Curl and Divergence:

"Definition": $\boxed{\vec{\nabla} := \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}}$

Remark: $\vec{\nabla} f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z}$

$$= \left(\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right)$$

"Definition": For a "nice" vector field on $\mathbb{R}^3$, $\vec{F} = (P, Q, R)$:

$$\boxed{\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla} \times \vec{F}}$$

Theorem: If f is a scalar function of three variables that has contiuous second order partial derivatives, then:

$$\boxed{\vec{\nabla} \times (\vec{\nabla} f) = 0}$$

Theorem: If $\vec{F}$ is a vector field on $\mathbb{R}^3$ whose component functions all have continuous first partials, then:

$$\boxed{\vec{\nabla} \times \vec{F} = 0 \implies \vec{F} \text{ is conservative}}$$

Definition: $\boxed{\operatorname{div}(\vec{F}) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla} \cdot \vec{F}}$

Theorem: $\boxed{\operatorname{div}(\operatorname{curl}(\vec{F})) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0}$

Theorem: If $\operatorname{div}(\vec{F}) = 0$, then there is a $\vec{G}$ such that $\vec{F} = \vec{\nabla} \times \vec{G}$.

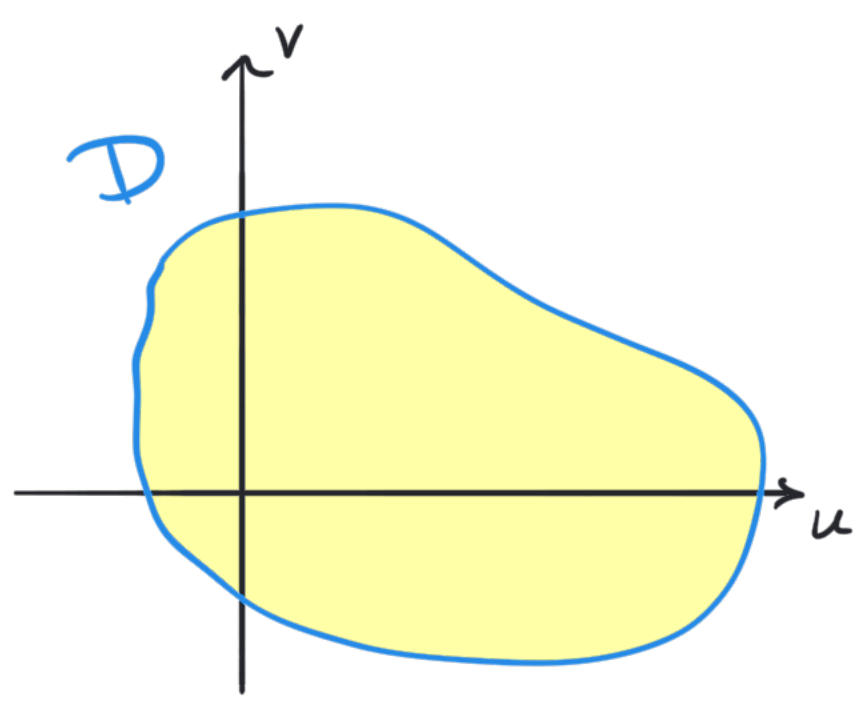
Intuition:

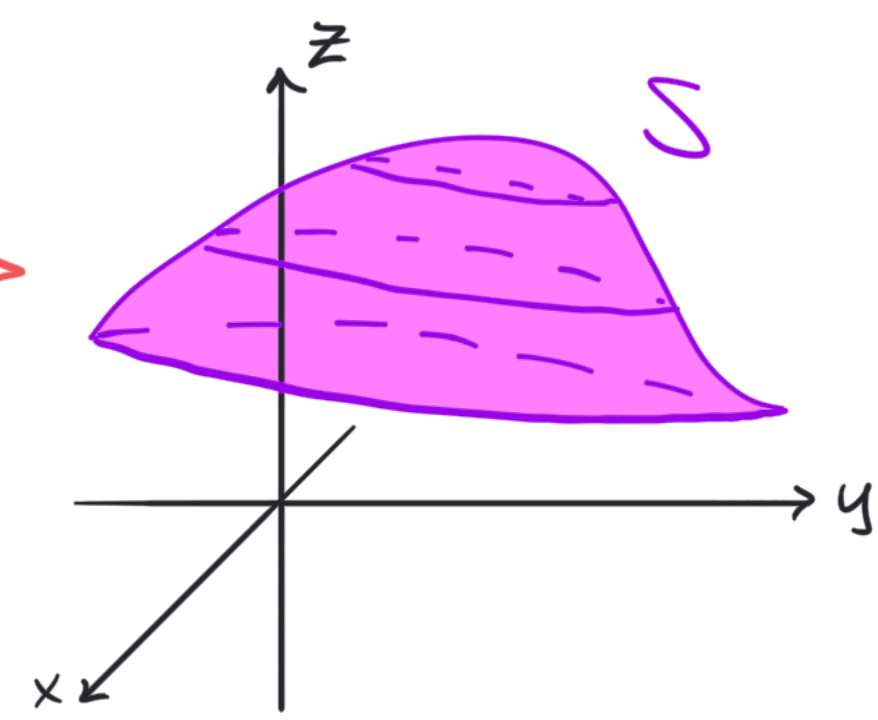
Example: Let $\vec{F}(x,y,z) = \left(e^{y}, zy, xy^2\right)$ and let $\vec{G}(x,y,z) = \left(z^2, \frac{x}{3}, xy\right)$.

Compute $\operatorname{div} \vec{F}$ and $\vec{\nabla} \times \vec{G}$.

Parametric Surfaces and their Area:

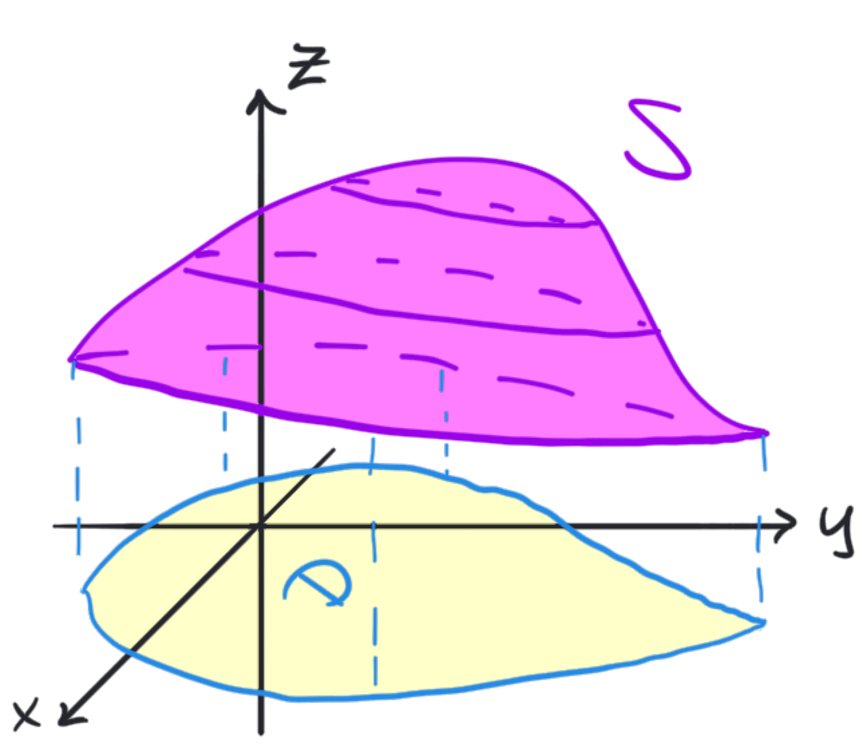
- Let $S \subset \mathbb{R}^3$ be a surface described by a vector-valued function : $\vec{r}(u,v) = \big(x(u,v), y(u,v), z(u,v) \big)$ for (u,v) in a region $D \subset \mathbb{R}^2$:





- Then we say : $\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right\}$ are the parametric equations of S.

