

Surface Integrals and Flux:

Last time:

↳ Parametric Surfaces

↳ Area of Parametric Surfaces

Goal for today:

↳ Integrate functions over surfaces:

$$\iint_S f \, dS$$

↳ Develop a notion of **Orientation**.

↳ Develop a notion of **Flux**.

Examples:

(a) If a surface S has density function δ , then the

$$\text{mass of } S, m(S) = \iint_S \delta \, dS.$$

(b) Rate at which water passes through a membrane or porous vessel.

(c) Rate at which heat energy is emitted from a metal object.

Surface Integrals:

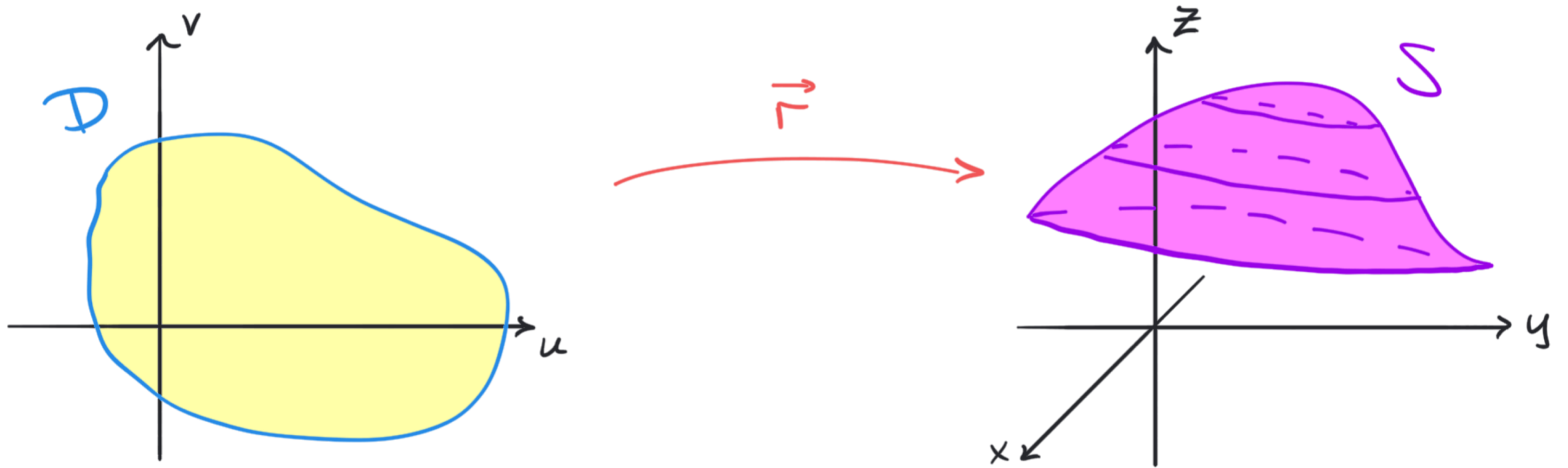
- We can think of the relationship :

Surface Area \longleftrightarrow Surface Integrals

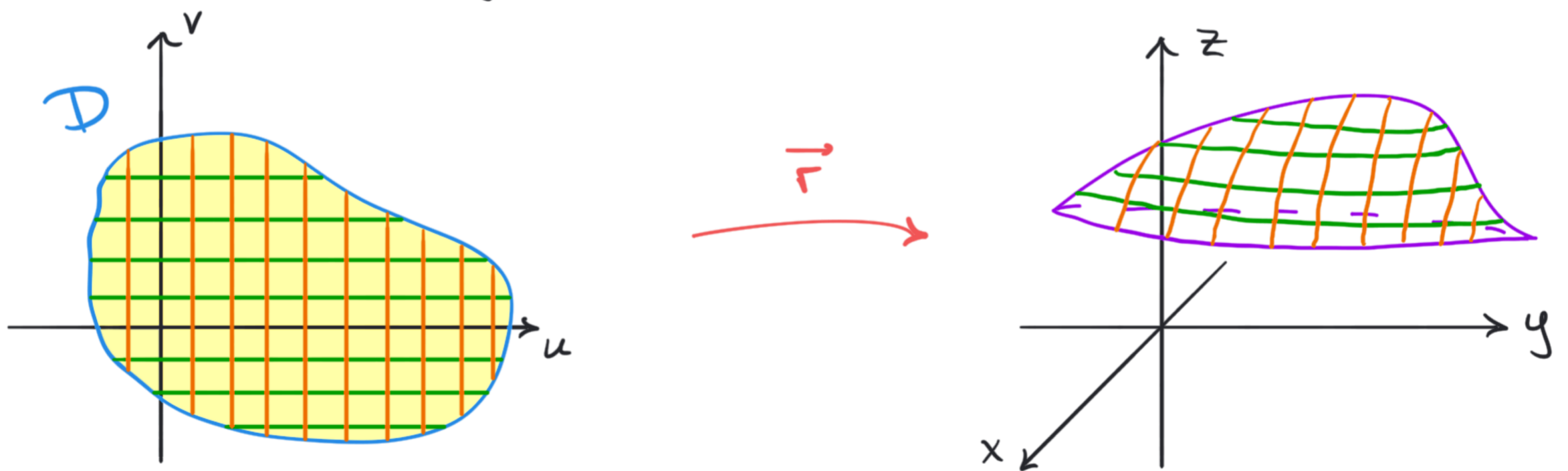
in a similar way to how we think of the relationship:

Arc Length \longleftrightarrow Line Integrals

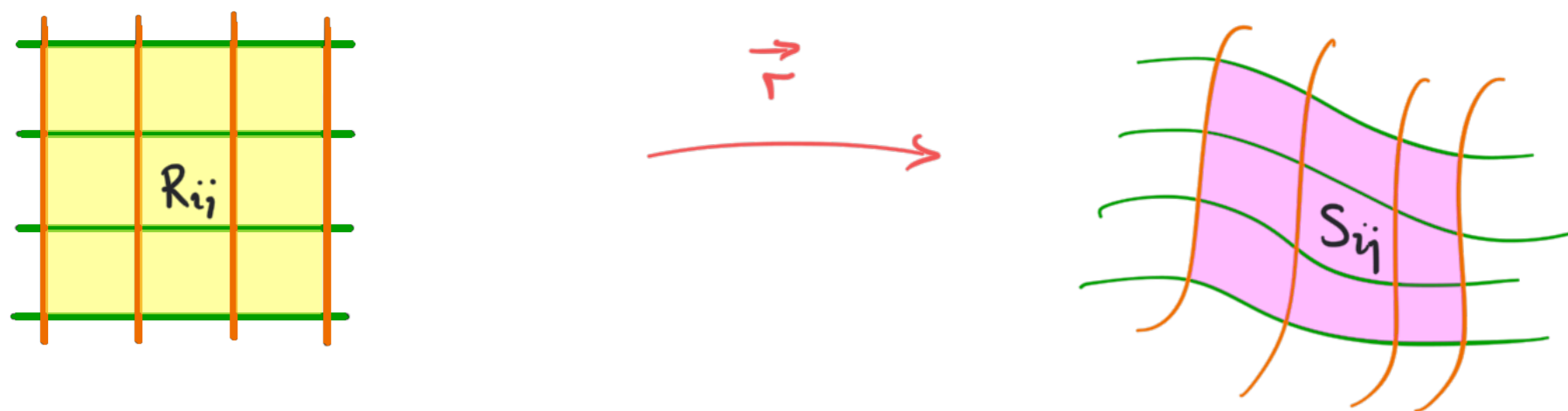
- If S is parametrized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$:



We can once again consider the grid lines :



If we zoom in on this picture:



We saw before that the area of S_{ij} :

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta R_{ij} = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \quad (*)$$

So if P_{ij} was a point in S_{ij} , and S had a mass density function δ , then the mass of the square S_{ij} would be:

$$\text{mass}(S_{ij}) \approx \delta(P_{ij}) \Delta S_{ij}$$

Doing this for each square:

$$\text{mass}(S) \approx \sum_{i=1}^m \sum_{j=1}^n \text{mass}(S_{ij}) \approx \sum_{i=1}^m \sum_{j=1}^n \delta(P_{ij}) \Delta S_{ij}$$

Using (*):

$$\text{mass}(S) \approx \sum_{i=1}^m \sum_{j=1}^n \delta(P_{ij}) |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

Taking finer and finer grids :

$$\begin{aligned} \text{mass}(S) &= \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \delta(P_{ij}) |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \\ &= \iint_D \delta(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| dA \end{aligned}$$

• In general :

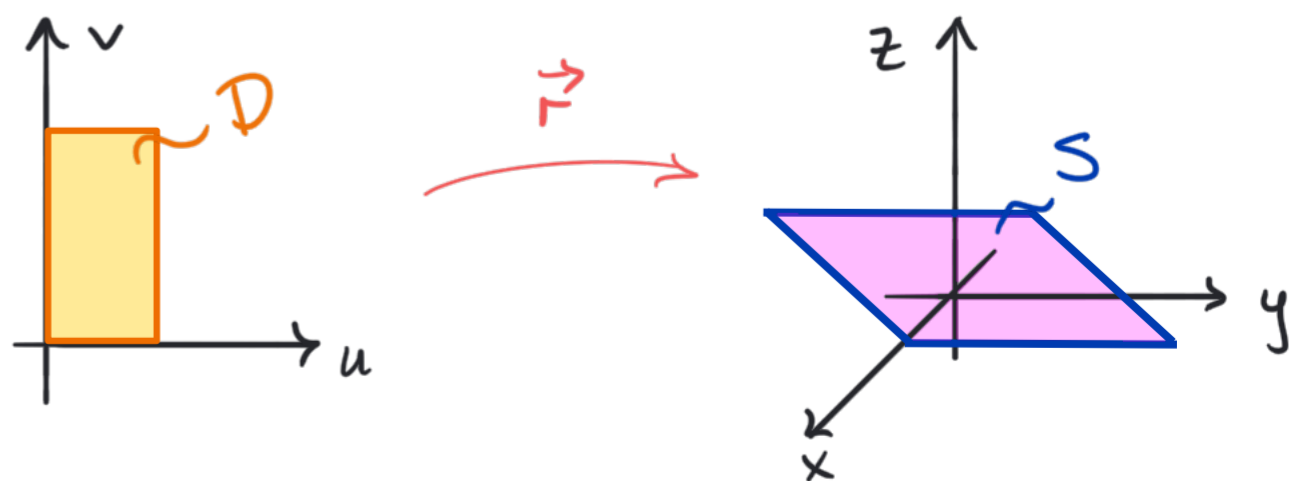
$$\iint_S f dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| dA$$

Example: Compute the mass of a sheet of metal (parallelogram), parametrised by:

$$\vec{r}(u,v) = (u+1, -u+v, u), \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2.$$

with mass density $\delta(x,y,z) = z^2$.

Solution:



Remark: We can develop center of mass formulas for a surface S with density function δ :

Center of mass = $(\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{1}{M} \iint_S x \delta(x, y, z) dS$$

$$\bar{y} = \frac{1}{M} \iint_S y \delta(x, y, z) dS$$

$$\bar{z} = \frac{1}{M} \iint_S z \delta(x, y, z) dS$$

Special Case: If S is the graph of a function: $z = g(x, y)$,

$\vec{r}(x, y) = (x, y, g(x, y))$, then, as before:

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

So we have:

$$\iint_S f dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dx dy$$

Example: Let S be the surface given by $z = y - x^2$ above the region $D = \{(x, y) ; 0 \leq x \leq 1, 0 \leq y \leq 3\}$.

