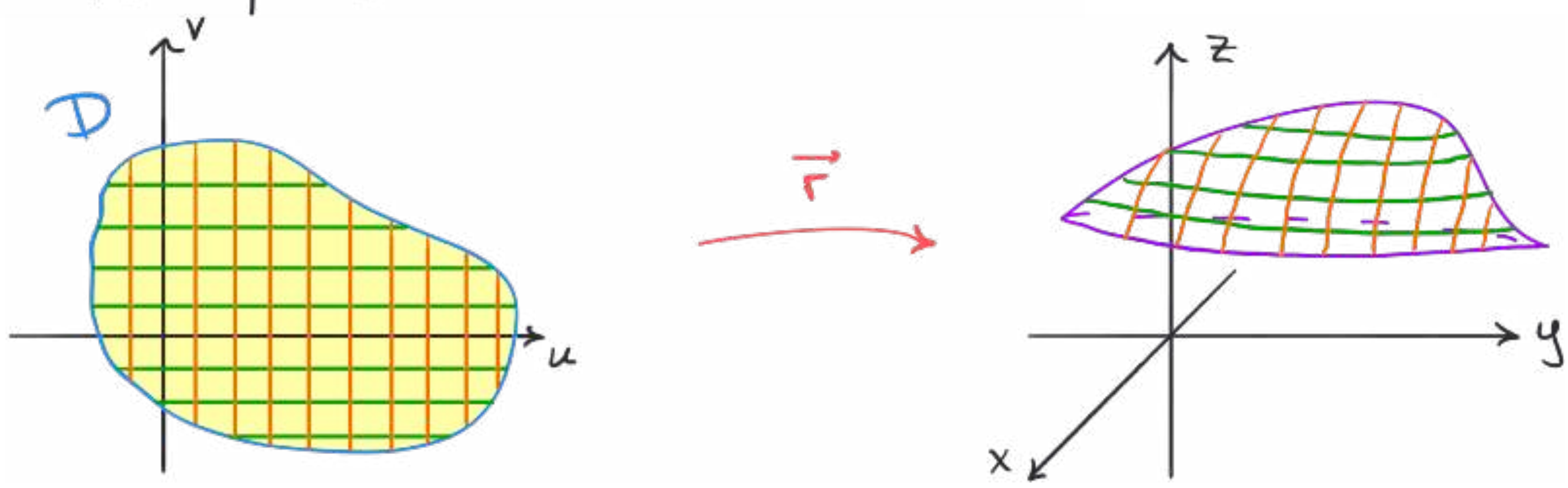
(e) $x + 2y + 3z = 14$

Surface Area:

If I now want to tackle the problem of finding the surface area of a surface S:



it is useful to return to our "gridline" picture:

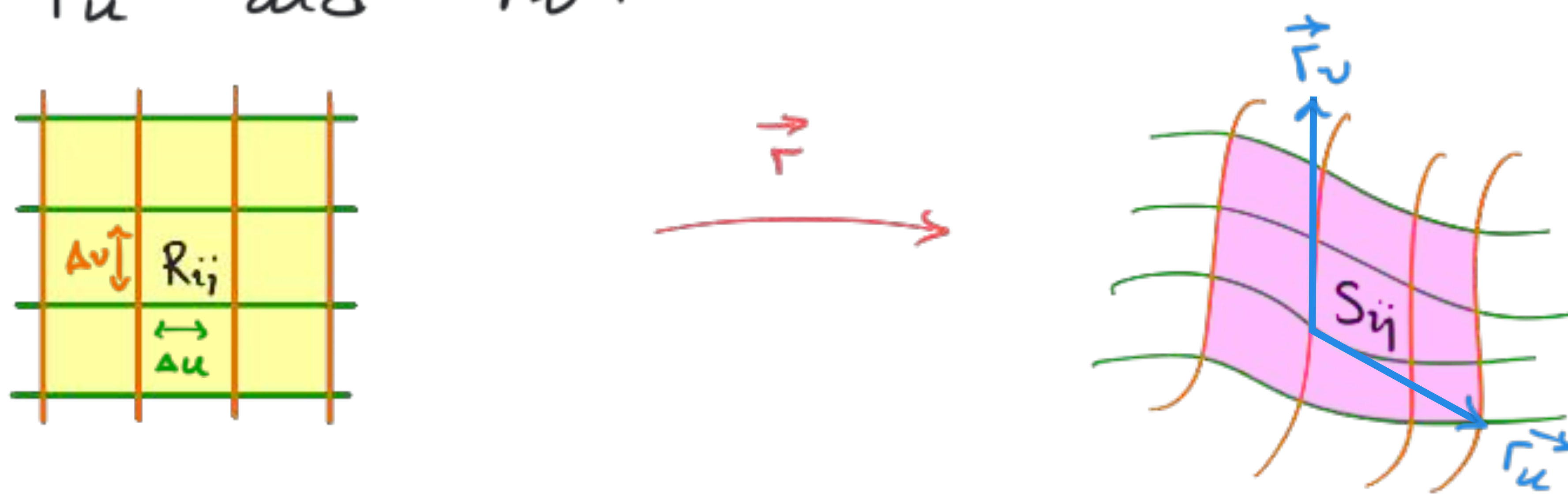


If we could approximate the area of each "piece" in this grid, we could add up all these approximations and get an approximation for the entire surface area.

So, let's "zoom in" on a piece of area:

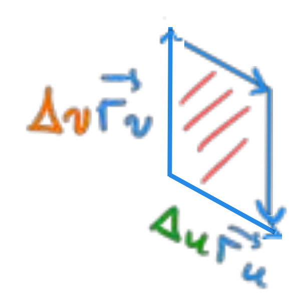
The area of R_{ij}: $A(R_{ij}) = \Delta u \, \Delta v$

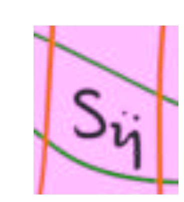
To approximate the area of S_{ij}, consider again $\vec{r}_u$ and $\vec{r}_v$:



We see that we can approximate the orange edge of S_{ij} by: $\Delta v \, \vec{r}_v$

and the green edge by: $\Delta u \, \vec{r}_u$

The area of the parallelogram they span approximates the area of S_{ij}: 

$\approx$ 

The area of this parallelogram is:

$$\left| \Delta u \, \vec{r}_u \times \Delta v \, \vec{r}_v \right| = \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \, \Delta v$$

We sum these up and take a limit of finer and finer grids to arrive at our formula for surface area:

$$\boxed{A(S) = \iint_D \left| \vec{r}_u \times \vec{r}_v \right| \, du \, dv}$$

Example :

16.(7 pts.) Which integral gives the surface area of the surface S parameterized by $\mathbf{r}(u,v) = \langle u^2 \cos v, u^2 \sin v, v \rangle$, where $0 \leq u \leq 1, 0 \leq v \leq \pi$.

(a) $\int_0^{\pi} \int_0^1 2u\sqrt{1+u^4}\, dudv$

(b) $\int_0^{\pi} \int_0^1 (4u^2 + 4u^6)\, dudv$

(c) $\int_0^{\pi} \int_0^1 4u^2(\sin v + \cos v) + 4u^4\, dudv$

(d) $\int_0^{\pi} \int_0^1 2u\sqrt{1+u^2}\, dudv$

(e) $\int_0^{\pi} \int_0^1 \sqrt{4u^2 \sin^2 v - \cos^2 v + 4u^6}\, dudv$

Special Case: If the surface S is the graph of a function $z = g(x,y)$ for (x,y) in some region $D \subset \mathbb{R}^2$, then:

$$\boxed{A(S) = \iint_D \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy}$$

Why?

Example: Let S be the graph of $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ for $0 \le x \le 1$, $0 \le y \le 1$. Set up the integral for $A(S)$.