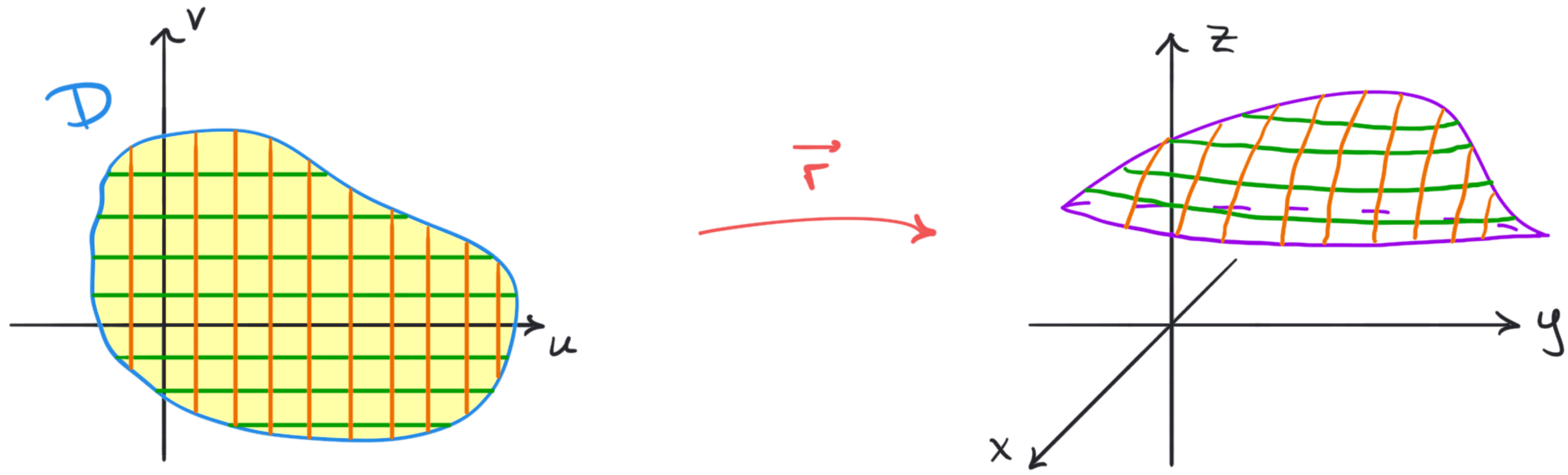
Special Case : If S is the graph of a function $z = g(x,y)$ over a region D in the xy plane :



we can parametrize S by :

$$\boxed{\vec{r}(x,y) = \big(x , y , g(x,y) \big)}$$

- The following picture illustrates grid curves :



The green lines / curves correspond to holding v constant.

The orange lines / curves correspond to holding u constant.

- We can think of restricting ourselves to a single grid curve : $\vec{r}(u, v_0)$ (Green) or $\vec{r}(u_0, v)$ (Orange).

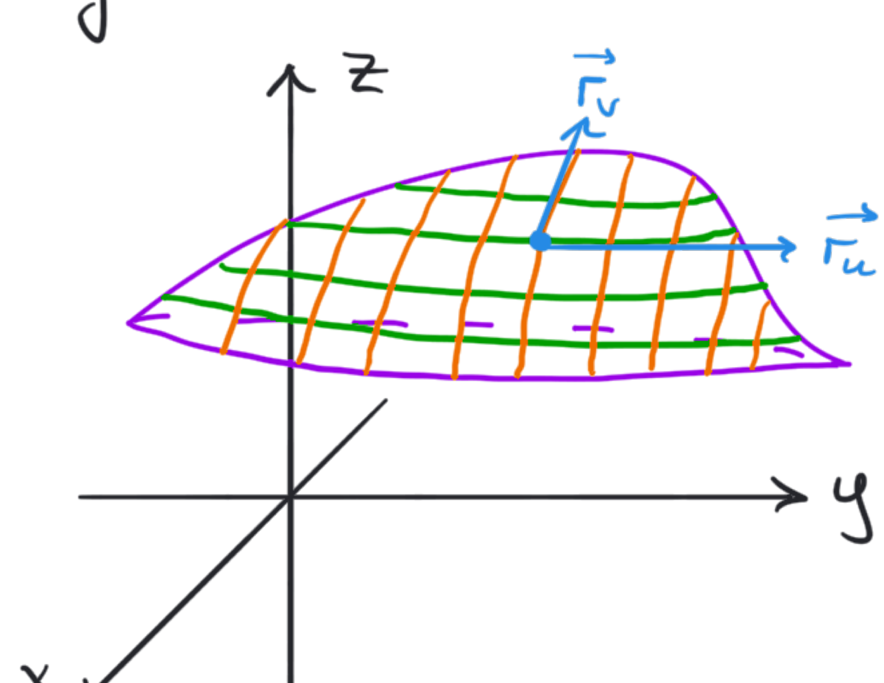
We can then think about limits like :

$$\lim_{h \to 0} \frac{r(u_0 + h, v_0) - r(u_0, v_0)}{h} = \frac{\partial r}{\partial u}(u_0, v_0) = \vec{r}_u(u_0, v_0)$$

or

$$\lim_{h \to 0} \frac{r(u_0, v_0 + h) - r(u_0, v_0)}{h} = \frac{\partial r}{\partial v}(u_0, v_0) = \vec{r}_v(u_0, v_0)$$

If we think about these geometrically, these correspond to tangent vectors to grid curves of S at $r(u_0, v_0)$:



Tangent Plane:

Hence, $r_u(u_0,v_0)$ and $r_v(u_0,v_0)$ span the tangent plane to S at $\vec{r}(u_0,v_0) = p = (x_0,y_0,z_0)$: T_pS.

So, if we wanted a normal vector to T_pS, we can compute $r_u(u_0,v_0) \times r_v(u_0,v_0)$.

This will allow us to find an equation for T_pS :

$$\boxed{(x-x_0\,,\ y-y_0\,,\ z-z_0)\cdot\left(r_u(u_0,v_0)\times r_v(u_0,v_0)\right) = 0}$$

Example:

1.(7 pts.) Compute the tangent plane to the surface parametrized by $\mathbf{r} = u\mathbf{i} + uv\mathbf{j} + (u+v)\mathbf{k}$ at the point $(1,2,3)$.