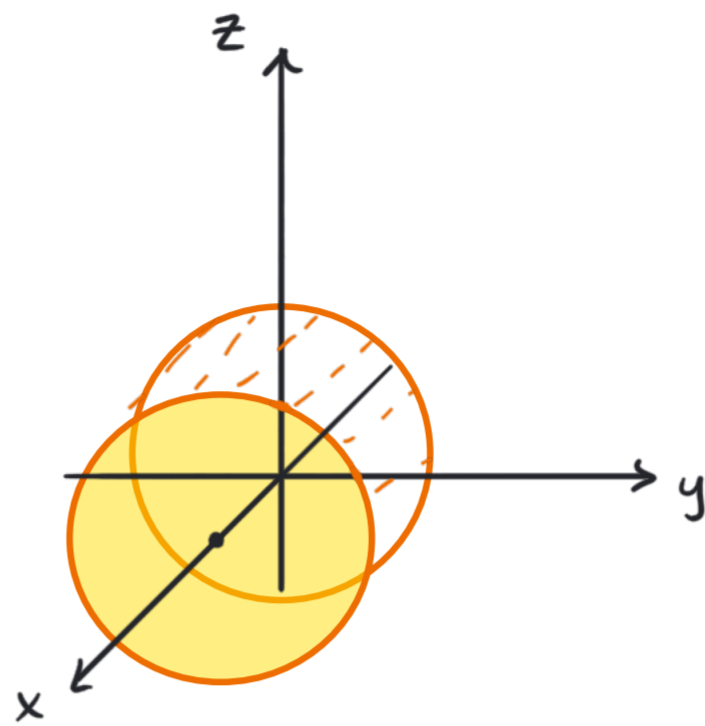
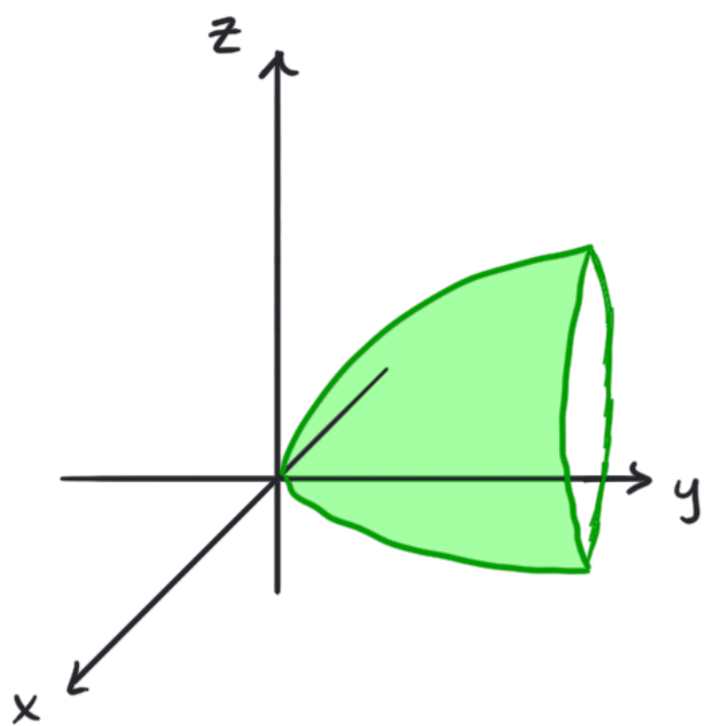
Let $f(x, y, z) = x^2 + x - y + z$.

Compute $\iint_S f \, dS$.

Solution:

$$\iint_S f \, dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dx \, dy$$

Remark: Some surfaces can be graphs of functions $y = h(x, z)$ or $x = j(y, z)$.



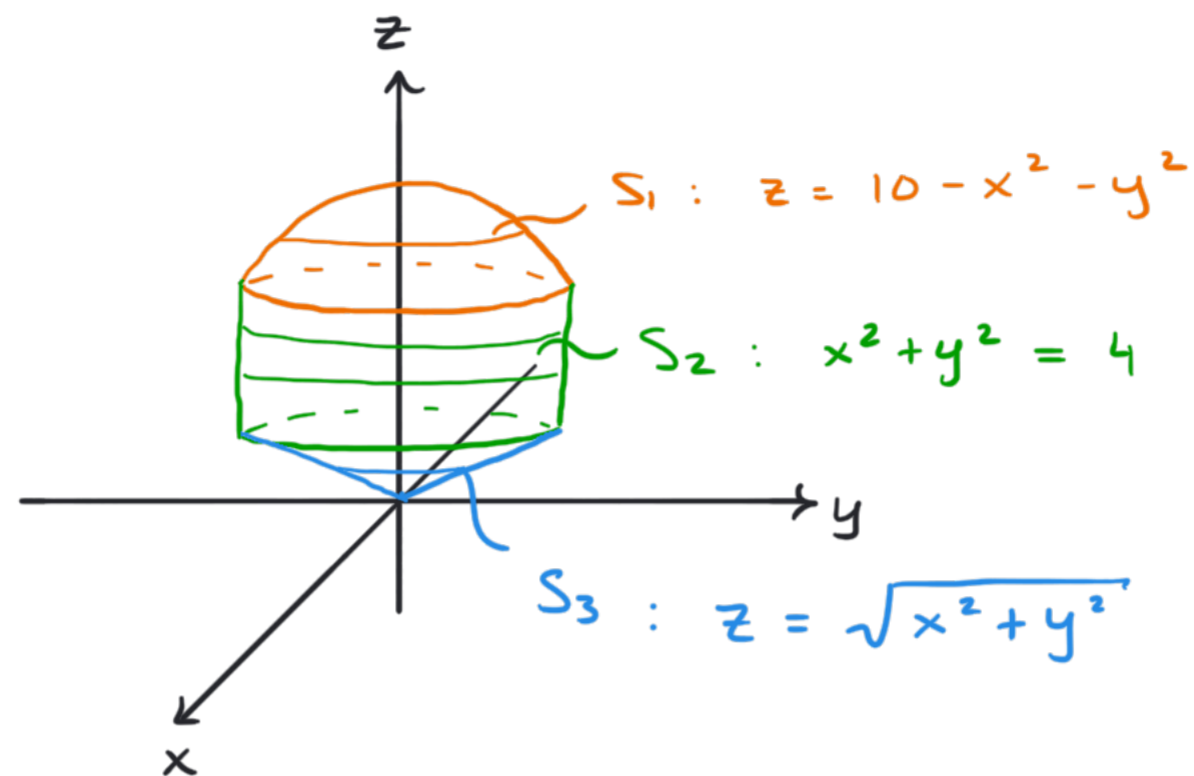
We have analogous formulas:

$$\iint_S f \, dS = \iint_D f(x, h(x, z), z) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2} \, dx \, dz$$

$$\iint_S f \, dS = \iint_D f(j(y, z), y, z) \sqrt{1 + \left(\frac{\partial j}{\partial y}\right)^2 + \left(\frac{\partial j}{\partial z}\right)^2} \, dy \, dz$$