$\oint$ Surface Integrals and Flux:

Last time:

↳ Parametric Surfaces

↳ Area of Parametric Surfaces

Goal for today:

↳ Integrate functions over surfaces: $\iint_S f \, dS$

↳ Develop a notion of Orientation.

↳ Develop a notion of Flux.

Examples:

(a) If a surface S has density function δ, then the mass of S, $m(S) = \iint_S \delta \, dS$.

(b) Rate at which water passes through a membrane or porous vessal.

(c) Rate at which heat energy is emitted from a metal object.

Surface Integrals:

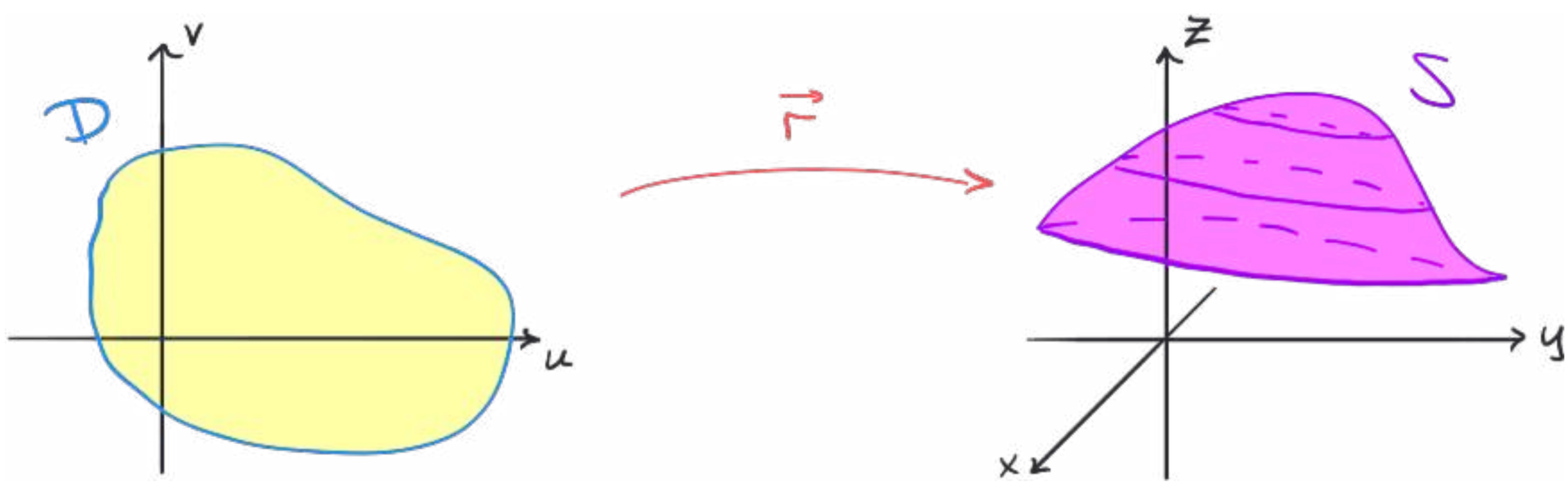
- We can think of the relationship:

Surface Area $\longleftrightarrow$ Surface Integrals

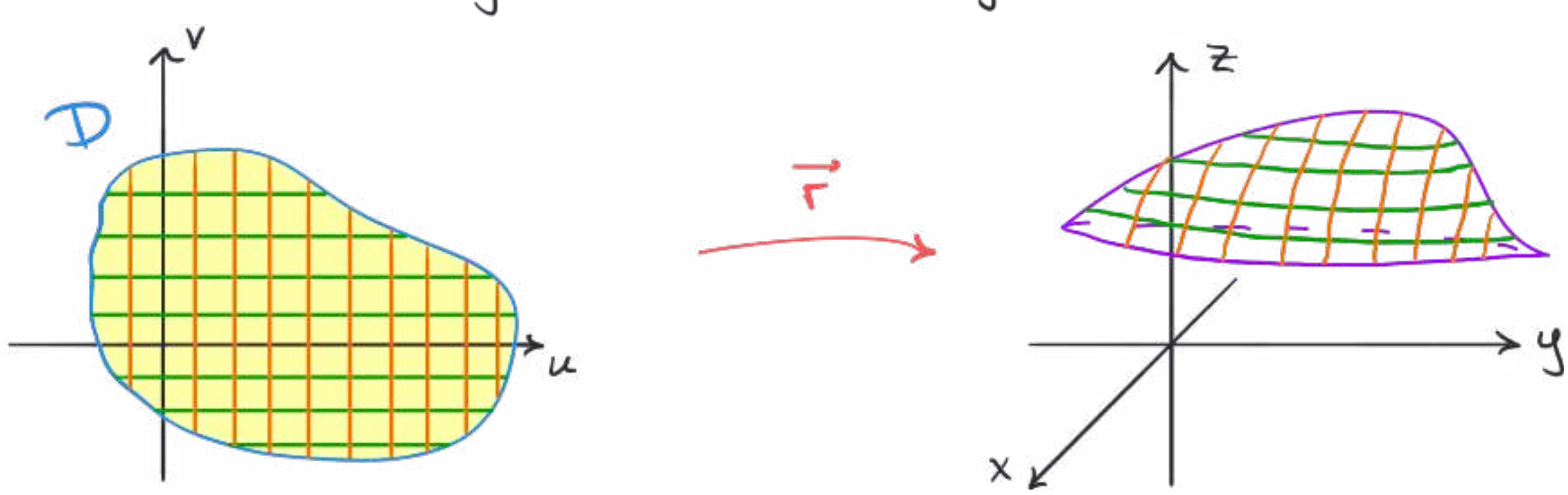
in a similar way to how we think of the relationship:

Arc Length $\longleftrightarrow$ Line Integrals

- If S is parametrized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$:



We can once again consider the grid lines:



If we zoom in on this picture:

We saw before that the area of S_{ij}:

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \, \Delta R_{ij} = |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v \qquad (*)$$

So if P_{ij} was a point in S_{ij}, and S had a mass density function δ, then the mass of the square S_{ij} would be:

$$\text{mass}(S_{ij}) \approx \delta(P_{ij}) \, \Delta S_{ij}$$

Doing this for each square:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \text{mass}(S_{ij}) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, \Delta S_{ij}$$

Using $(*)$:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v$$

Taking finer and finer grids :

$$\boxed{\begin{aligned} \text{mass}(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \delta(P_{ij})\,|\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v \\ &= \iint_D \delta(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA \end{aligned}}$$

- In general :

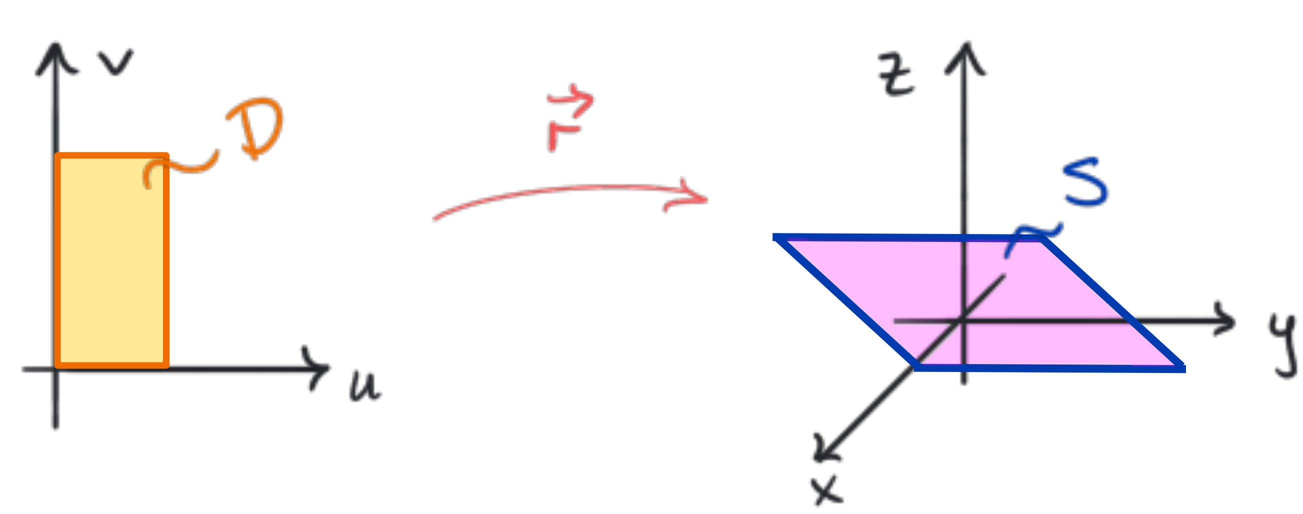
$$\boxed{\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA}$$