(a) $3x + 2y + z = 10$

(b) $\langle x,y,z\rangle = \langle 1+u, 2+uv, 3+u+v\rangle$

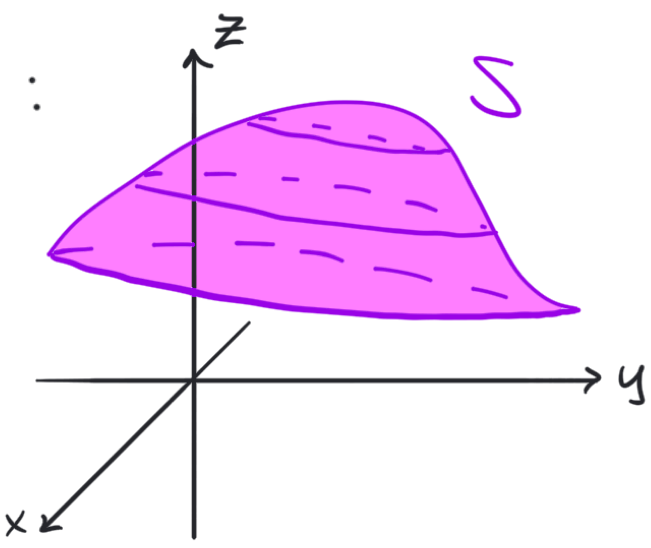
(c) $x - y + z = 2$

(d) $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$

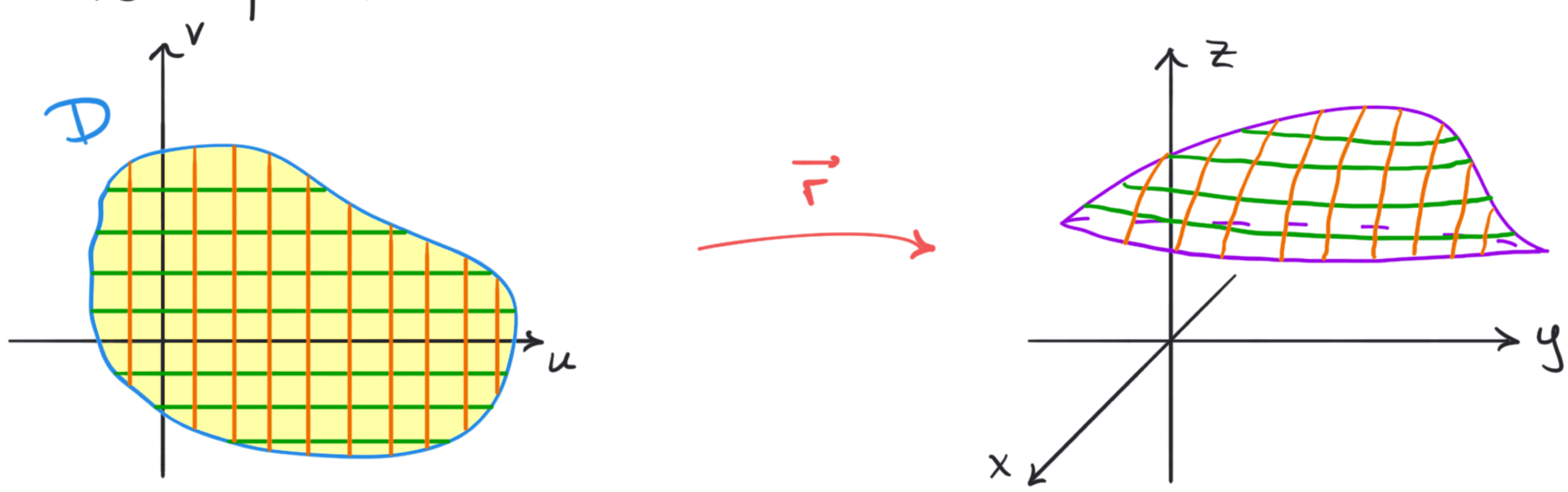
(e) $x + 2y + 3z = 14$

Surface Area:

If I now want to tackle the problem of finding the surface area of a surface S:

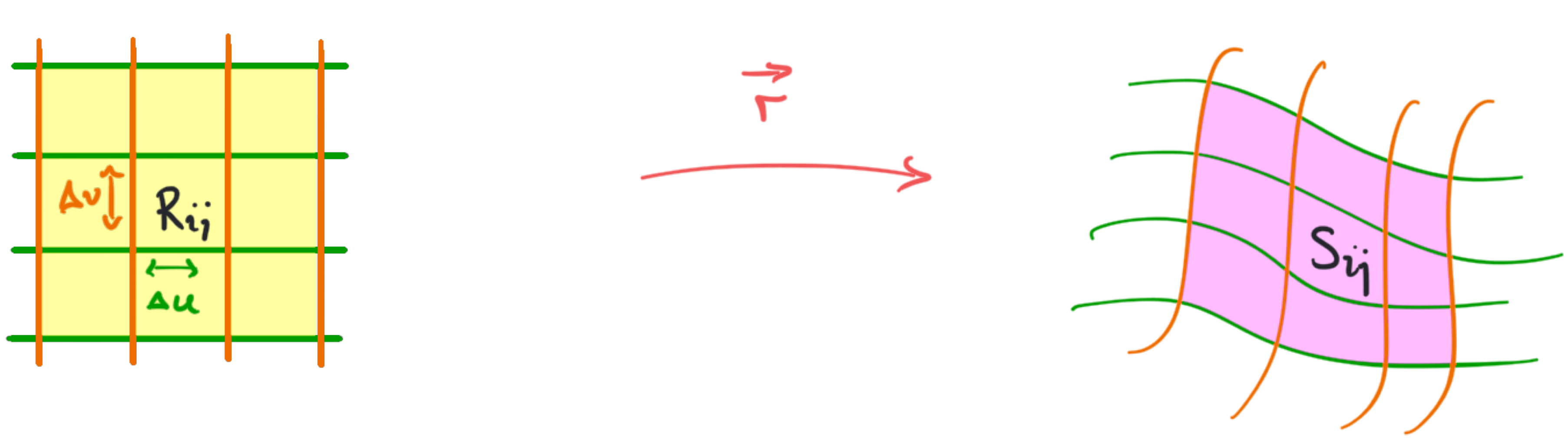


it is useful to return to our "gridline" picture:



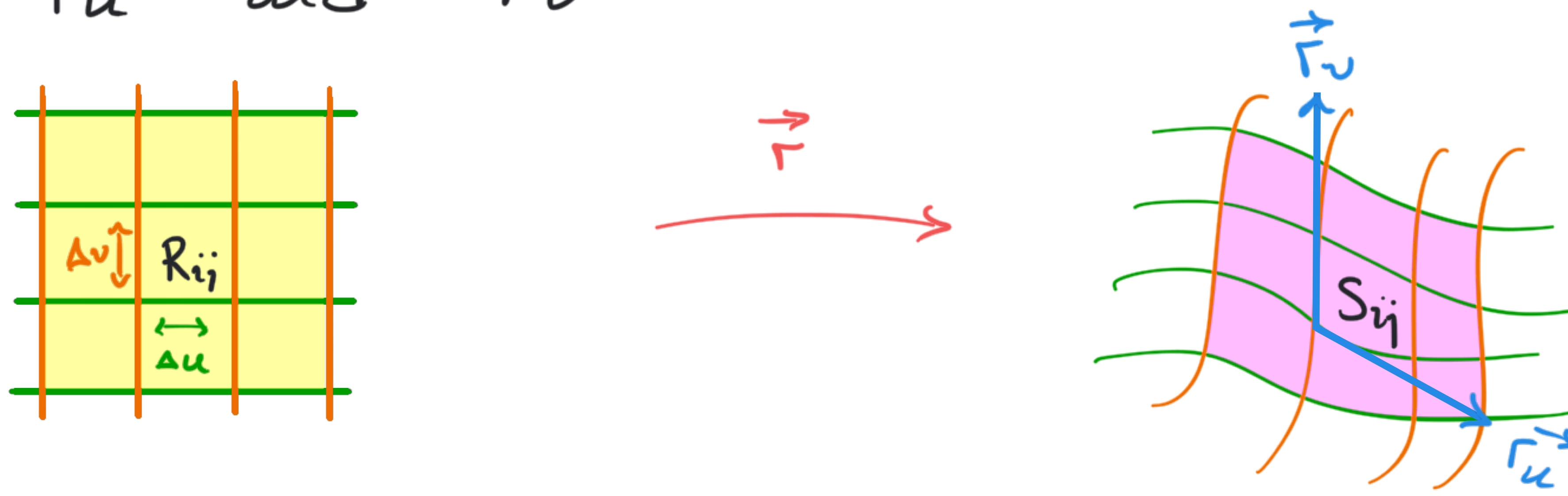
If we could approximate the area of each "piece" in this grid, we could add up all these approximations and get an approximation for the entire surface area.