- We say that S is a piecewise-smooth surface if it is a finite union of smooth surfaces S_1, \dots, S_n that are joined together along their boundaries:

E.g.:



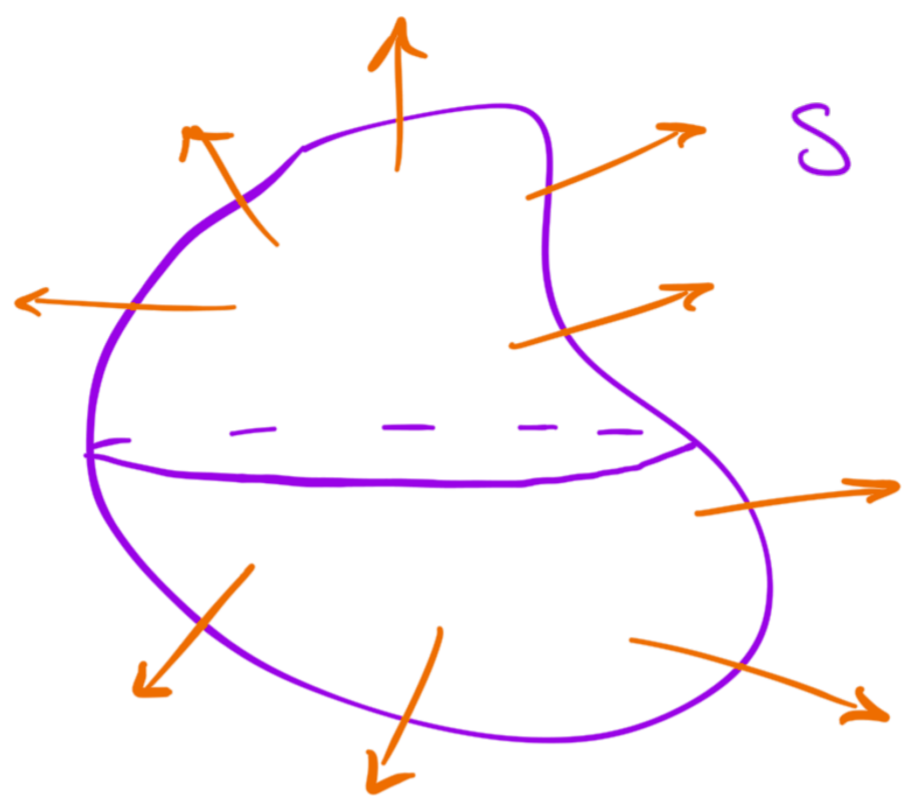
Then :

$$\iint_S f \, dS = \sum_{i=1}^n \iint_{S_i} f \, dS$$

Orientation:

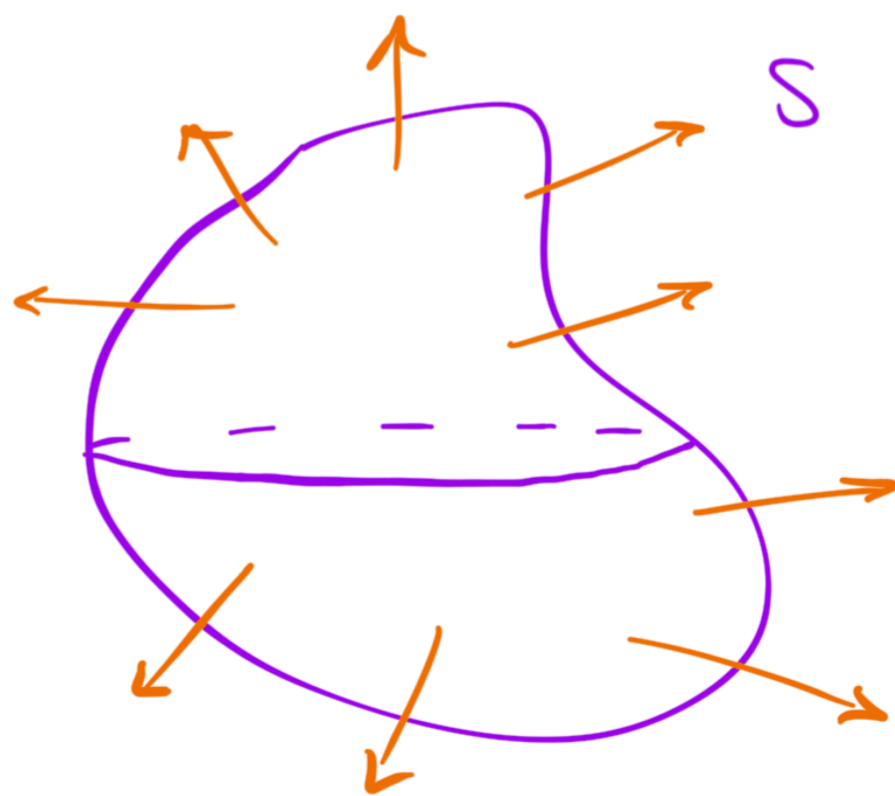
Motivation:

Say I have a metal object, S , which is emitting heat energy:



If we computed this flux, should it be a positive or negative quantity?

Now say I have a metal object, S , which is losing heat energy:



If we computed this flux, should it be a positive or negative quantity?

