$\oint$ Surface Integrals and Flux:

Last time:

↳ Parametric Surfaces

↳ Area of Parametric Surfaces

Goal for today:

↳ Integrate functions over surfaces: $\boxed{\iint_S f\, dS}$

↳ Develop a notion of **Orientation**.

↳ Develop a notion of **Flux**.

Examples:

(a) If a surface S has density function δ, then the mass of S, $m(S) = \iint_S \delta\, dS$.

(b) Rate at which water passes through a membrane or porous vessal.

(c) Rate at which heat energy is emitted from a metal object.

Surface Integrals:

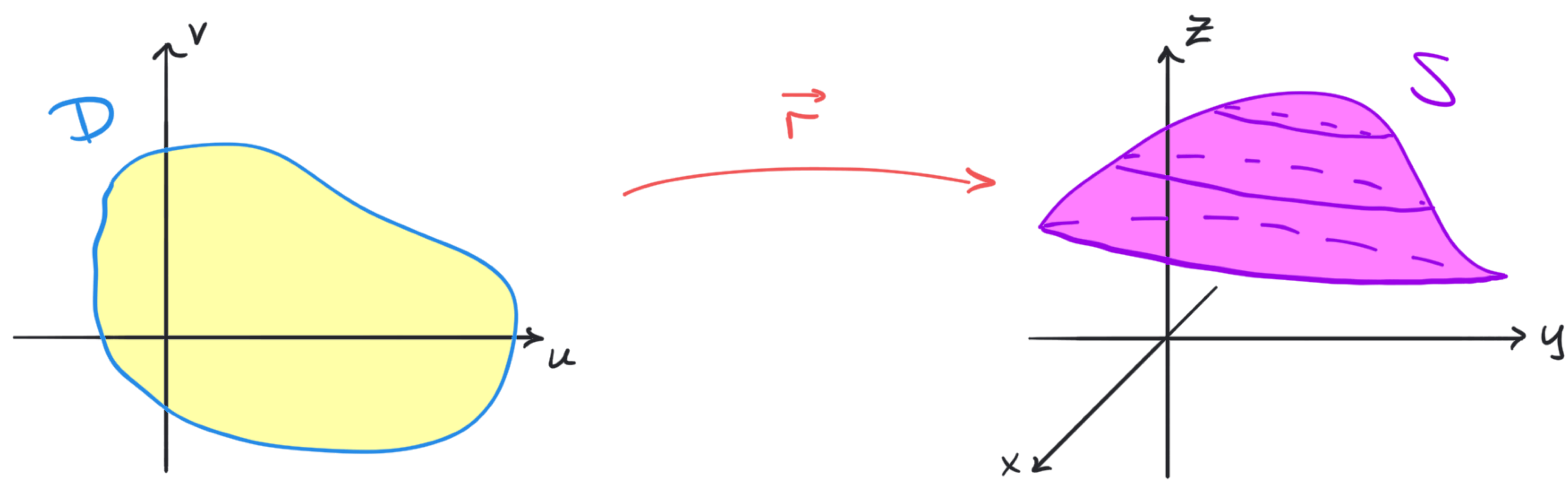
- We can think of the relationship:

  Surface Area $\longleftrightarrow$ Surface Integrals

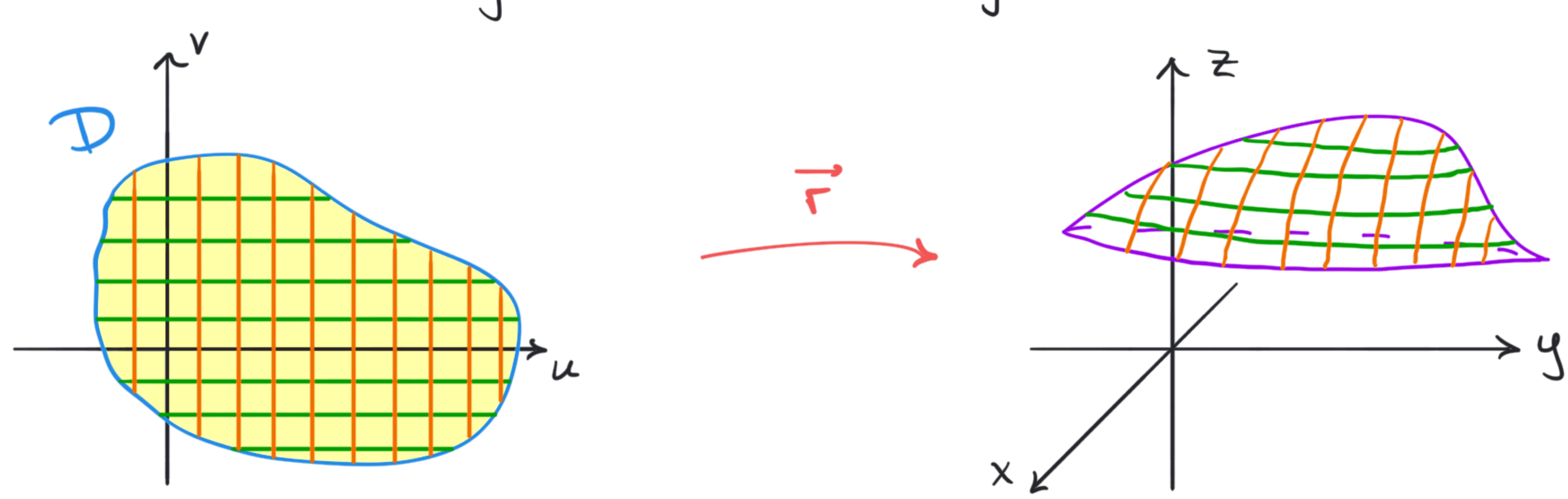
  in a similar way to how we think of the relationship:

  Arc Length $\longleftrightarrow$ Line Integrals

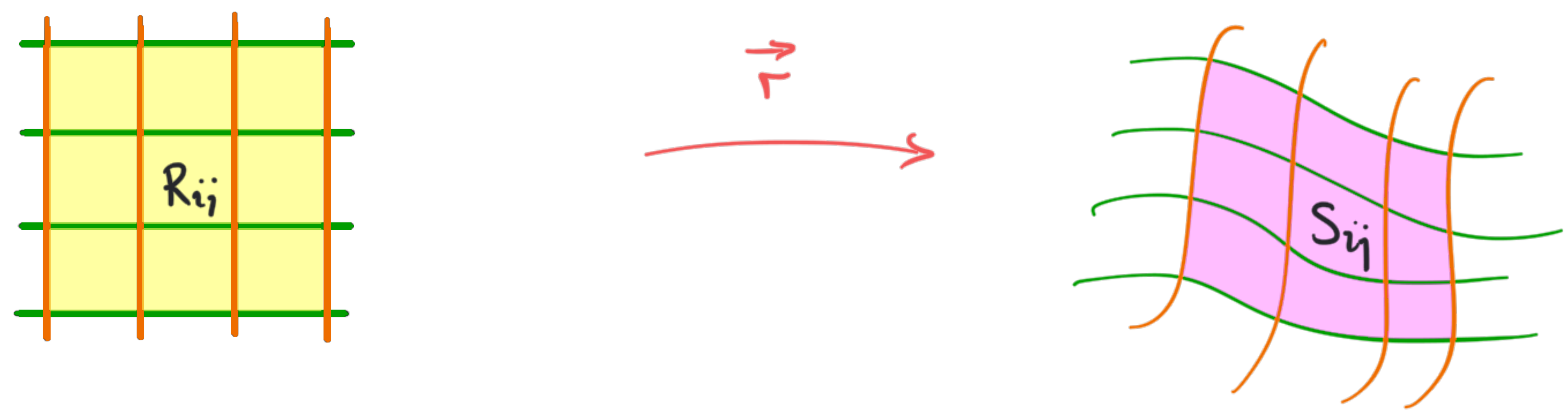
- If S is parametrized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$:



We can once again consider the grid lines:



If we zoom in on this picture:



We saw before that the area of S_{ij}:

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \, \Delta R_{ij} = |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v \qquad (*)$$

So if P_{ij} was a point in S_{ij}, and S had a mass density function δ, then the mass of the square S_{ij} would be:

$$\text{mass}(S_{ij}) \approx \delta(P_{ij}) \, \Delta S_{ij}$$

Doing this for each square:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \text{mass}(S_{ij}) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, \Delta S_{ij}$$

Using $(*)$:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$$

Taking finer and finer grids :

$$\boxed{\begin{aligned} \text{mass}(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \Delta v \\ &= \iint_D \delta(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| dA \end{aligned}}$$

- In general :

$$\boxed{\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| dA}$$

<u>Example :</u> Compute the mass of a sheet of metal (parallelogram), parametrised by :

$$\vec{r}(u,v) = (u+1, -u+v, u), \quad 0 \le u \le 1, \quad 0 \le v \le 2.$$

with mass density $\delta(x,y,z) = z^2$.

Remark: We can develop center of mass formulas for a surface S with density function δ:

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{1}{m} \iint_S x\, \delta(x,y,z)\, dS$$

$$\bar{y} = \frac{1}{m} \iint_S y\, \delta(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \delta(x,y,z)\, dS$$

Special Case: If S is the graph of a function: $z = g(x,y)$, $\vec{r}(x,y) = (x, y, g(x,y))$, then, as before:

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

So we have:

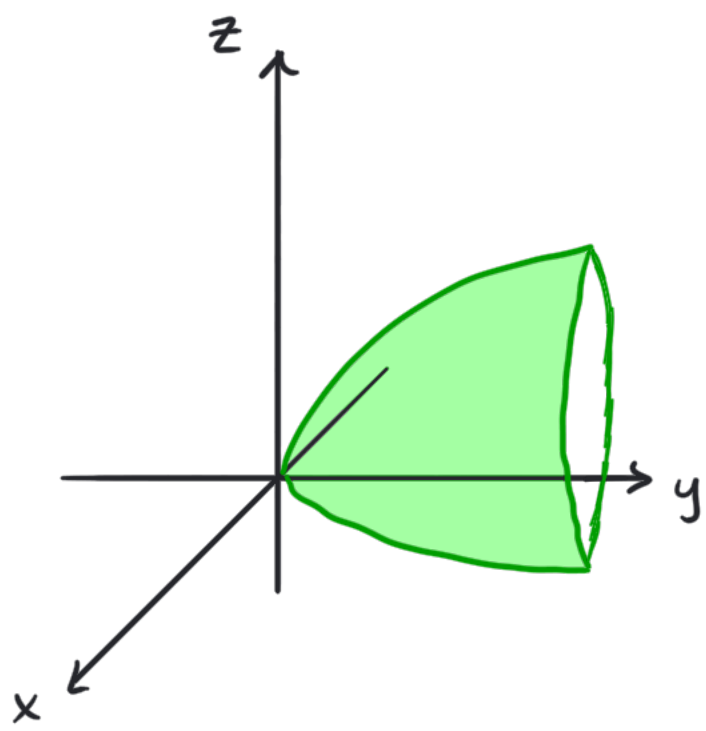
$$\iint_S f\, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\,dy$$