So, let's "zoom in" on a piece of area:



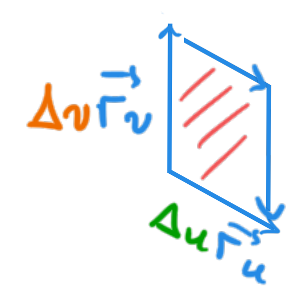
The area of R_{ij} : $A(R_{ij}) = \Delta u \, \Delta v$

To approximate the area of S_{ij}, consider again $\vec{r}_u$ and $\vec{r}_v$:



We see that we can approximate the orange edge of S_{ij} by: $\Delta v\, \vec{r}_v$

and the green edge by: $\Delta u\, \vec{r}_u$

The area of the parallelogram they span approximates the area of S_{ij}: 

$\approx$ 

The area of this parallelogram is:

$$\left| \Delta u\, \vec{r}_u \times \Delta v\, \vec{r}_v \right| = \left| \vec{r}_u \times \vec{r}_v \right| \Delta u\, \Delta v$$

We sum these up and take a limit of finer and finer grids to arrive at our formula for surface area:

$$\boxed{A(S) = \iint_D \left| \vec{r}_u \times \vec{r}_v \right| \, du\, dv}$$

Example:

16.(7 pts.) Which integral gives the surface area of the surface S parameterized by $\mathbf{r}(u,v) = \langle u^2 \cos v, u^2 \sin v, v \rangle$, where $0 \leq u \leq 1, 0 \leq v \leq \pi$.

(a) $\int_0^\pi \int_0^1 2u\sqrt{1+u^4}\, dudv$

(b) $\int_0^\pi \int_0^1 (4u^2 + 4u^6)\, dudv$

(c) $\int_0^\pi \int_0^1 4u^2(\sin v + \cos v) + 4u^4\, dudv$

(d) $\int_0^\pi \int_0^1 2u\sqrt{1+u^2}\, dudv$

(e) $\int_0^\pi \int_0^1 \sqrt{4u^2 \sin^2 v - \cos^2 v + 4u^6}\, dudv$

Special Case: If the surface S is the graph of a function $z = g(x,y)$ for (x,y) in some region $D \subset \mathbb{R}^2$, then:

$$\boxed{A(S) = \iint_D \sqrt{1 + g_x^2 + g_y^2}\, dx\, dy}$$

Why?

Example: Let S be the graph of $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ for $0 \le x \le 1$, $0 \le y \le 1$. Set up the integral for $A(S)$.

$\oint$ Surface Integrals and Flux :

Last time :

↳ Parametric Surfaces

↳ Area of Parametric Surfaces

Goal for today :

↳ Integrate functions over surfaces : $\iint_S f\, dS$

↳ Develop a notion of Orientation.

↳ Develop a notion of Flux.

Examples :

(a) If a surface S has density function δ, then the mass of S, $m(S) = \iint_S \delta\, dS$.

(b) Rate at which water passes through a membrane or porous vessal.

(c) Rate at which heat energy is emitted from a metal object.

Surface Integrals:

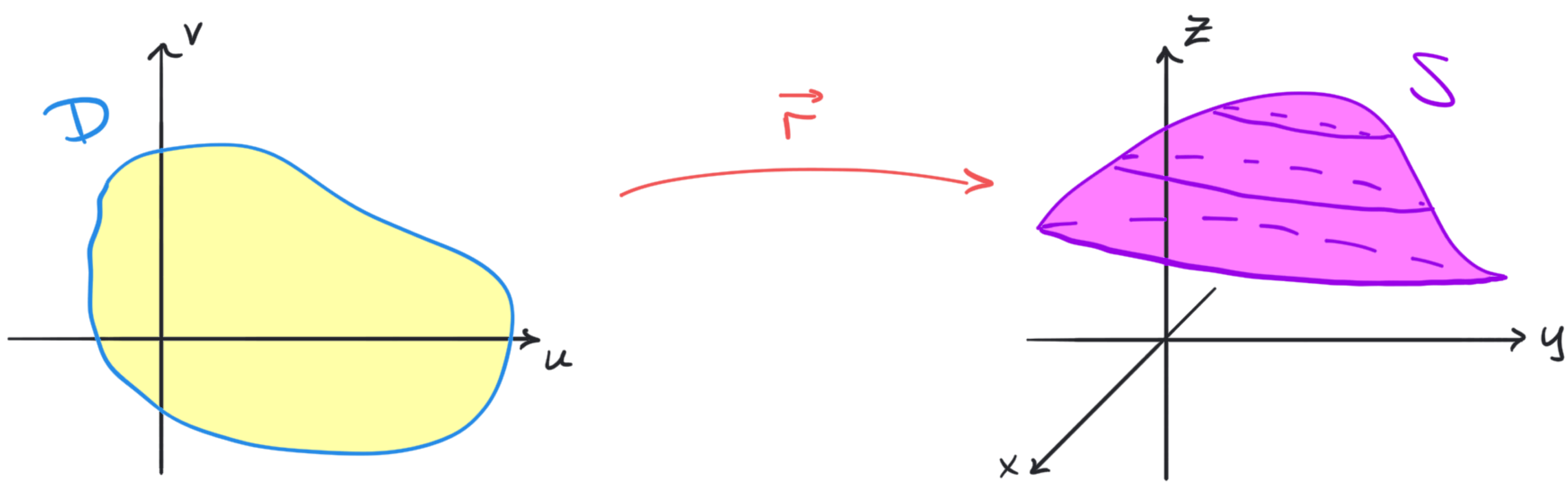
- We can think of the relationship:

  Surface Area $\longleftrightarrow$ Surface Integrals

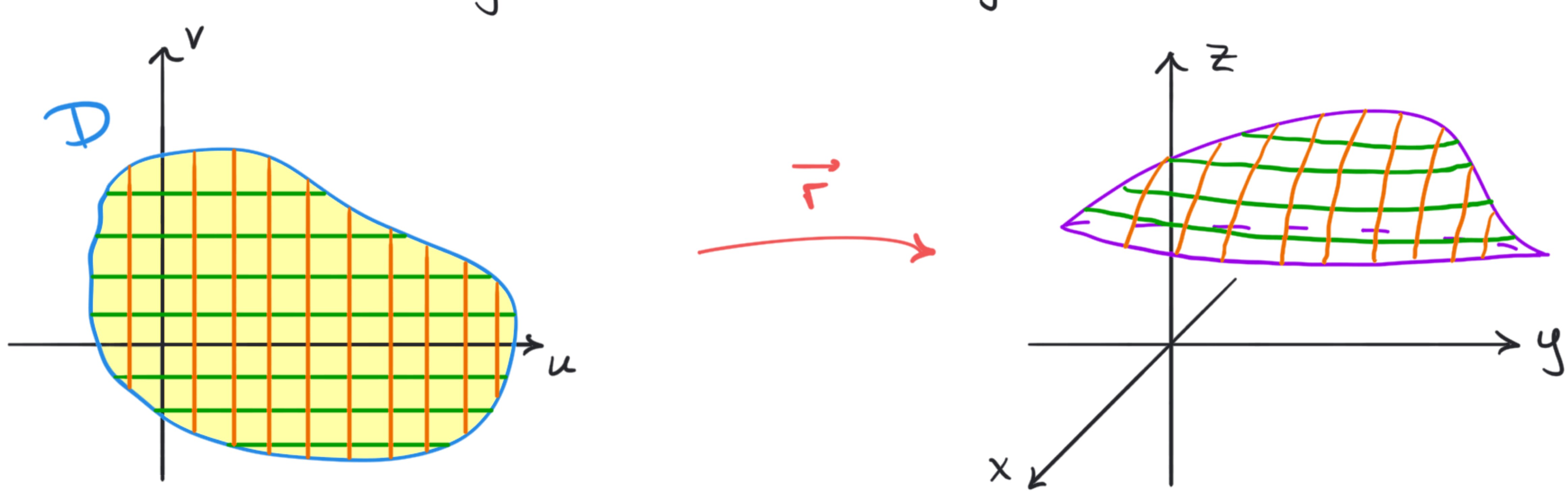
  in a similar way to how we think of the relationship:

  Arc Length $\longleftrightarrow$ Line Integrals

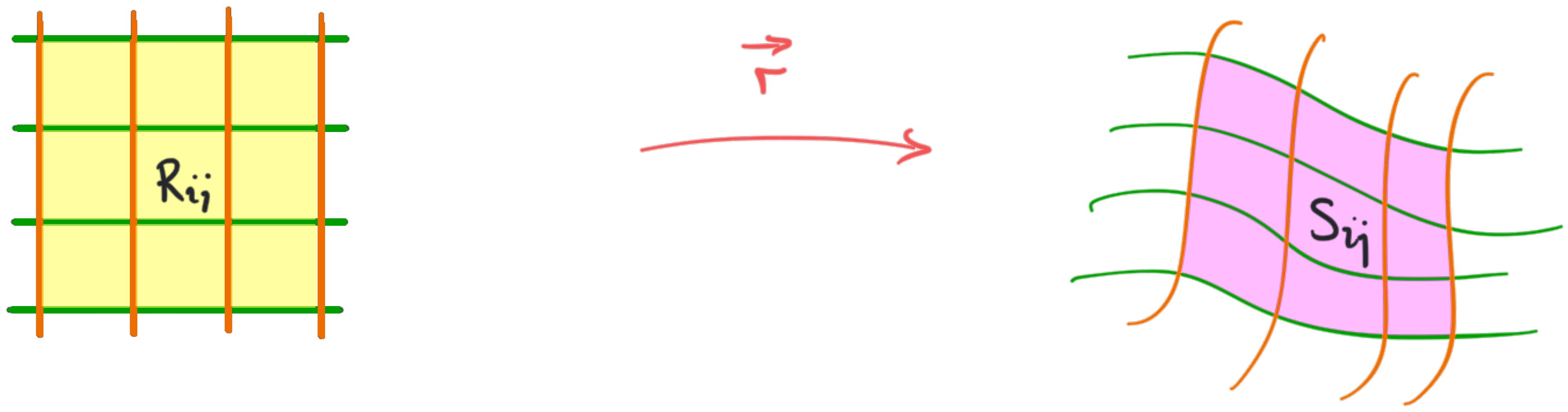
- If S is parametrized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$:



We can once again consider the grid lines:



If we zoom in on this picture:



We saw before that the area of S_{ij}:

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \, \Delta R_{ij} = |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v \qquad (*)$$

So if P_{ij} was a point in S_{ij}, and S had a mass density function δ, then the mass of the square S_{ij} would be:

$$\text{mass}(S_{ij}) \approx \delta(P_{ij}) \, \Delta S_{ij}$$

Doing this for each square:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \text{mass}(S_{ij}) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, \Delta S_{ij}$$

Using $(*)$:

$$\text{mass}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \delta(P_{ij}) \, |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v$$

Taking finer and finer grids :

$$\boxed{\begin{aligned} \text{mass}(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \delta(P_{ij}) \left|\vec{r}_u \times \vec{r}_v\right| \Delta u \Delta v \\ &= \iint_D \delta(\vec{r}(u,v)) \left|\vec{r}_u \times \vec{r}_v\right| dA \end{aligned}}$$

- In general :

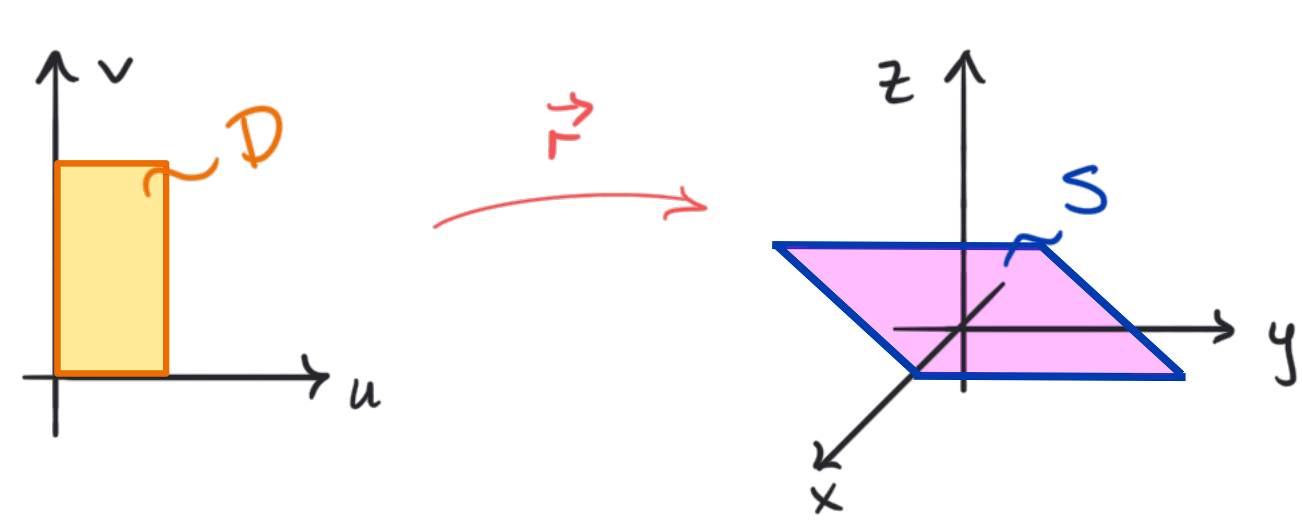
$$\boxed{\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \left|\vec{r}_u \times \vec{r}_v\right| dA}$$

Example : Compute the mass of a sheet of metal (parallelogram), parametrised by :

$\vec{r}(u,v) = (u+1, -u+v, u)$, $0 \leq u \leq 1$, $0 \leq v \leq 2$.

with mass density $\delta(x,y,z) = z^2$.

Solution :



Remark: We can develop center of mass formulas for a surface S with density function δ:

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$, where:

$$\bar{x} = \frac{1}{m} \iint_S x\, \delta(x,y,z)\, dS$$

$$\bar{y} = \frac{1}{m} \iint_S y\, \delta(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m} \iint_S z\, \delta(x,y,z)\, dS$$

Special Case: If S is the graph of a function: $z = g(x,y)$, $\vec{r}(x,y) = (x, y, g(x,y))$, then, as before:

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

So we have:

$$\iint_S f\, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$

Example: Let S be the surface given by $z = y - x^2$ above the region $D = \{(x,y) ; 0 \leq x \leq 1 , 0 \leq y \leq 3\}$.

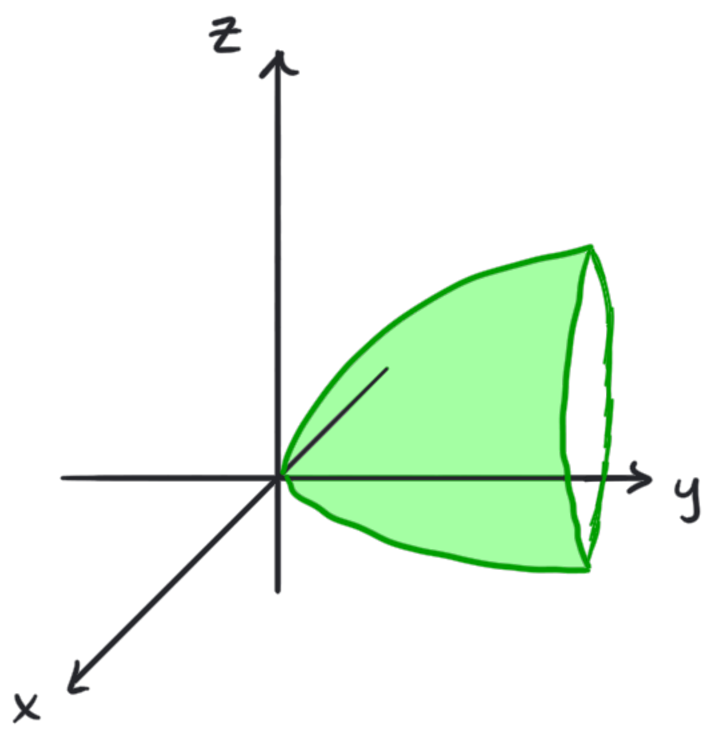
Let $f(x,y,z) = x^2 + x - y + z$.