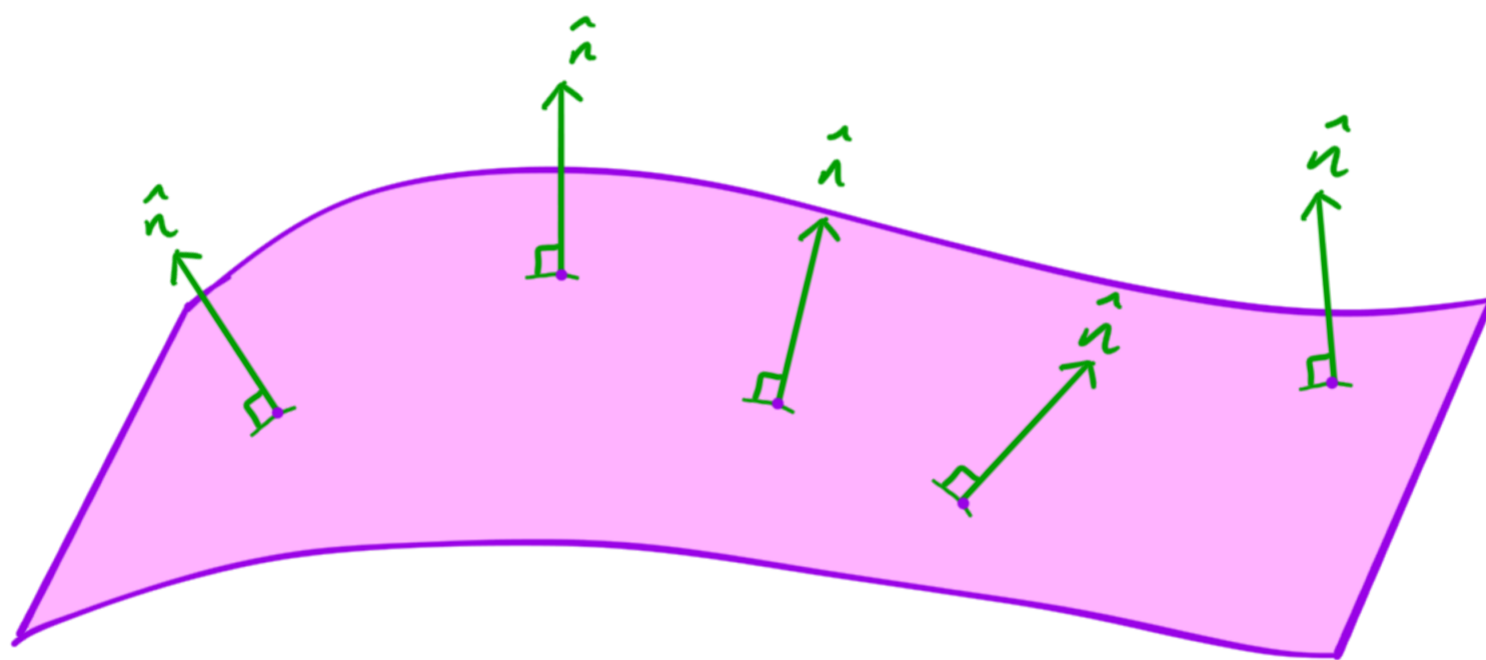
Conclusion:

There is no overall "should".

You have to make a choice.

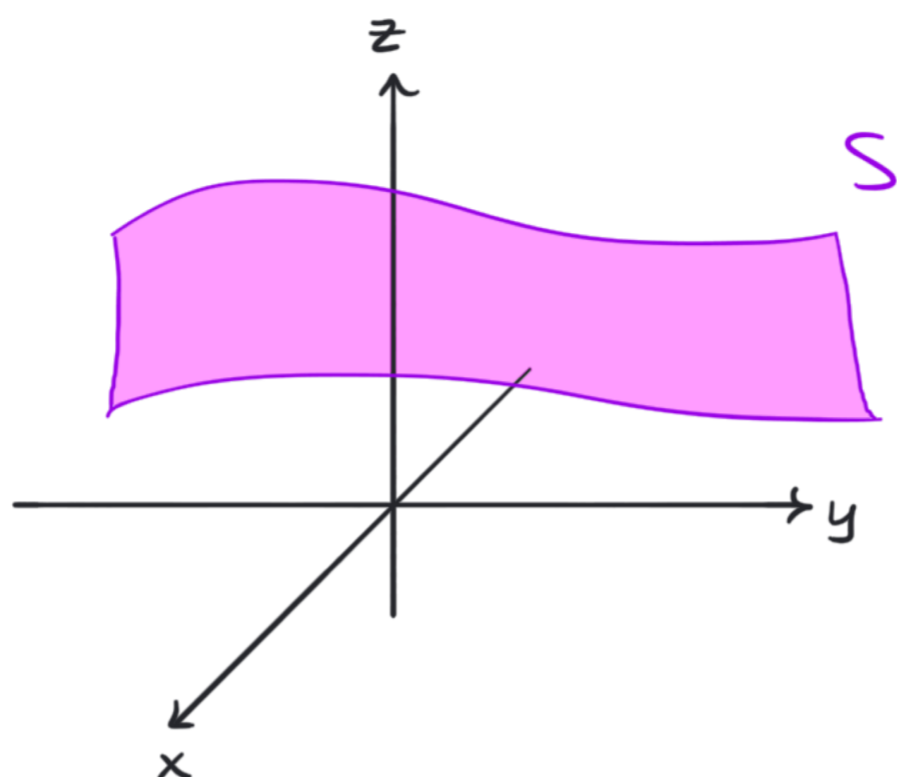
You have to make a choice of Orientation.

So orientation is technically a choice of continuously varying unit normal vector : \hat{n} .