Example : Compute the mass of a sheet of metal (parallelogram), parametrised by :

$\vec{r}(u,v) = (u+1, -u+v, u)$, $0 \le u \le 1$, $0 \le v \le 2$.

with mass density $\delta(x,y,z) = z^2$.

Solution :



Remark: We can develop center of mass formulas for a surface S with density function δ:

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{1}{m} \iint_S x\, \delta(x,y,z)\, dS$$

$$\bar{y} = \frac{1}{m} \iint_S y\, \delta(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \delta(x,y,z)\, dS$$

Special Case: If S is the graph of a function: $z = g(x,y)$,

$\vec{r}(x,y) = (x, y, g(x,y))$, then, as before:

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

So we have:

$$\iint_S f\, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$

Example: Let S be the surface given by $z = y - x^2$ above the region $D = \{(x,y) ; 0 \le x \le 1 , 0 \le y \le 3\}$.

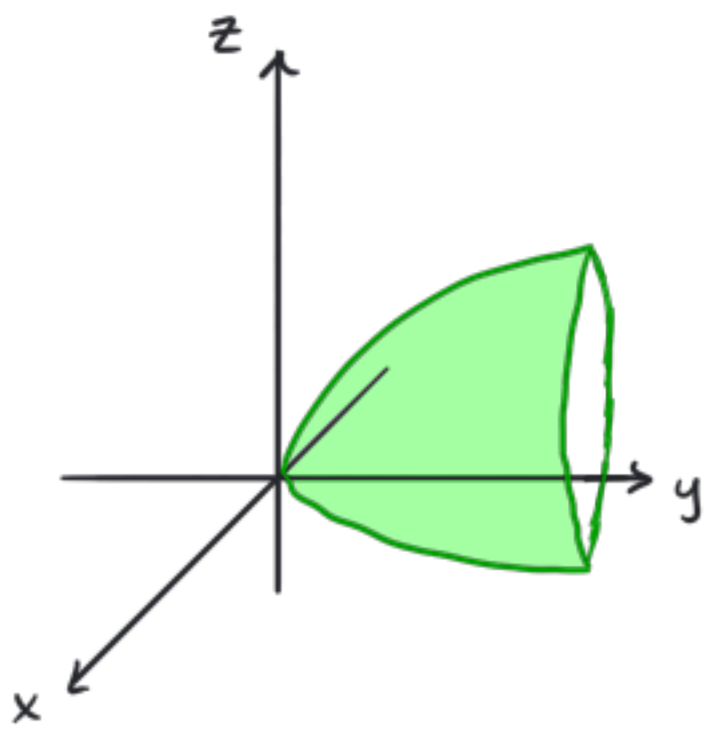
Let $f(x,y,z) = x^2 + x - y + z$.

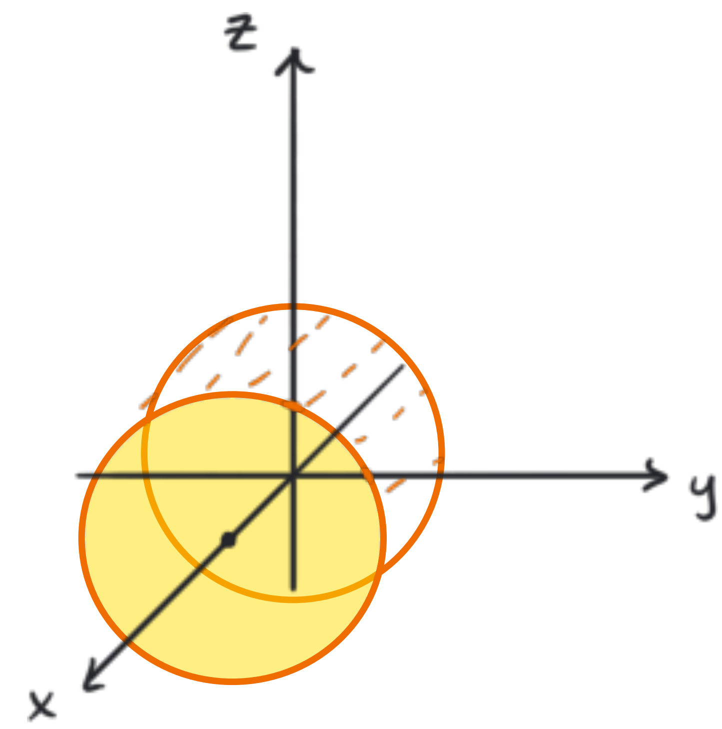
Compute $\displaystyle\iint_S f \, dS$.

Solution:

$$\iint_S f \, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dx\,dy$$

Remark: Some surfaces can be graphs of functions $y = h(x,z)$ or $x = j(y,z)$.





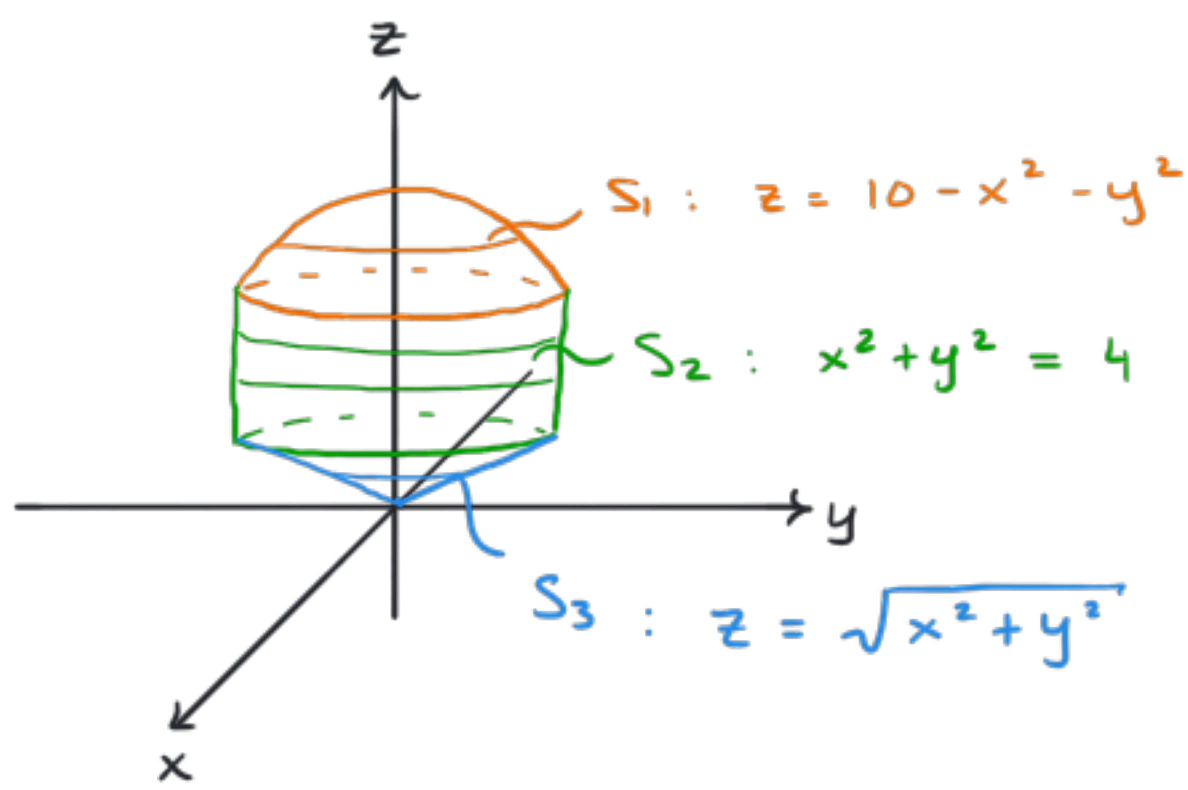
We have analogous formulas:

$$\iint_S f\, dS = \iint_D f(x, h(x,z), z) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}\, dx\, dz$$

$$\iint_S f\, dS = \iint_D f(j(y,z), y, z) \sqrt{1 + \left(\frac{\partial j}{\partial y}\right)^2 + \left(\frac{\partial j}{\partial z}\right)^2}\, dy\, dz$$

- We say that S is a piecewise-smooth surface if it is a finite union of smooth surfaces $S_1, \ldots, S_n$ that are joined together along their boundaries:

E.g.:

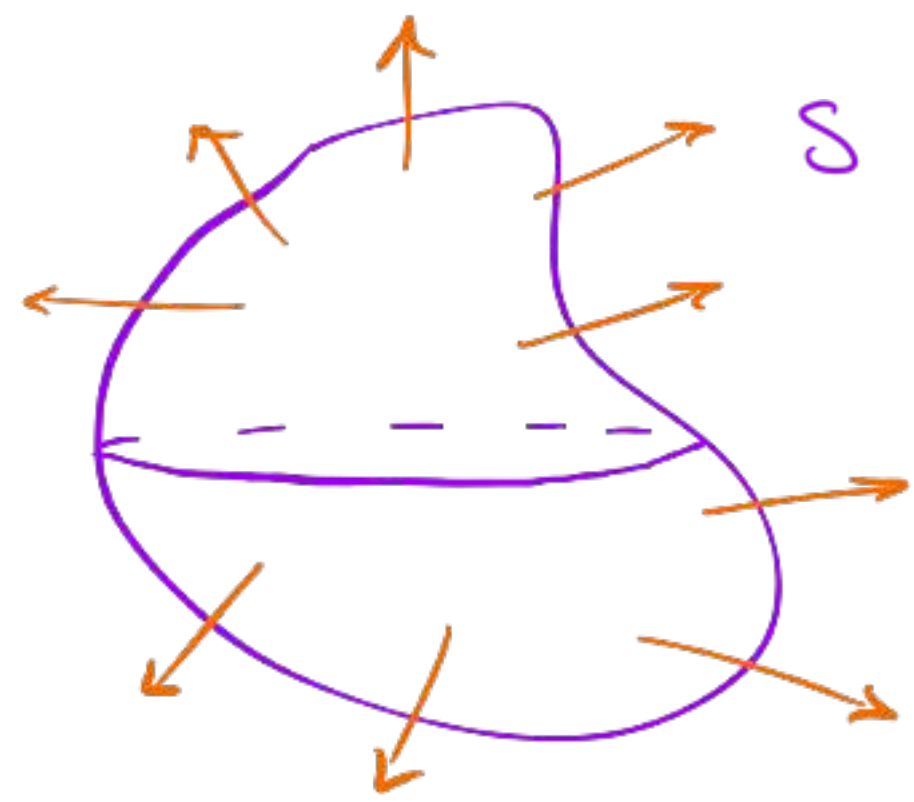


Then:

$$\boxed{\iint_S f \, dS = \sum_{i=1}^{n} \iint_{S_i} f \, dS}$$

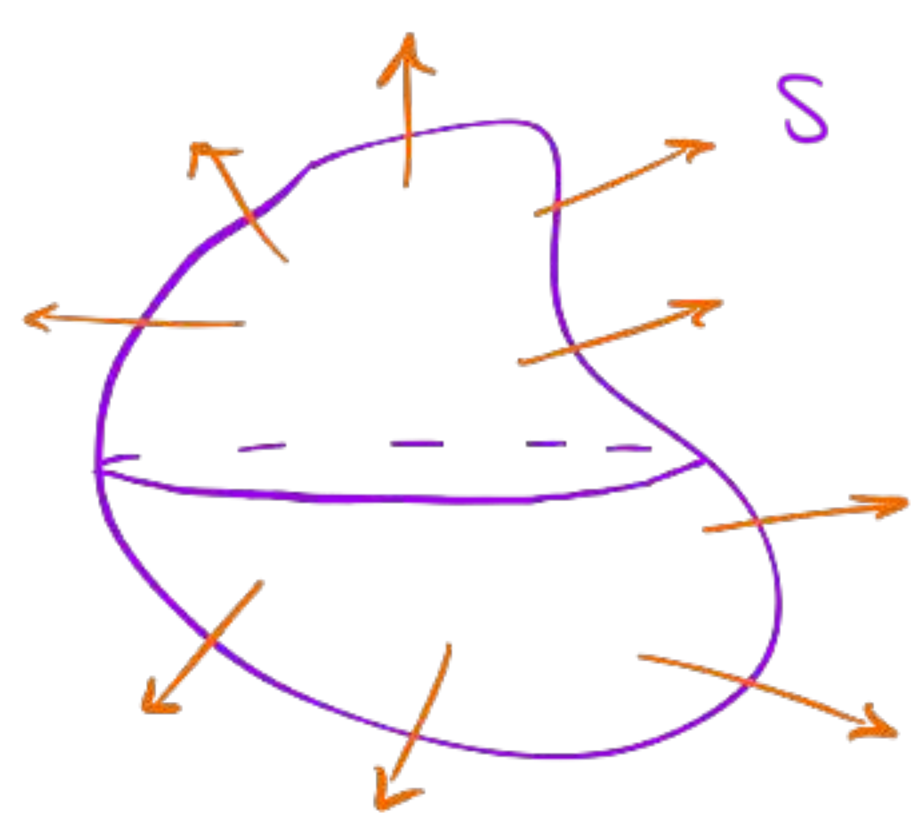
Orientation:

Motivation: Say I have a metal object, S, which is emitting heat energy:



If we computed this flux, should it be a positive or negative quantity?

Now say I have a metal object, S, which is losing heat energy:



If we computed this flux, should it be a positive or negative quantity?