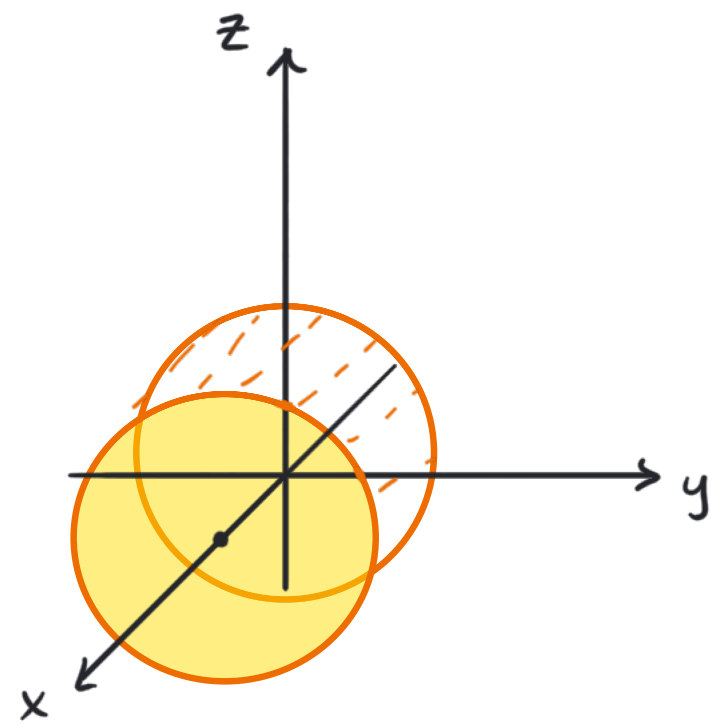
Compute $\displaystyle\iint_S f \, dS$.

Solution:

$$\iint_S f \, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dx\,dy$$

Remark: Some surfaces can be graphs of functions $y = h(x,z)$ or $x = j(y,z)$.





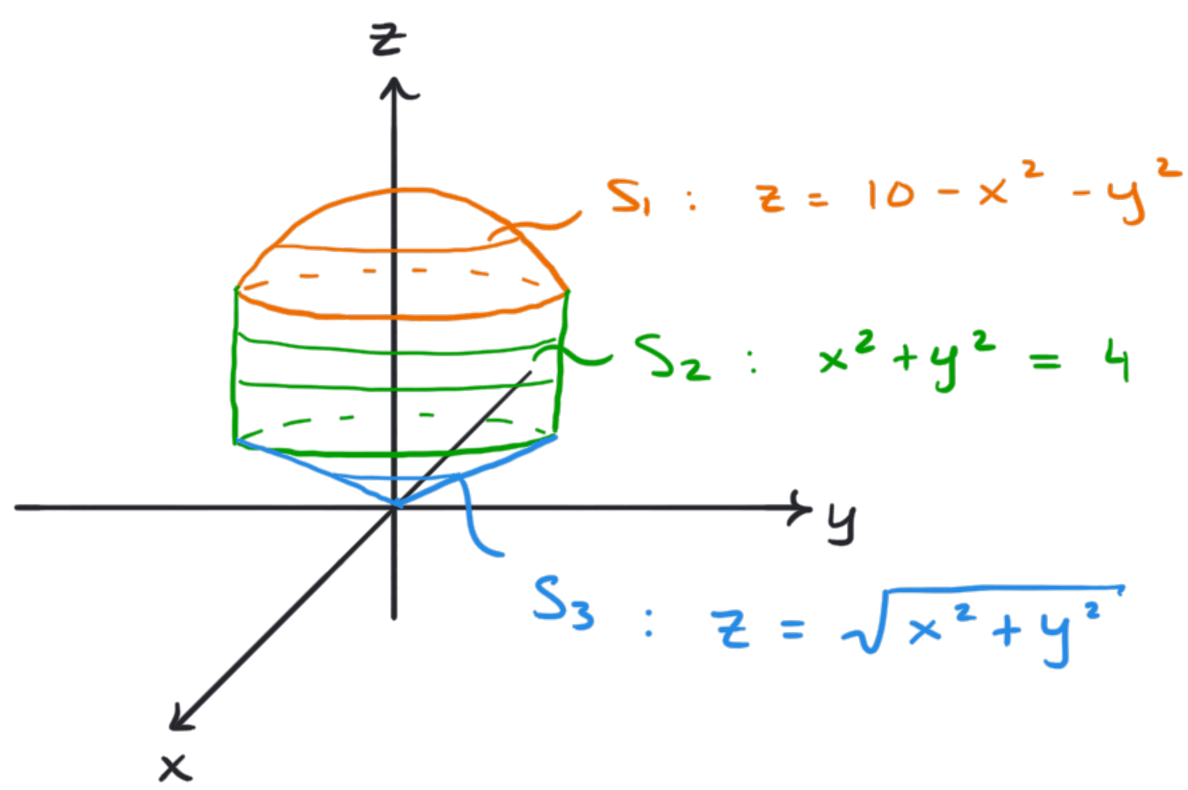
We have analogous formulas:

$$\iint_S f \, dS = \iint_D f(x, h(x,z), z) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2} \, dx\,dz$$

$$\iint_S f \, dS = \iint_D f(j(y,z), y, z) \sqrt{1 + \left(\frac{\partial j}{\partial y}\right)^2 + \left(\frac{\partial j}{\partial z}\right)^2} \, dy\,dz$$

• We say that S is a piecewise-smooth surface if it is a finite union of smooth surfaces $S_1, \dots, S_n$ that are joined together along their boundaries:

E.g.:

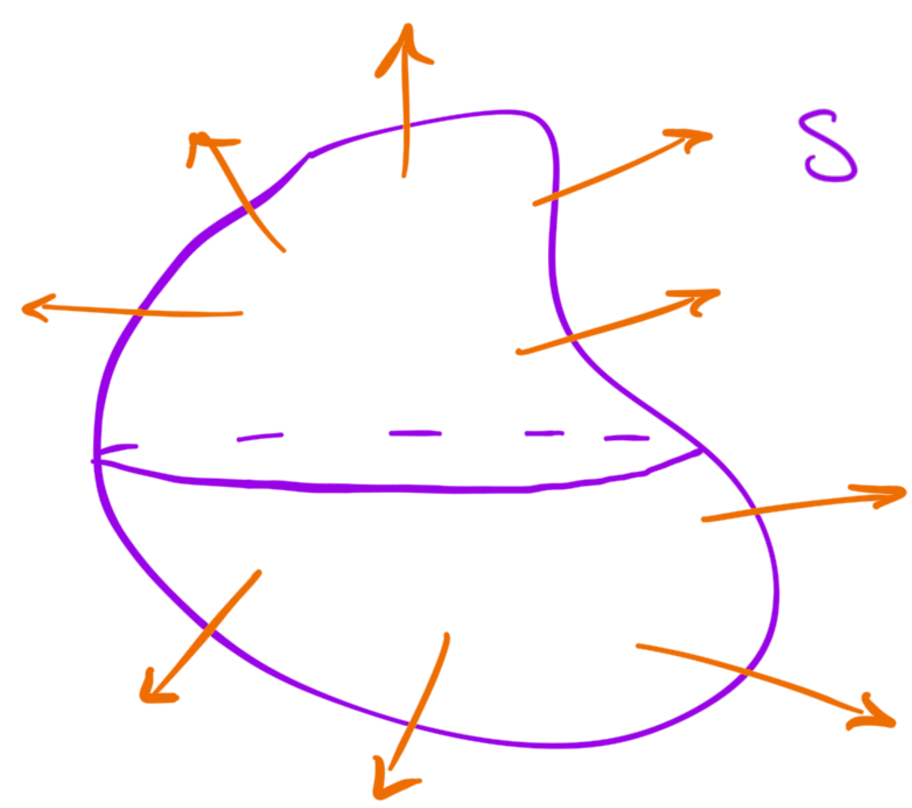


Then:

$$\boxed{\iint_S f\, dS = \sum_{i=1}^{n} \iint_{S_i} f\, dS}$$

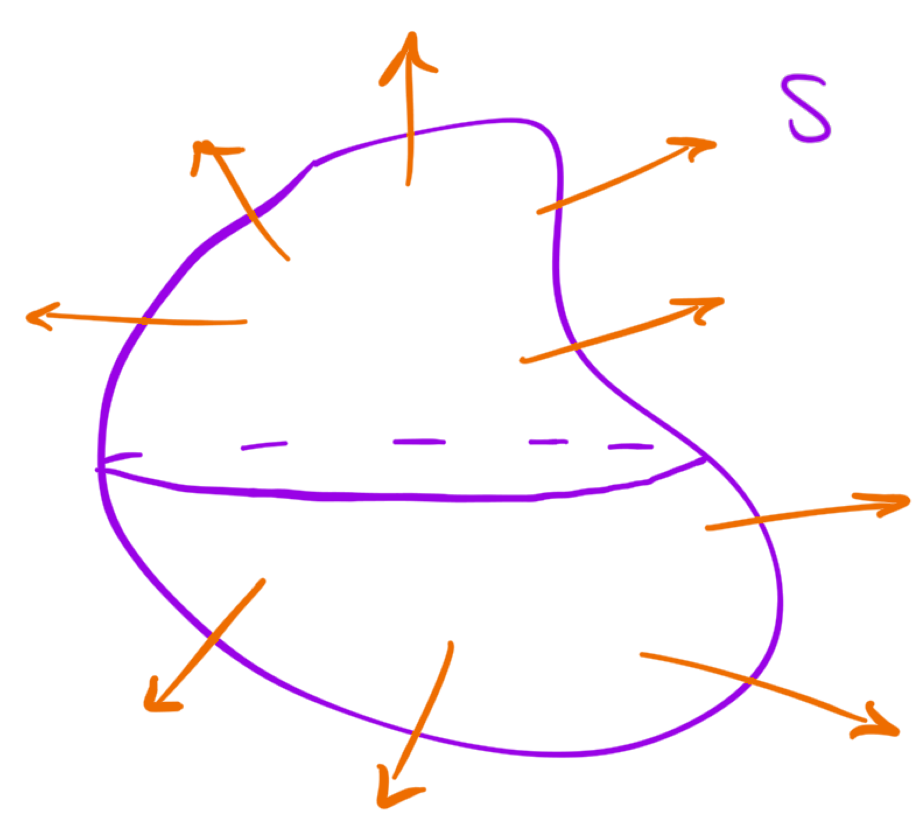
Orientation:

Motivation: Say I have a metal object, S, which is emitting heat energy:



If we computed this flux, should it be a positive or negative quantity?

Now say I have a metal object, S, which is losing heat energy:



If we computed this flux, should it be a positive or negative quantity?

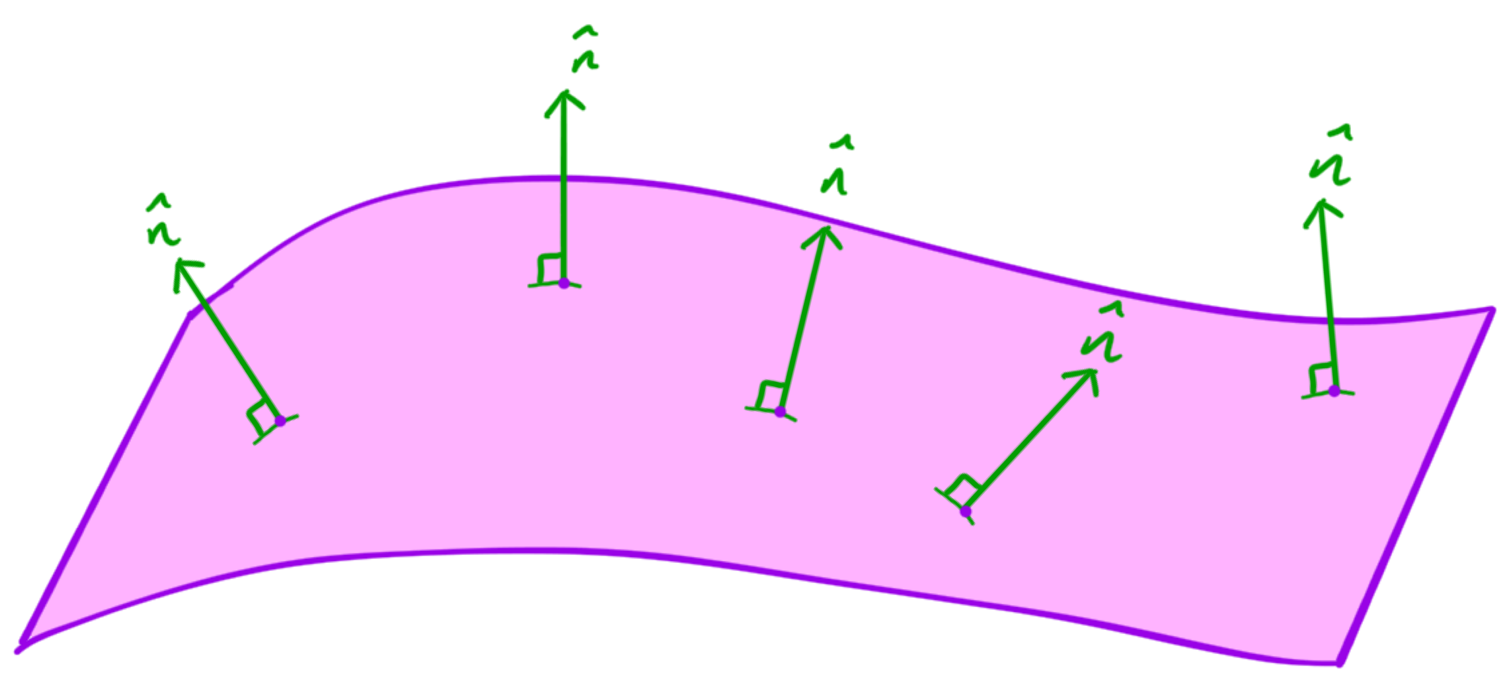
Conclusion:

There is no overall "should".

You have to make a choice.

You have to make a choice of Orientation.

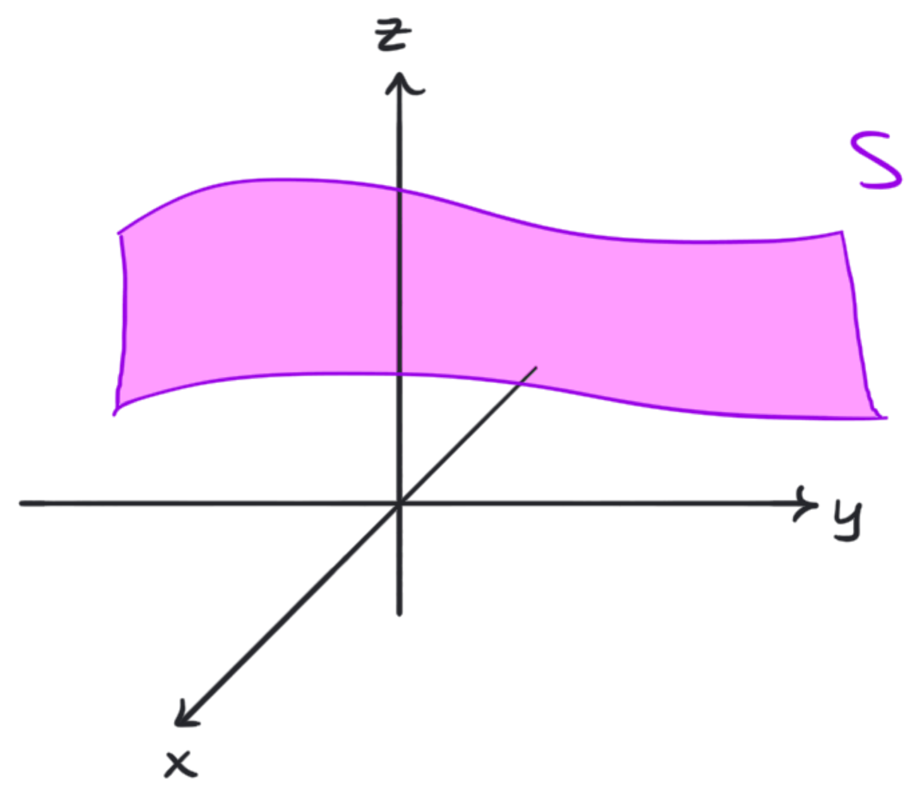
So orientation is technically a choice of continuously varying unit normal vector: $\hat{n}$.