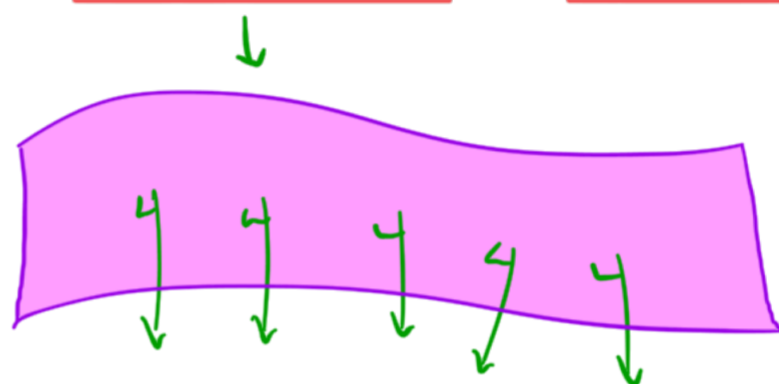
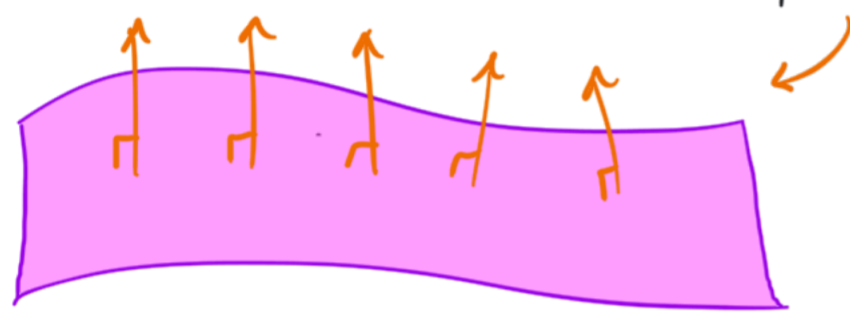


We say a surface S , equipped with an orientation \hat{n} is an oriented surface.

- If S is the graph of a function: $z = g(x, y)$



We can think of "upward" or "downward" orientation:



We can find an explicit formula for the upward pointing unit normal to this graph:

$$\hat{n} = \frac{\left(-\left(\frac{\partial g}{\partial x}\right), -\left(\frac{\partial g}{\partial y}\right), 1 \right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

Exercise: Find the upward pointing unit normal to the surface given by $z = x^2 + y^2$.

- If S is a parametric surface represented by $\vec{r}(u,v)$,

then:

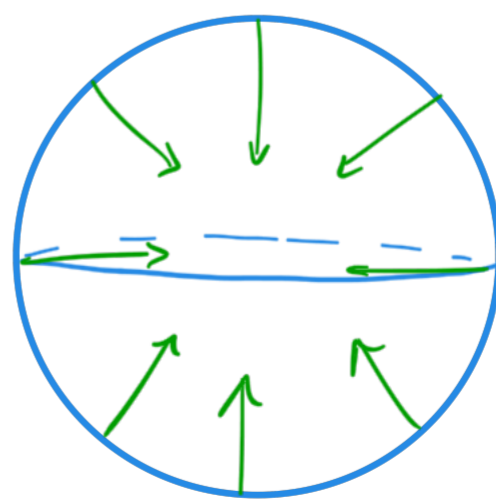
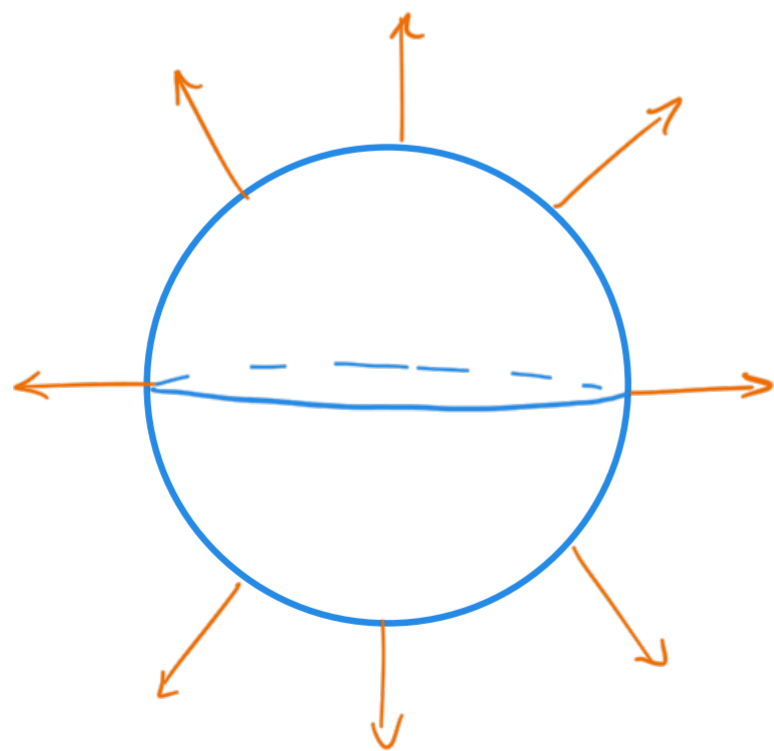
$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

is a unit normal vector.

Remark: The opposite orientation is given by $-\hat{n}$.

Exercise: Find a unit normal to the surface parametrized by $\vec{r}(u,v) = (u+1, -u+v, u)$.