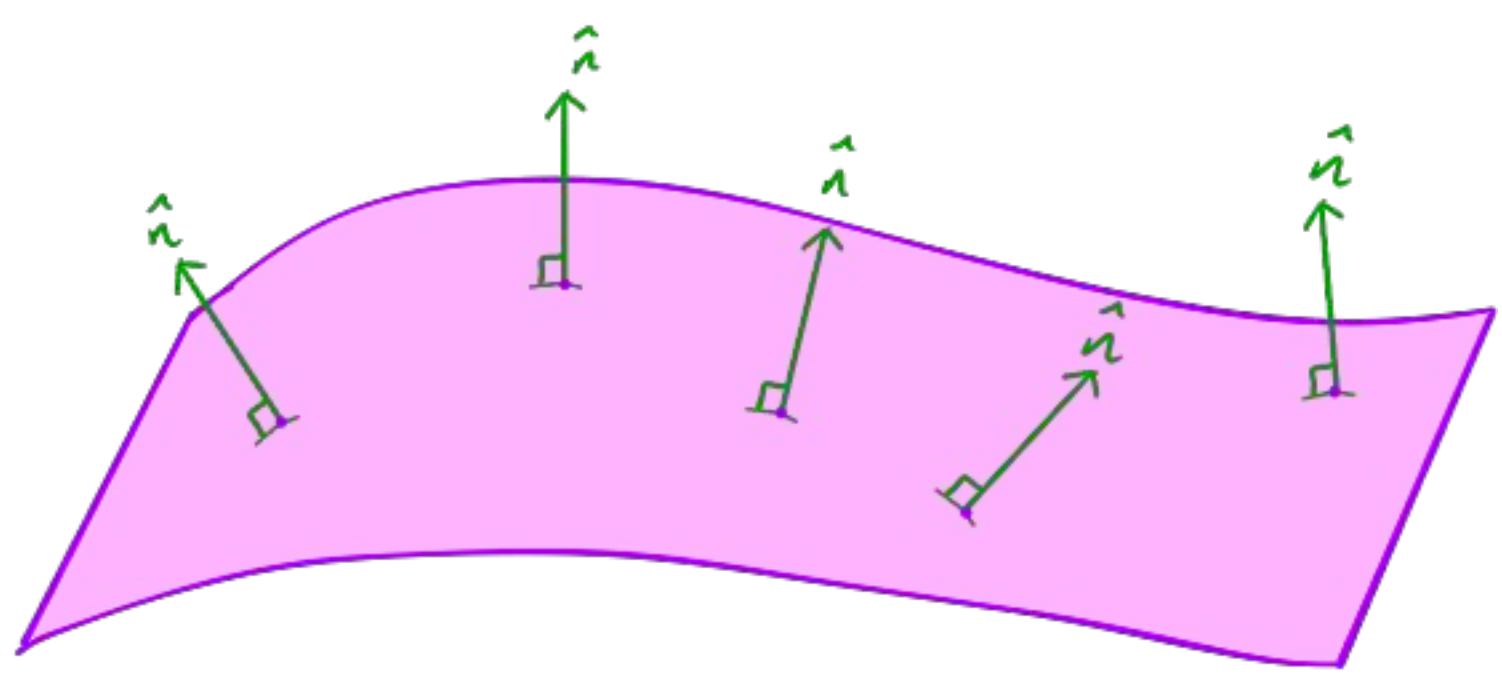
Conclusion:

There is no overall "should".

You have to make a choice.

You have to make a choice of Orientation.

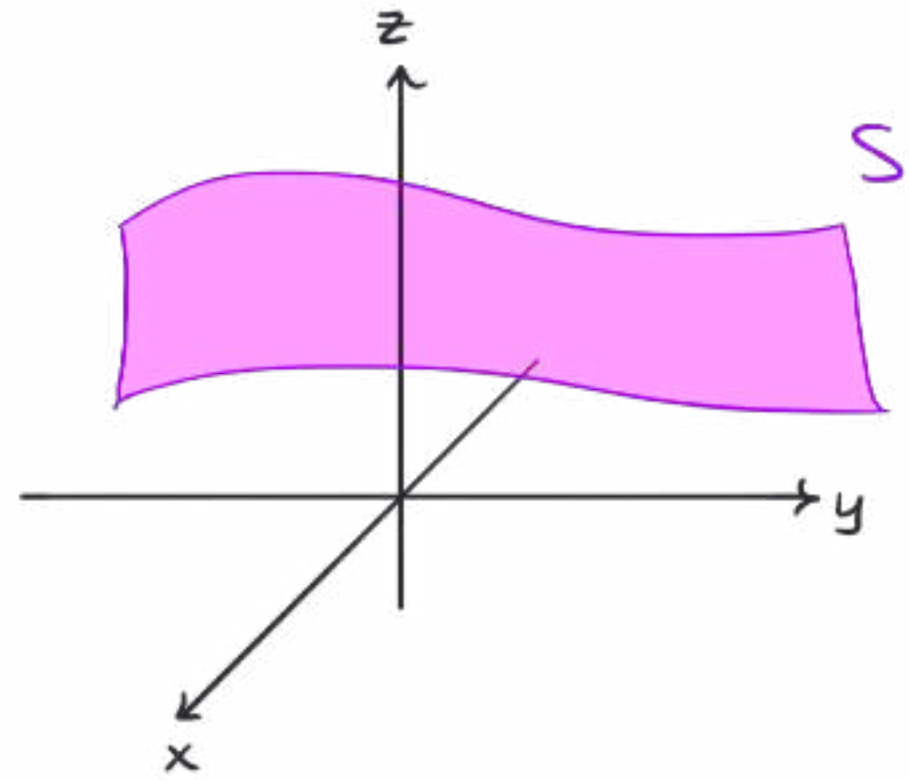
So orientation is technically a choice of continuously varying unit normal vector: $\hat{n}$.



We say a surface S, equipped with an orientation $\hat{n}$ is an oriented surface.

Special Case :

If S is the graph of a function : $z = g(x,y)$



We can think of "upward" or "downward" orientation :

We can find an explicit formula for the upward pointing unit normal to this graph :

$$\boxed{\hat{n} = \frac{\left(-\left(\frac{\partial g}{\partial x}\right), -\left(\frac{\partial g}{\partial y}\right), 1 \right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}}$$

Exercise : Find the upward pointing unit normal to the surface given by $z = x^2 + y^2$.

• If S is a parametric surface represented by $\vec{r}(u,v)$,

then:

$$\boxed{\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

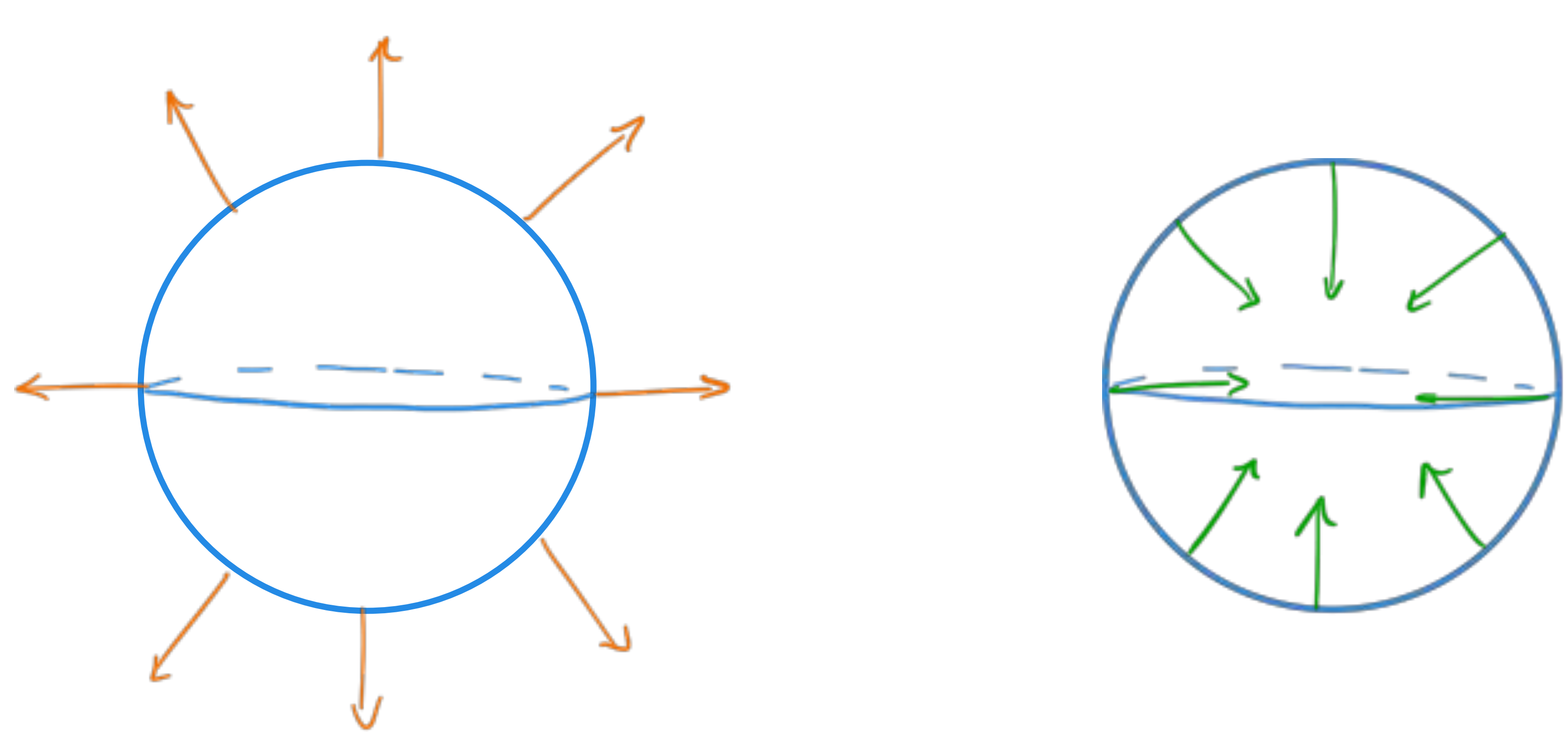
This may be upward or downward – you have to check the sign of the z-component!