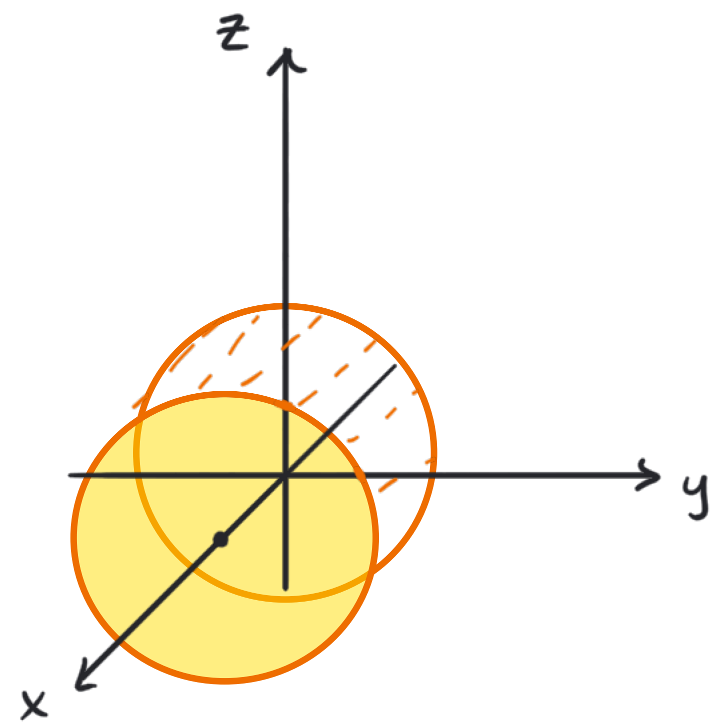
Example: Let S be the surface given by $z = y - x^2$ above the region $D = \{(x,y) ; 0 \le x \le 1 , 0 \le y \le 3\}$.

Let $f(x,y,z) = x^2 + x - y + z$.

Compute $$\iint_S f \, dS .$$

Remark: Some surfaces can be graphs of functions $y = h(x,z)$ or $x = j(y,z)$.





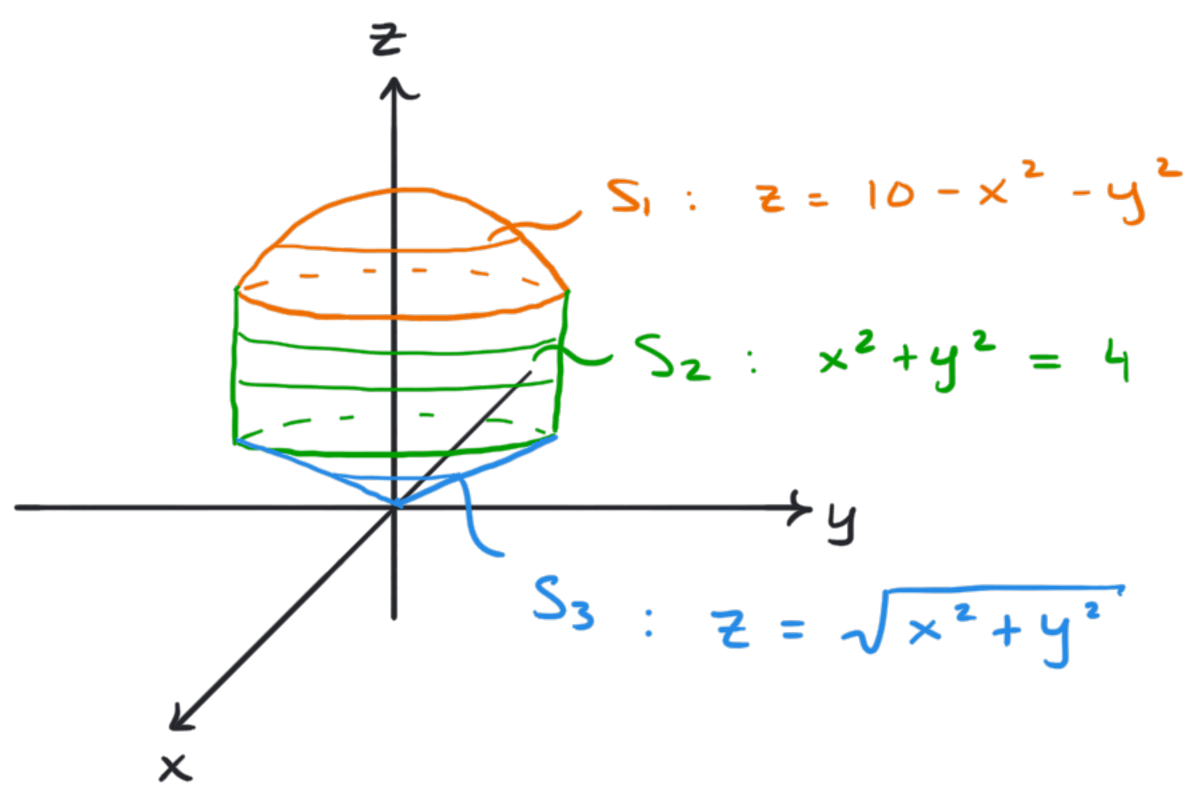
We have analogous formulas:

$$\iint_S f\, dS = \iint_D f(x, h(x,z), z) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}\, dx\, dz$$

$$\iint_S f\, dS = \iint_D f(j(y,z), y, z) \sqrt{1 + \left(\frac{\partial j}{\partial y}\right)^2 + \left(\frac{\partial j}{\partial z}\right)^2}\, dy\, dz$$

- We say that S is a piecewise-smooth surface if it is a finite union of smooth surfaces $S_1, \dots, S_n$ that are joined together along their boundaries:

E.g.:

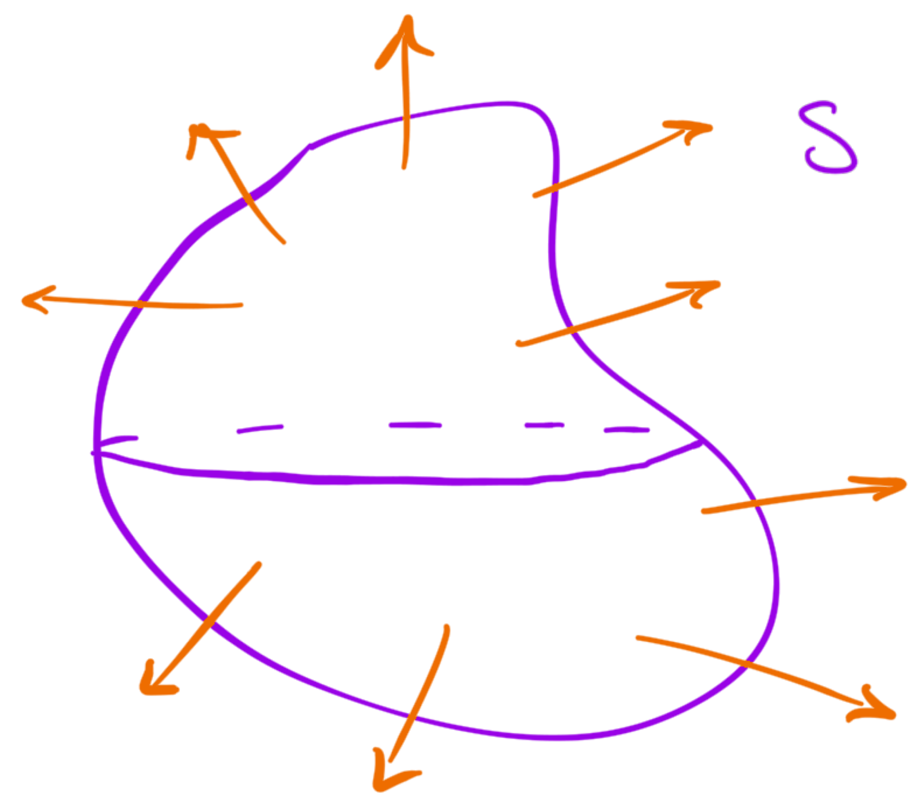


Then:

$$\boxed{\iint_S f \, dS = \sum_{i=1}^{n} \iint_{S_i} f \, dS}$$

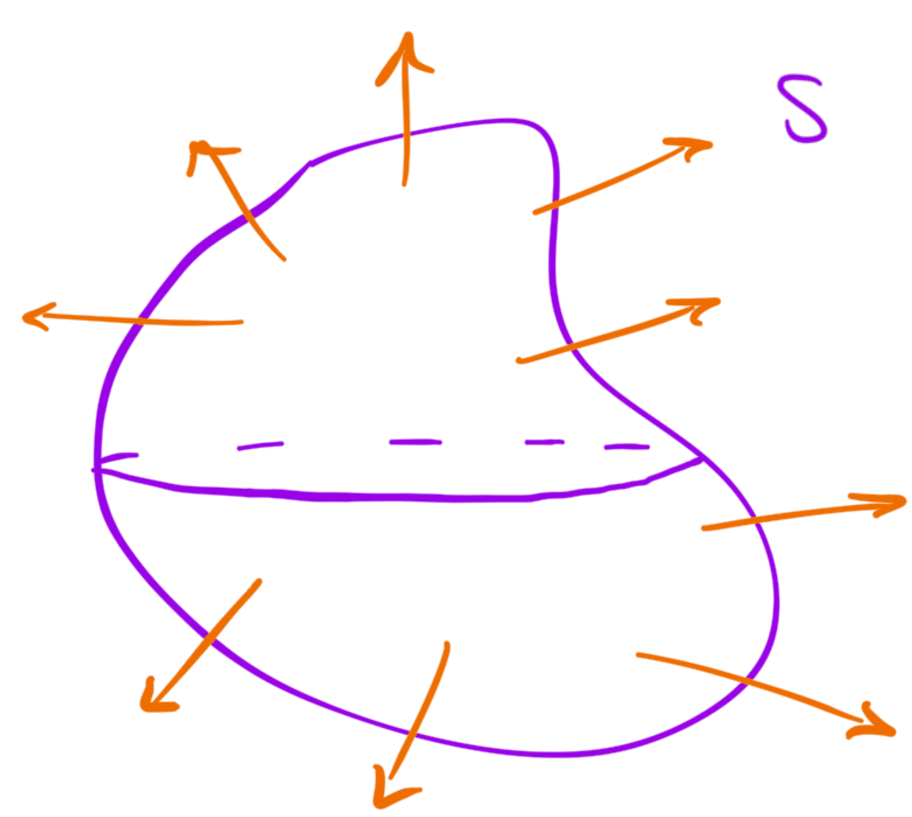
Orientation:

Motivation: Say I have a metal object, S, which is emitting heat energy:



If we computed this flux, should it be a positive or negative quantity?

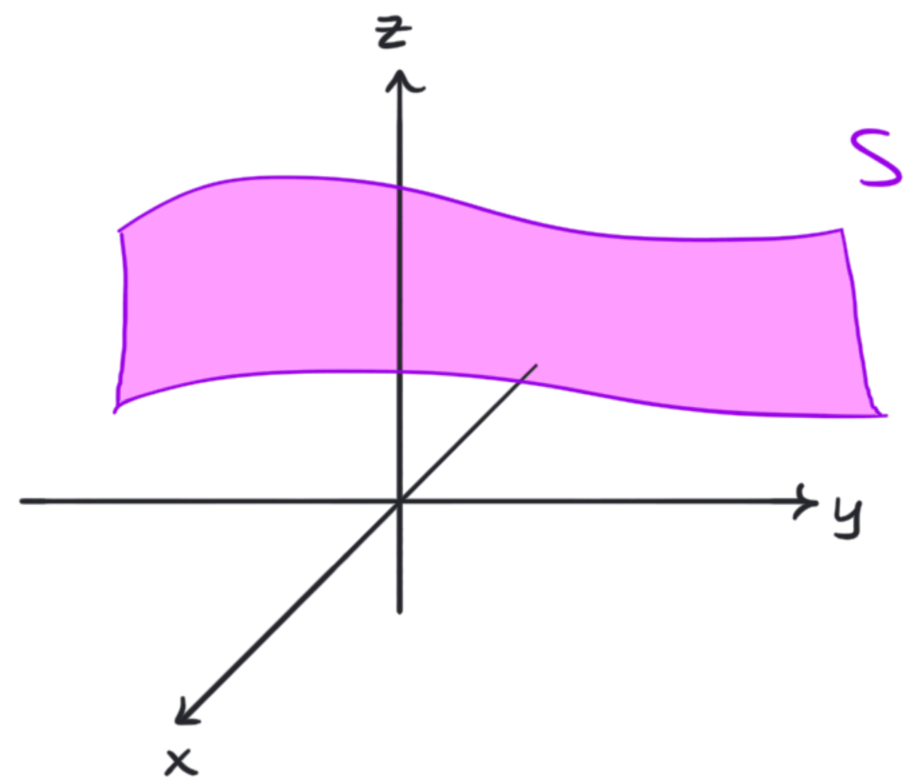
Now say I have a metal object, S, which is losing heat energy:



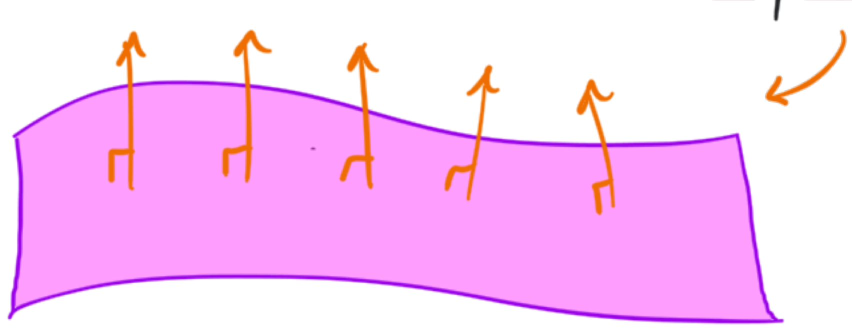
If we computed this flux, should it be a positive or negative quantity?