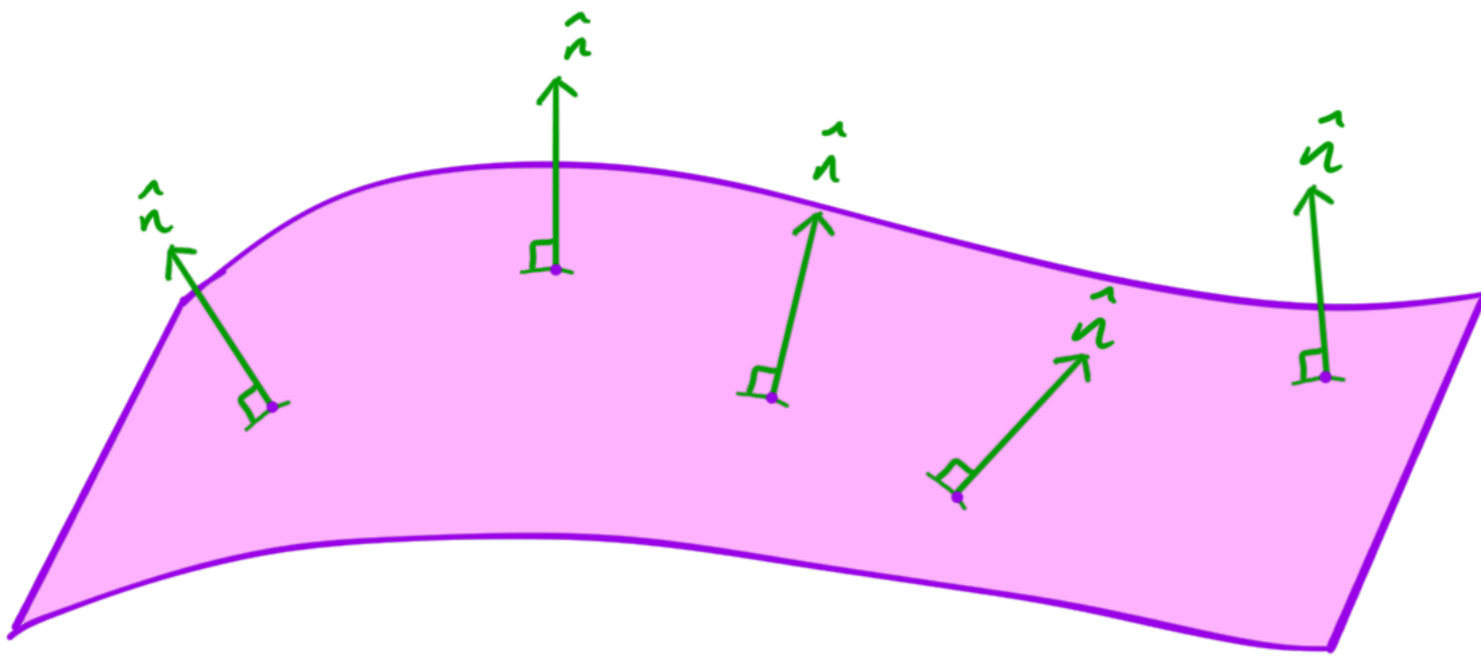
- For a closed surface we can define "outward" and "inward" pointing unit normals:



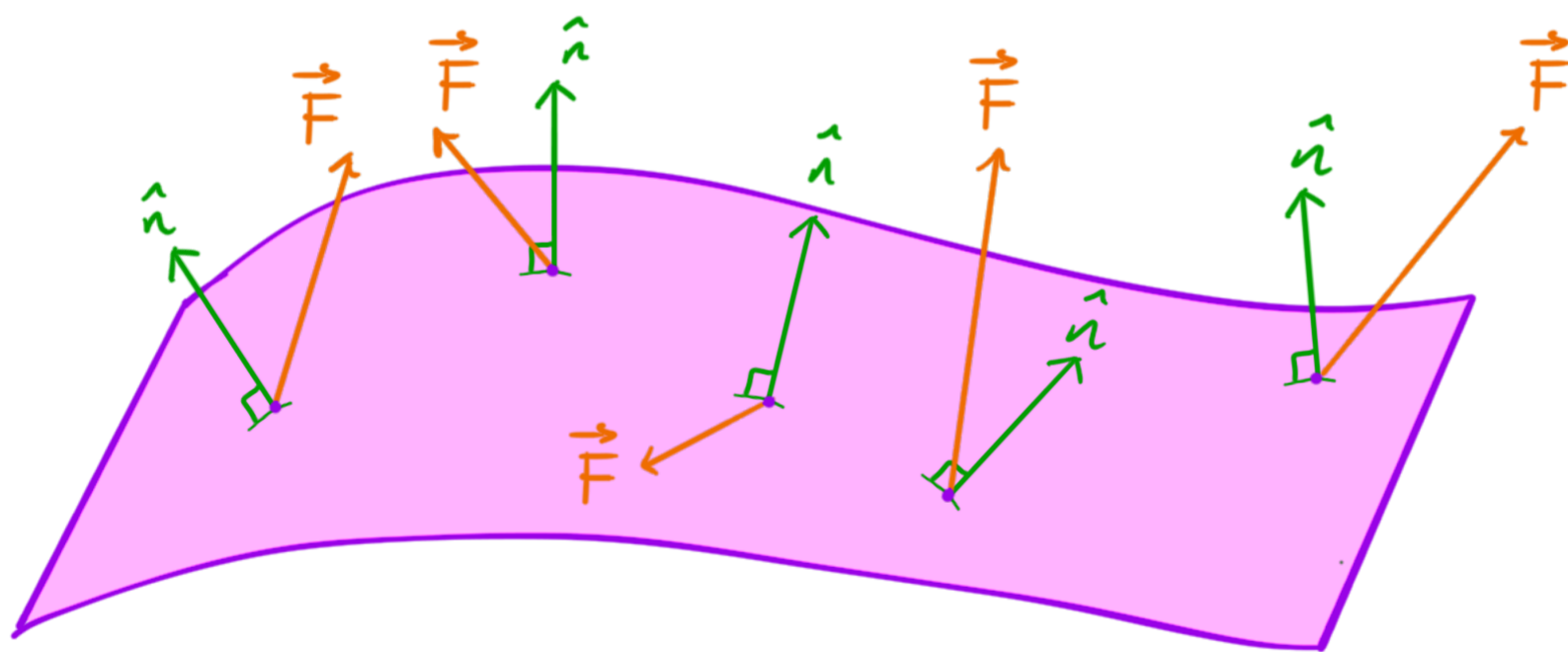
Example: The outward pointing unit normal to a sphere of radius R is $\hat{n} = \frac{1}{R} \langle x, y, z \rangle$.

Surface Integrals of Vector Fields:

Suppose we have an oriented surface S with unit normal \hat{n} .