is a unit normal vector.

Remark: The opposite orientation is given by $-\hat{n}$.

Exercise: Find a unit normal to the surface parametrized by $\vec{r}(u,v) = (u+1, -u+v, u)$.

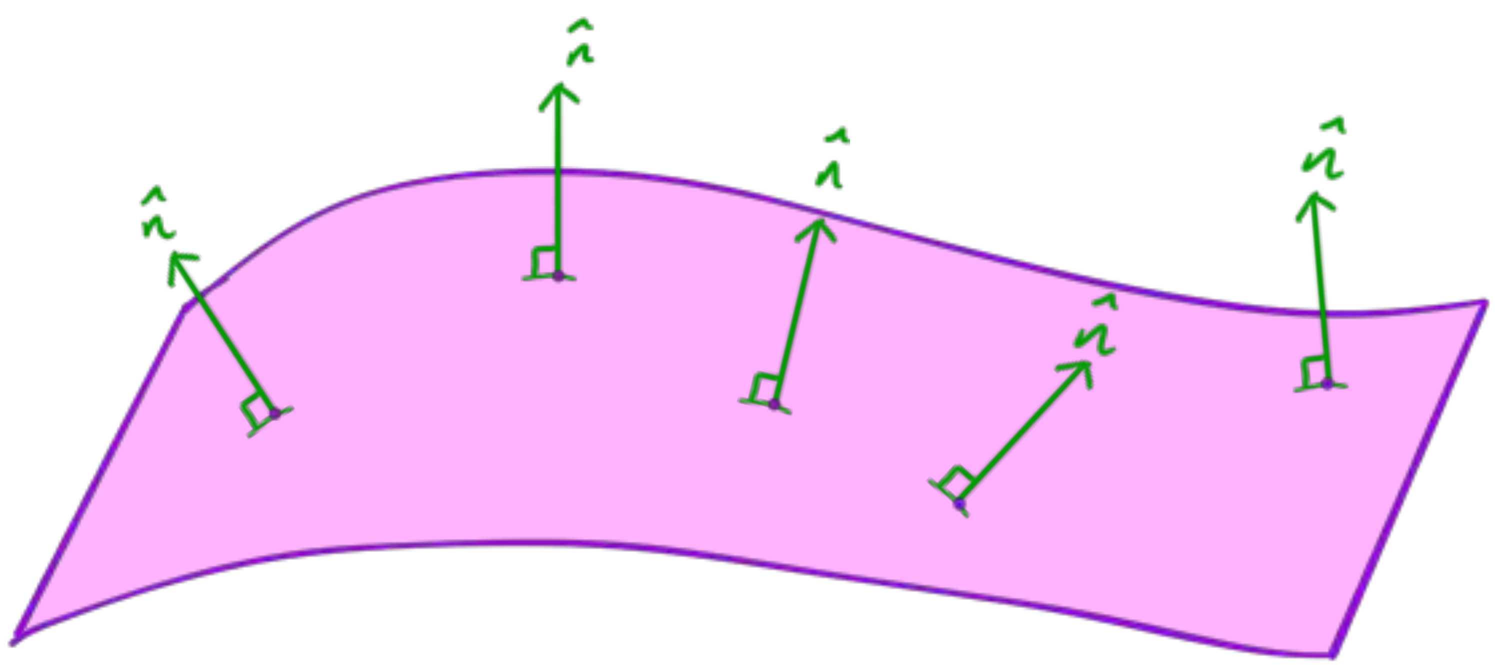
• For a closed surface we can define "outward" and "inward" pointing unit normals:



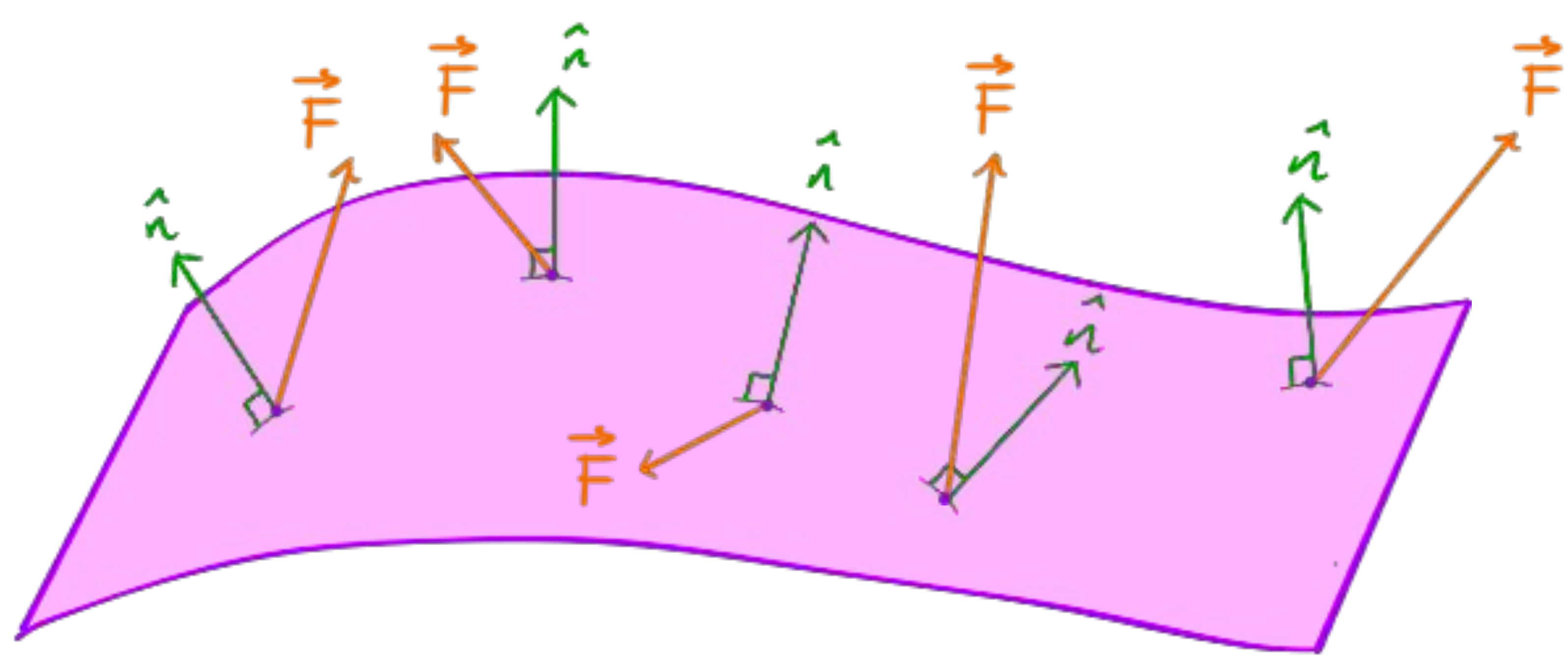
Example: The outward pointing unit normal to a sphere of radius R is $\hat{n} = \frac{1}{R} \langle x, y, z \rangle$.