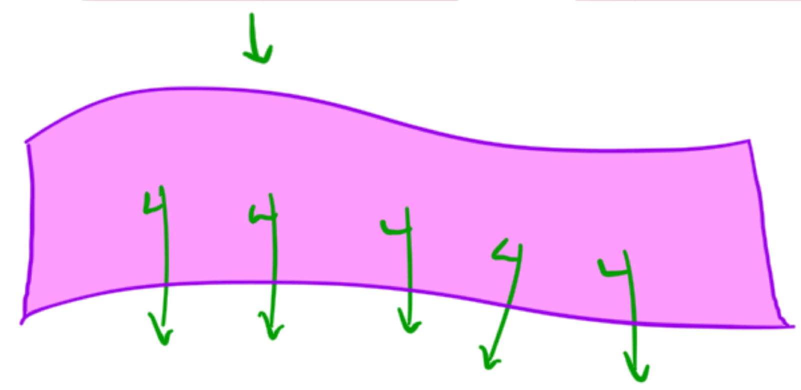
Conclusion :

- If S is the graph of a function : $z = g(x,y)$



We can think of "upward" or "downward" orientation :





We can find an explicit formula for the upward pointing unit normal to this graph :

$$\hat{n} = \frac{\left(-\left(\frac{\partial g}{\partial x}\right), -\left(\frac{\partial g}{\partial y}\right), 1 \right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

Exercise : Find the upward pointing unit normal to the surface given by $z = x^2 + y^2$.

If S is a parametric surface represented by $\vec{r}(u,v)$,

then :

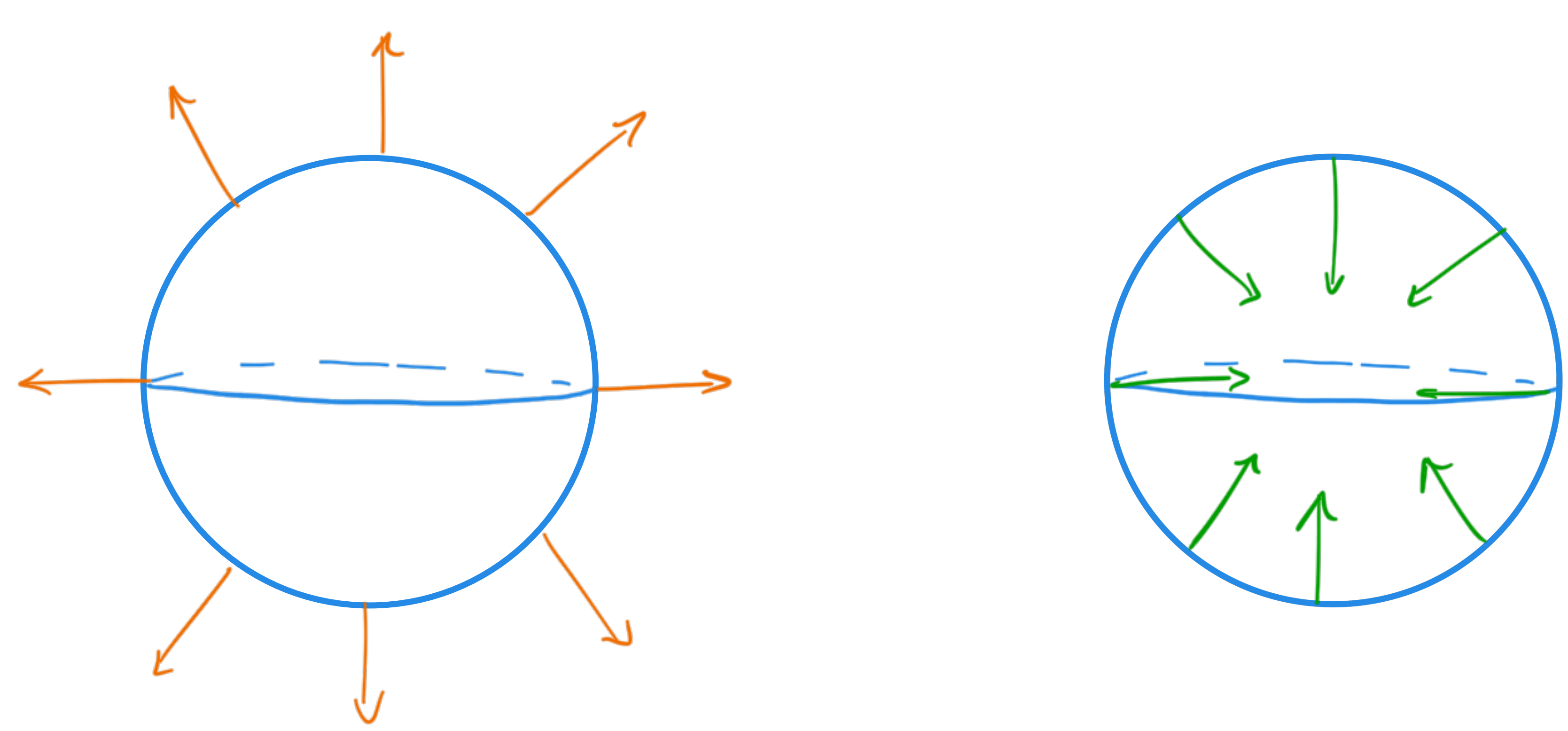
$$\boxed{\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

is a unit normal vector .

Remark : The opposite orientation is given by $-\hat{n}$.

Exercise : Find a unit normal to the surface parametrized by $\vec{r}(u,v) = (u+1, -u+v, u)$.