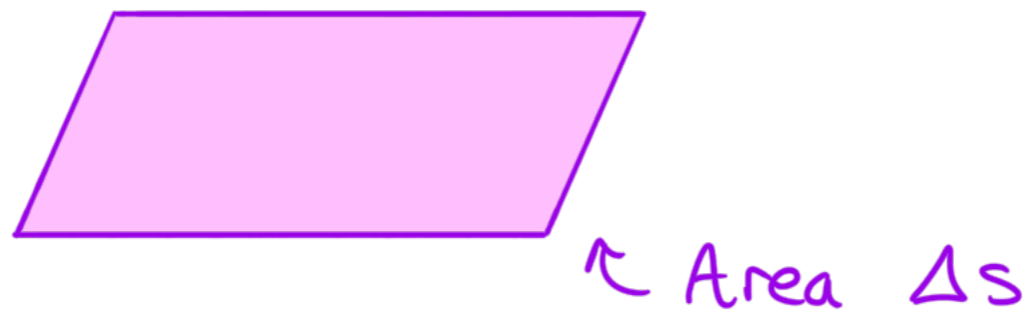


Let's say we have a vector field \vec{F} on S :

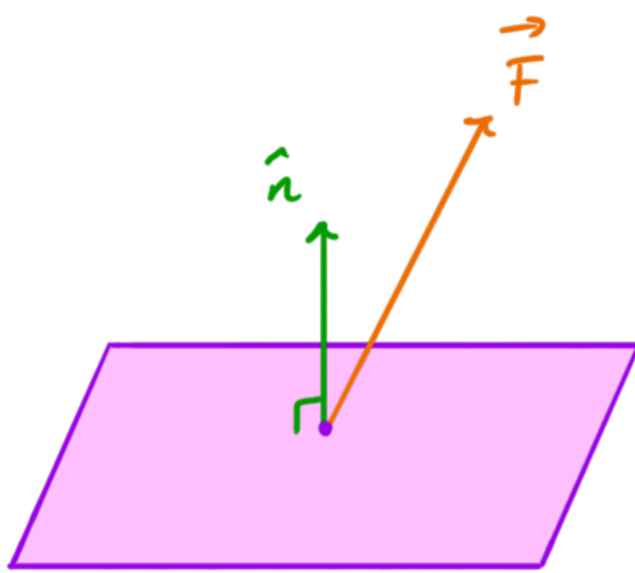


Recall: Our intuition told us that Flux should capture how much \vec{F} is "flowing through" S .

Let's zoom in on a small patch of area ΔS :

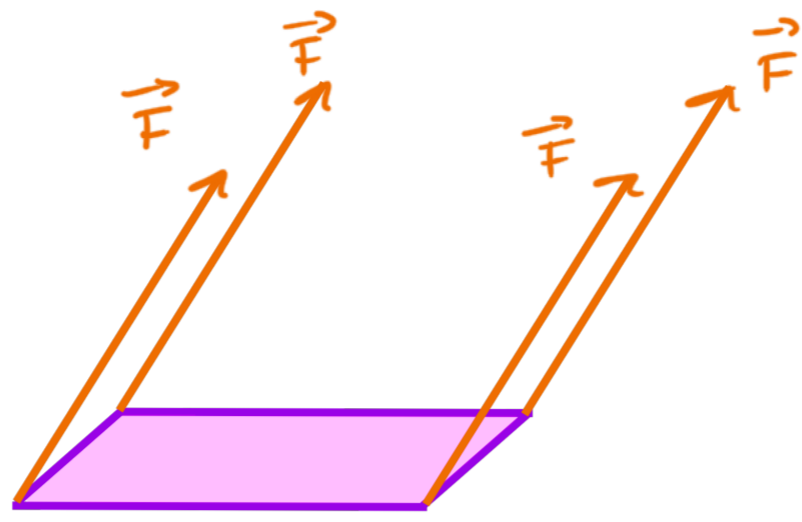


On this small patch, as our vector fields \vec{F} and \hat{n} are "well behaved", they should look pretty much constant on this small patch:

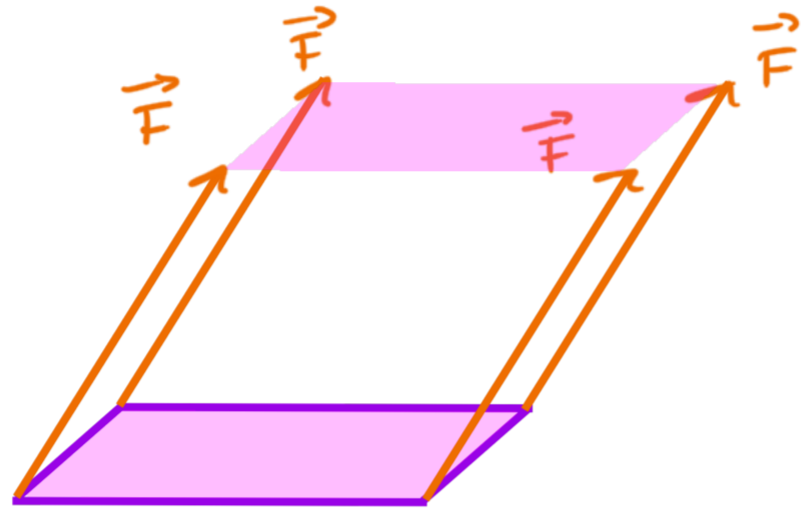


If we think of \vec{F} as the rate at which water is flowing across the points in this patch, what volume of water will flow through per unit time?

If we think of \vec{F} as a velocity field for some fluid, and assume that on this patch it's moving at v meters per second:



What is the volume of this "box"?



Definition: If \vec{F} is a continuous vector field on an oriented surface S with unit normal \hat{n} , then the surface integral of \vec{F} over S or the Flux of \vec{F} across S is given by:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS$$

Example:

Let S be the unit sphere : $x^2 + y^2 + z^2 = 1$.

Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$

Solution:

• If S is given by $\vec{r}(u,v)$ then :

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS \\ &= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| \, dA\end{aligned}$$

Hence we have :

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

• If S is the graph of a function : $z = g(x,y)$:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

Why?

Example: Let S be the surface given by $z = x^2 + y^2$

above the region $D: x^2 + y^2 \leq 1$.

Let $F(x, y, z) = \langle -x, -y, x^2 + y^2 \rangle$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

Office Hours:

Prof. Edelen: 3pm-4pm, Today

P: 3pm - 4pm and 6pm-7pm, Thursday

Problem Session: 6pm-7pm, Today