Surface Integrals of Vector Fields:

Suppose we have an oriented surface S with unit normal $\hat{n}$.

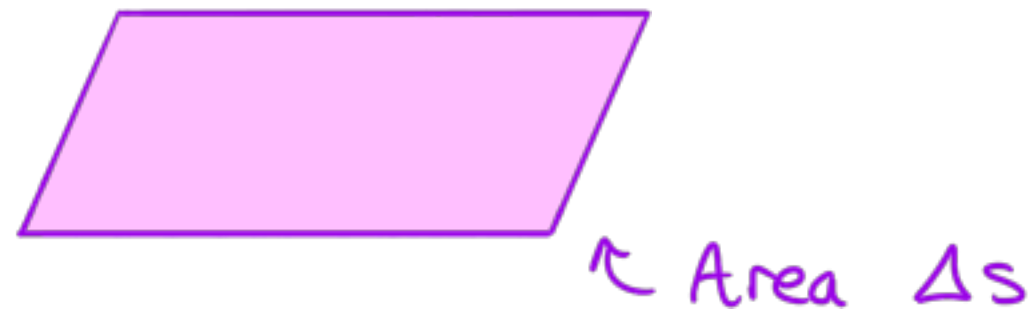


Let's say we have a vector field $\vec{F}$ on S:

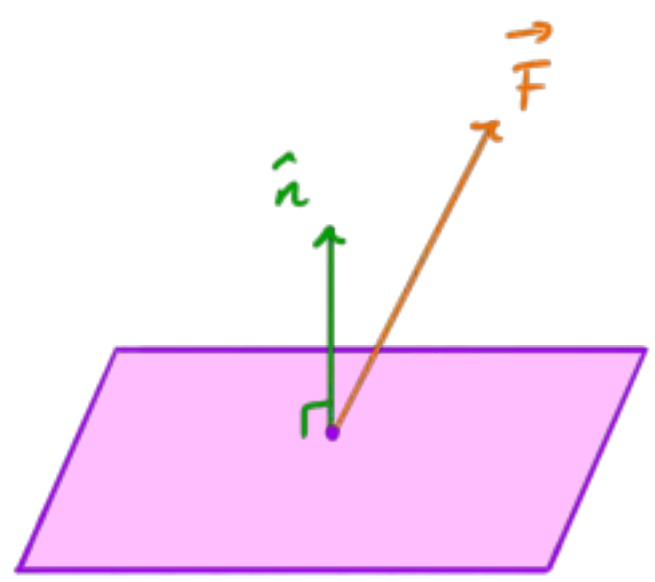


Recall: Our intuition told us that Flux should capture how much $\vec{F}$ is "flowing through" S.

Let's zoom in on a small patch of area ΔS:

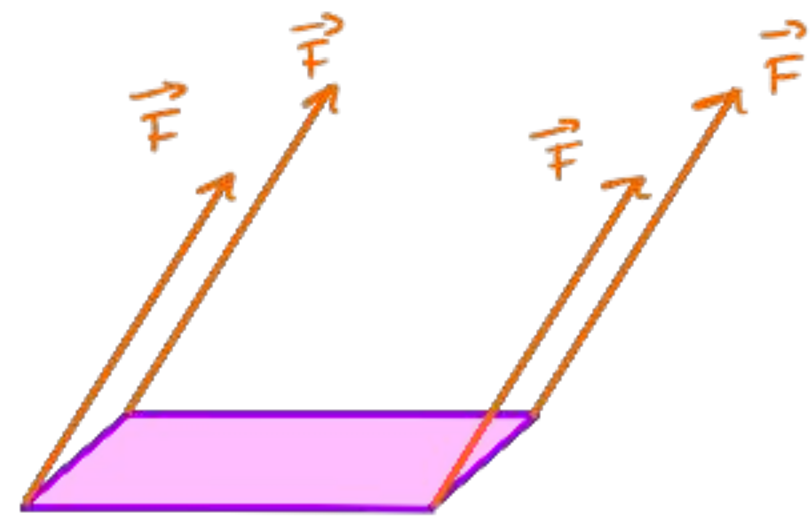


On this small patch, as our vector fields $\vec{F}$ and $\hat{n}$ are "well behaved", they should look pretty much constant on this small patch:

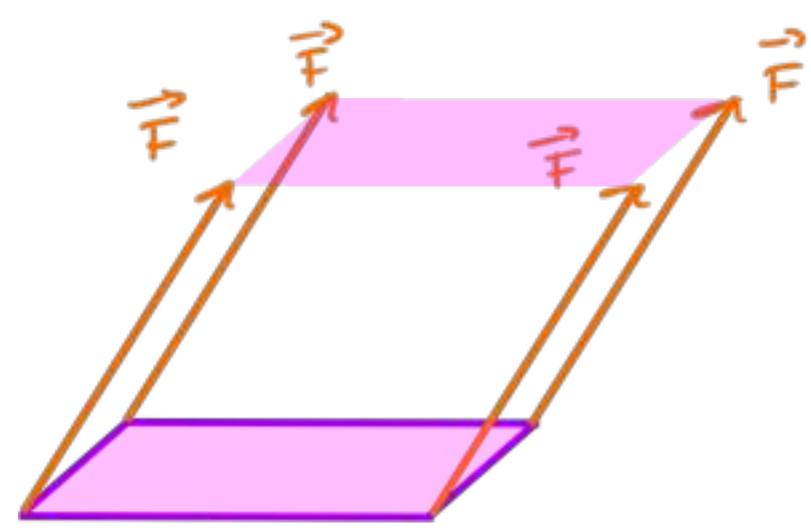


If we think of $\vec{F}$ as the rate at which water is flowing across the points in this patch, what volume of water will flow through per unit time?

If we think of $\vec{F}$ as a velocity field for some fluid, and assume that on this patch it's moving at v meters per second:



What is the volume of this "box"?



Definition: If $\vec{F}$ is a continuous vector field on an oriented surface S with unit normal $\hat{n}$, then the surface integral of $\vec{F}$ over S or the Flux of $\vec{F}$ across S is given by:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS}$$

Example:

Let S be the unit sphere : $x^2 + y^2 + z^2 = 1$.

Let $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$

Solution:

If S is given by $\vec{r}(u,v)$ then :

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,dS$$

$$= \iint_D \vec{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\cdot|\vec{r}_u\times\vec{r}_v|\,dA$$

Hence we have :

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv}$$

Example :

20.(7 pts.) Find the flux of the vector field

$$\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$$

over a surface with **downward** orientation, whose parametric equation is given by

$$\mathbf{r}(u,v) = 2u\mathbf{i} + 2v\mathbf{j} + (5-u^2-v^2)\mathbf{k}$$

with $u^2+v^2\leq 1$.

(a) $-\dfrac{56\pi}{3}$ (b) $\dfrac{112\pi}{3}$ (c) -18π (d) -36π (e) 9π

Special Case:

If S is the graph of a function: $z = g(x,y)$:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

Why?

Example: Let S be the surface given by $z = x^2 + y^2$ above the region $D: x^2 + y^2 \le 1$.