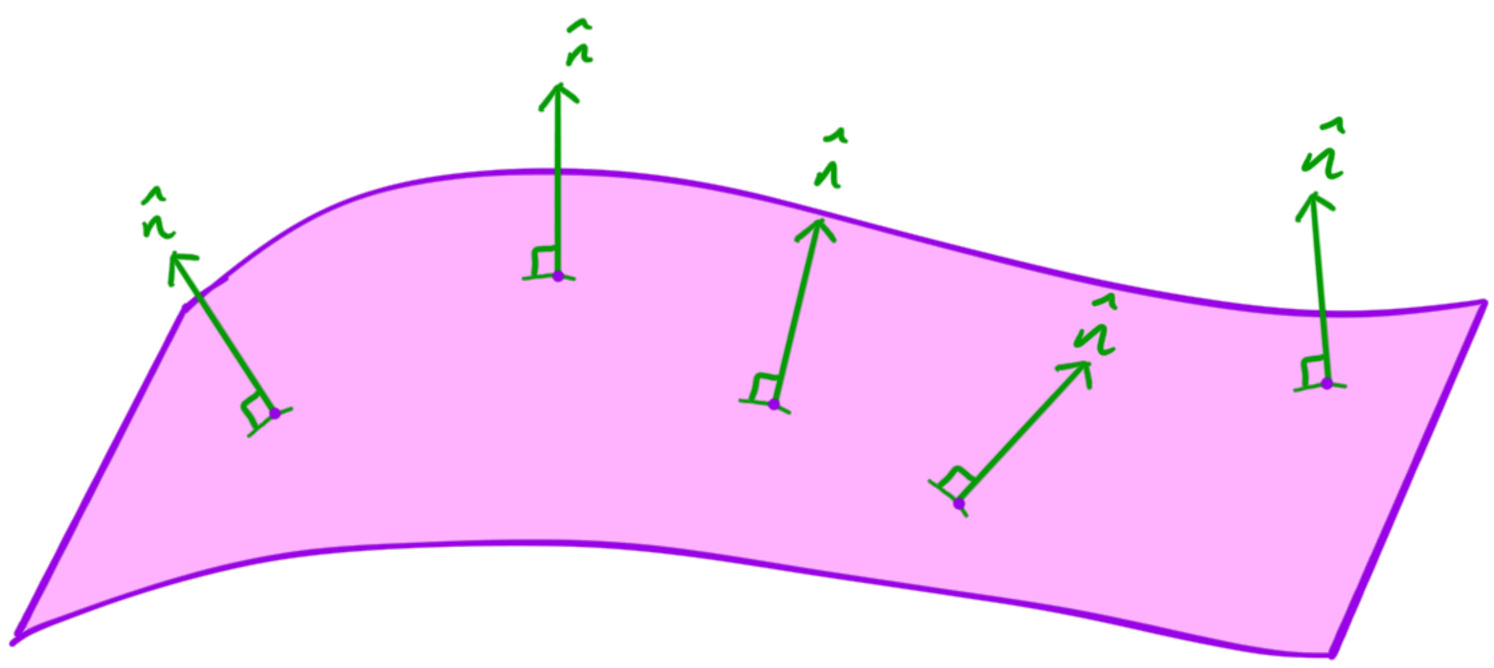
- For a closed surface we can define "outward" and "inward" pointing unit normals :



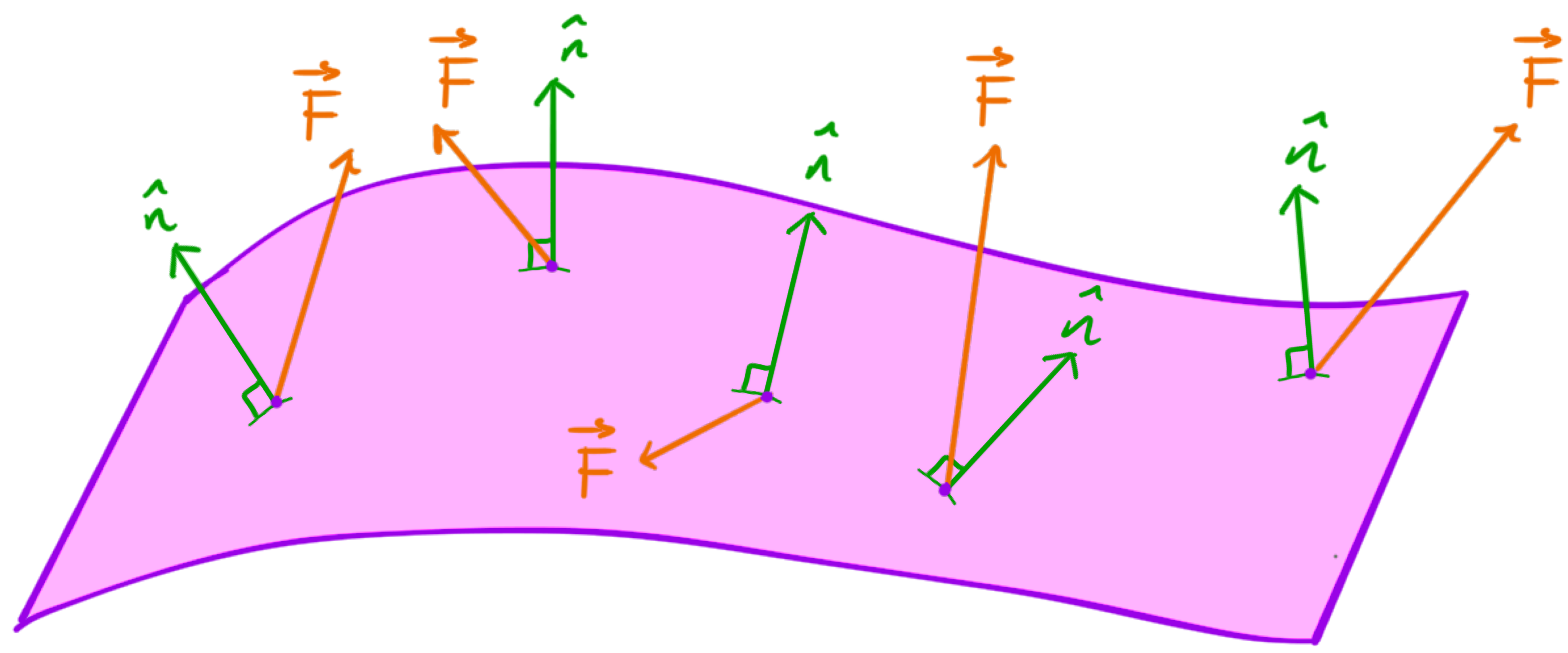
Example :