Let $F(x,y,z) = \langle -x, -y, x^2 + y^2 \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$

Solution:

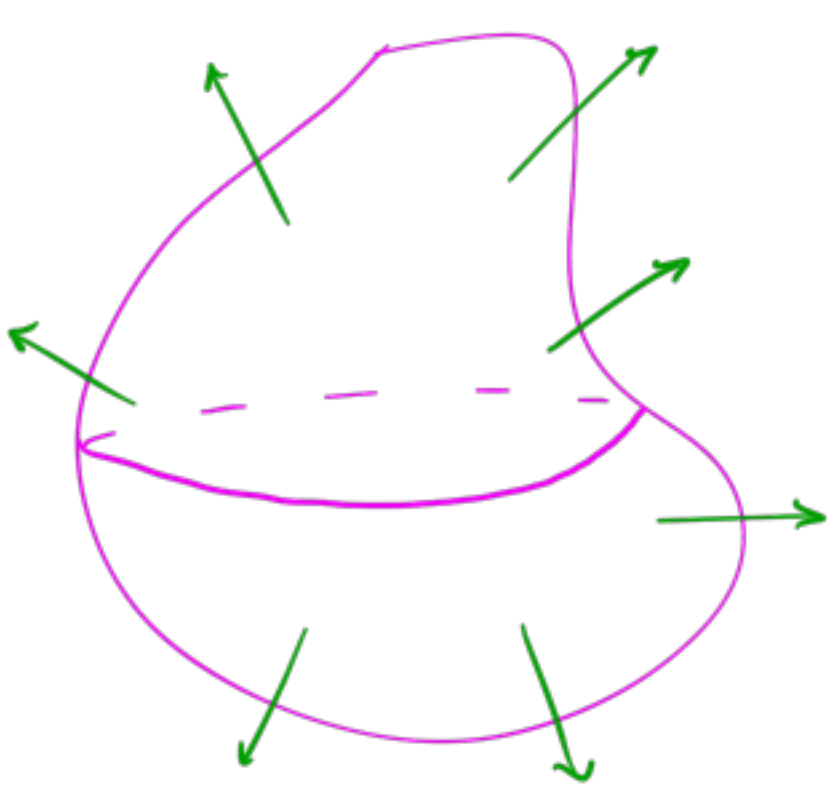
$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

The Divergence Theorem:

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation.

Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV}$$



S = Shell (Hollow)

E = Solid (Filled In)

Intuition:

Example:

4.(7 pts.) Use the Divergence theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is calculate the flux of $\mathbf{F}$ across S.

$$\mathbf{F} = \langle e^y, zy, xy^2 \rangle,$$

S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2$ and $z = 4$ with outward orientation.

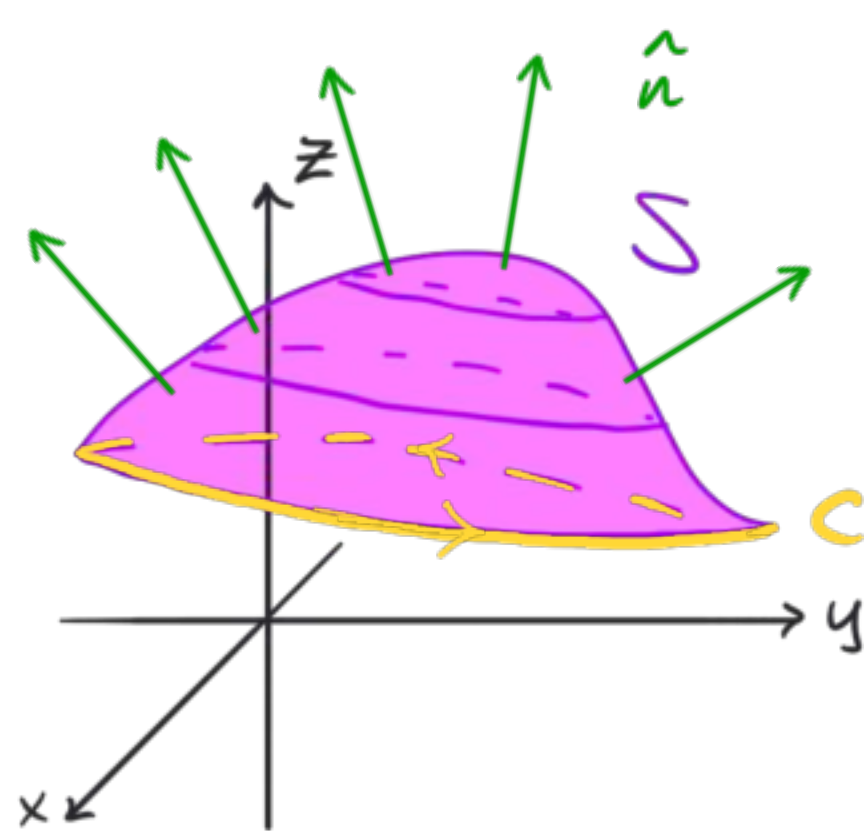
(a) $\frac{3\pi}{2}$ (b) 6π (c) 4π (d) 2π (e) π

Stokes' Theorem:

Let S be an oriented piecewise smooth surface that is bounded by a simple closed piecewise smooth curve C with positive orientation.

Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on open region in $\mathbb{R}^3$ that contains S. Then:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F})\, d\vec{S}}$$



Intuition:

Remark: If S is a closed surface (no boundary curve), what does Stoke's Theorem say?

Question: Can we relate Stoke's Theorem to Green's Theorem?

Example:

8.(7 pts.) Let C be the rectangle in the $z = 1$ plane with verticies $(0, 0, 1)$, $(1, 0, 1)$,$(1, 3, 1)$, and $(0, 3, 1)$ oriented counterclockwise when viewed from above. Use Stokes' Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + \frac{x}{3}\mathbf{j} + xy\mathbf{k}$.

(a) 1 (b) 9/2 (c) 0 (d) 6 (e) $-3/2$

Name: Harder Problems Date: /12/2019

Main Questions

4. Let S be the portion of the graph $z = 4 - 2x^2 - 3y^2$ that lies over the region in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = 1$. Write the integral that computes $\iint_S (x^2 + y^2 + z)\, \mathrm{d}S$.

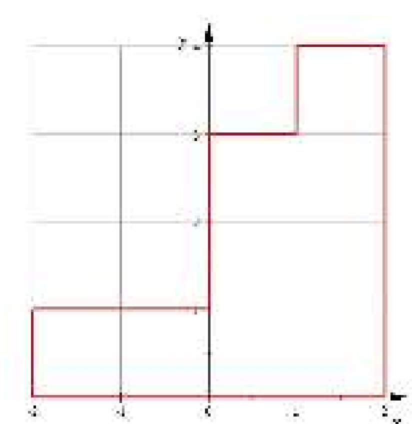
5. Compute $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by

$$x = 2u, \quad y = 2v, \quad z = 5 - u^2 - v^2,$$

where $u^2 + v^2 \leq 1$. S has downward orientation.

6. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

7. Evaluate $\int_C (x^4 e^{5y} - 3y)\mathrm{d}x + (4x + x^5 e^{5y})\mathrm{d}y$ where C is the curve below and C is oriented in clockwise direction.



8. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

9. Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, z, 1 \rangle$ across the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with orientation away from the origin.

10. Let S be the boundary surface of the region bounded by $z = \sqrt{36 - x^2 - y^2}$ and $z = 0$, with outward orientation. Find $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$.

11. Let C be the boundary curve of the part of the plane $x + y + 2z = 2$ in the first octant. C has counterclockwise orientation when viewing from above. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{\sin x^2}, z, 3y \rangle$.

12. Evaluate

$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\, dz$$

where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$ with counterclockwise direction when viewed from above. (*Hint*: the curve C lies on the surface $z = 2xy$.)