Surface Integrals of Vector Fields:

Suppose we have an oriented surface S with unit normal $\hat{n}$.

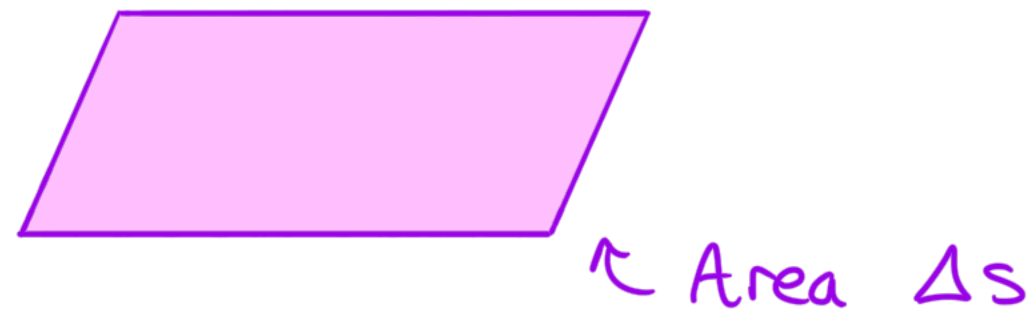


Let's say we have a vector field $\vec{F}$ on S:

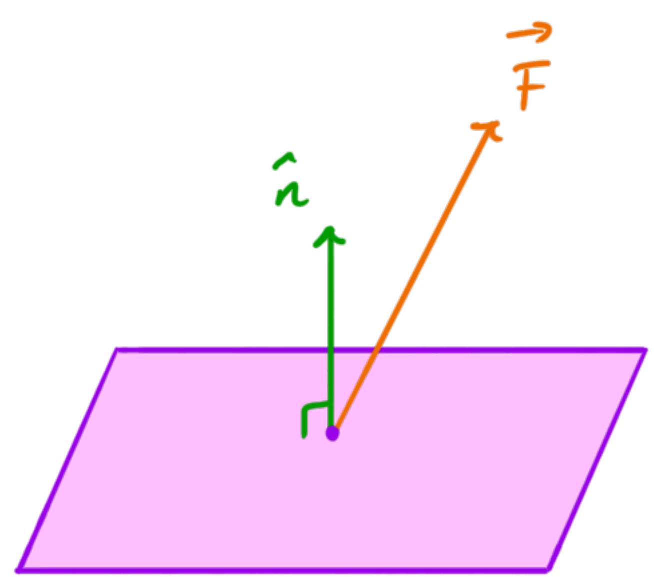


Recall: Our intuition told us that Flux should capture how much $\vec{F}$ is "flowing through" S.

Let's zoom in on a small patch of area ΔS:



On this small patch, as our vector fields $\vec{F}$ and $\hat{n}$ are "well behaved", they should look pretty much constant on this small patch:



If we think of $\vec{F}$ as the rate at which water is flowing across the points in this patch, what volume of water will flow through per unit time?

Definition: If $\vec{F}$ is a continuous vector field on an oriented surface S with unit normal $\hat{n}$, then the surface integral of $\vec{F}$ over S or the Flux of $\vec{F}$ across S is given by:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS}$$

Example:

Let S be the unit sphere : $x^2 + y^2 + z^2 = 1$.

Let $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$

- If S is given by $\vec{r}(u,v)$ then :

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS$$

$$= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v|\, dA$$

Hence we have :

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv}$$

- If S is the graph of a function : $z = g(x,y)$:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA}$$

Why?

Example: Let S be the surface given by $z = x^2 + y^2$

above the region D: $x^2 + y^2 \leq 1$.

Let $F(x, y, z) = \langle -x, -y, x^2 + y^2 \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$