We say a surface S, equipped with an orientation $\hat{n}$ is an oriented surface.

Special Case :

If S is the graph of a function : $z = g(x,y)$



We can think of "upward" or "downward" orientation :

We can find an explicit formula for the upward pointing unit normal to this graph :

$$\boxed{\hat{n} = \frac{\left(-\left(\frac{\partial g}{\partial x}\right), -\left(\frac{\partial g}{\partial y}\right), 1 \right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}}$$

Exercise : Find the upward pointing unit normal to the surface given by $z = x^2 + y^2$.

- If S is a parametric surface represented by $\vec{r}(u,v)$, then:

$$\boxed{\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

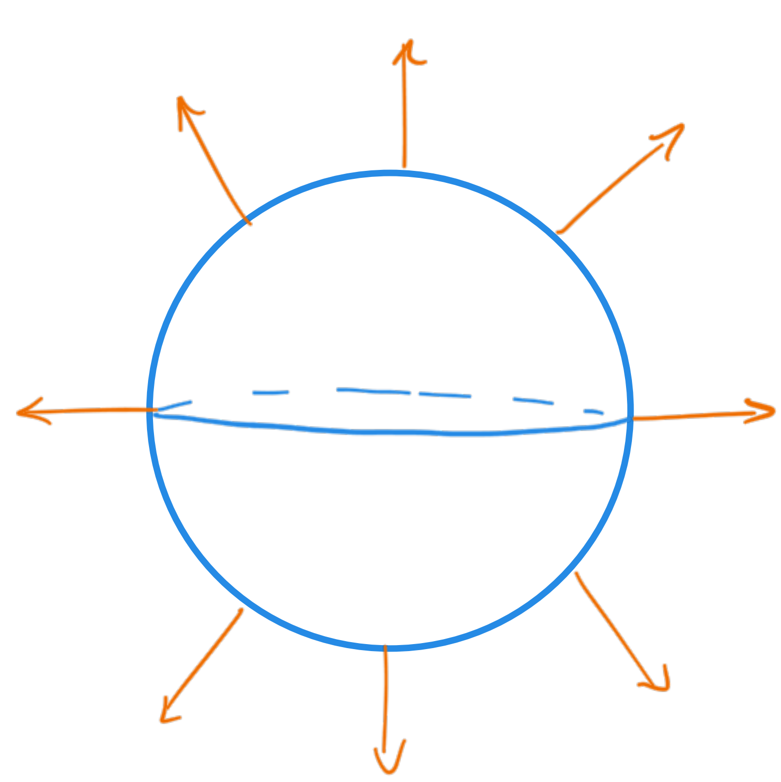
This may be upward or downward – you have to check the sign of the z-component!

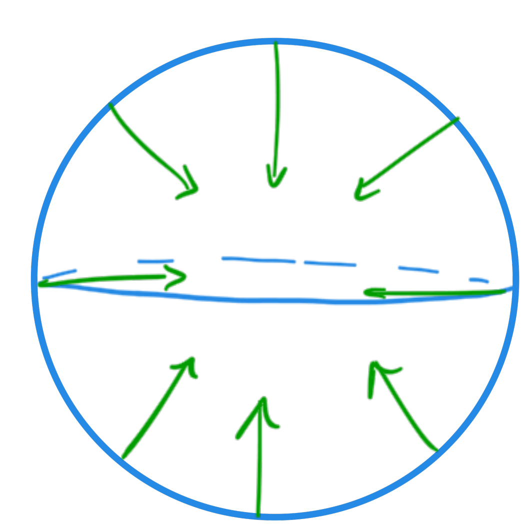
is a unit normal vector.

Remark: The opposite orientation is given by $-\hat{n}$.

Exercise: Find a unit normal to the surface parametrized by $\vec{r}(u,v) = (u+1, -u+v, u)$.

- For a closed surface we can define "outward" and "inward" pointing unit normals:

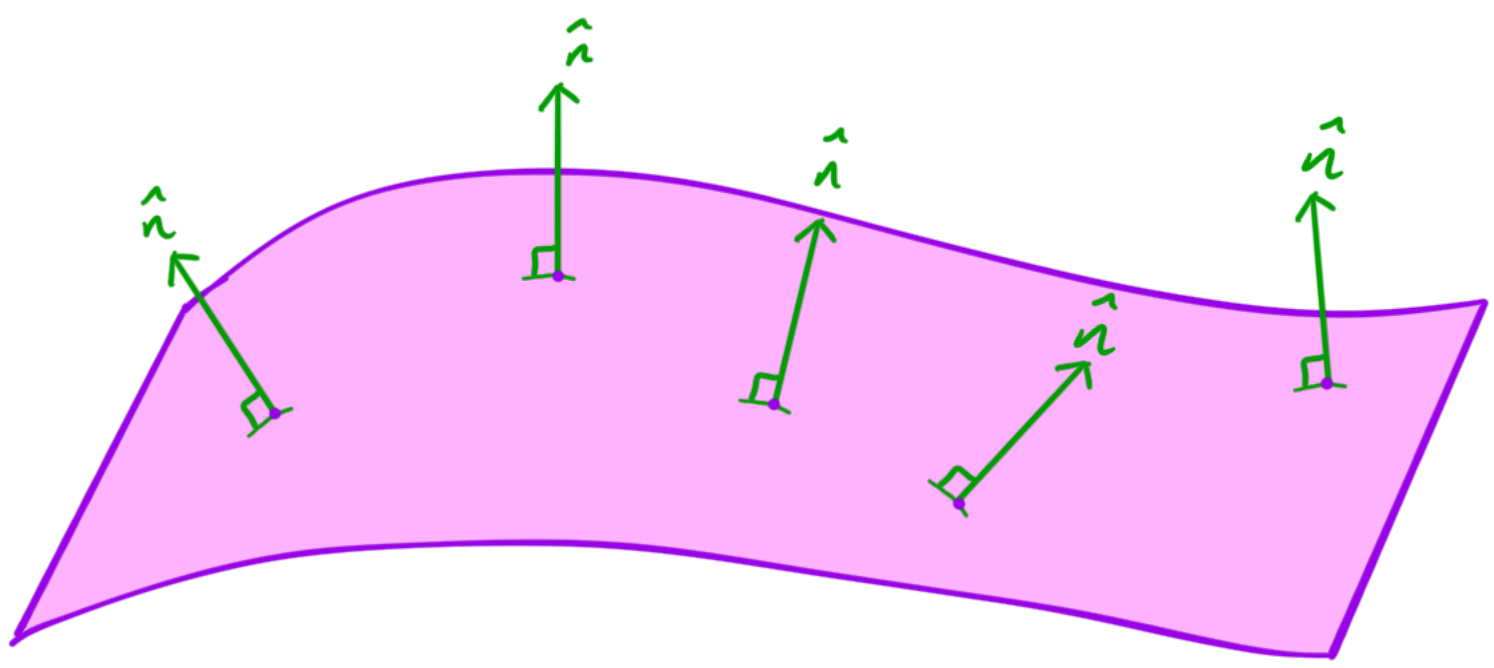




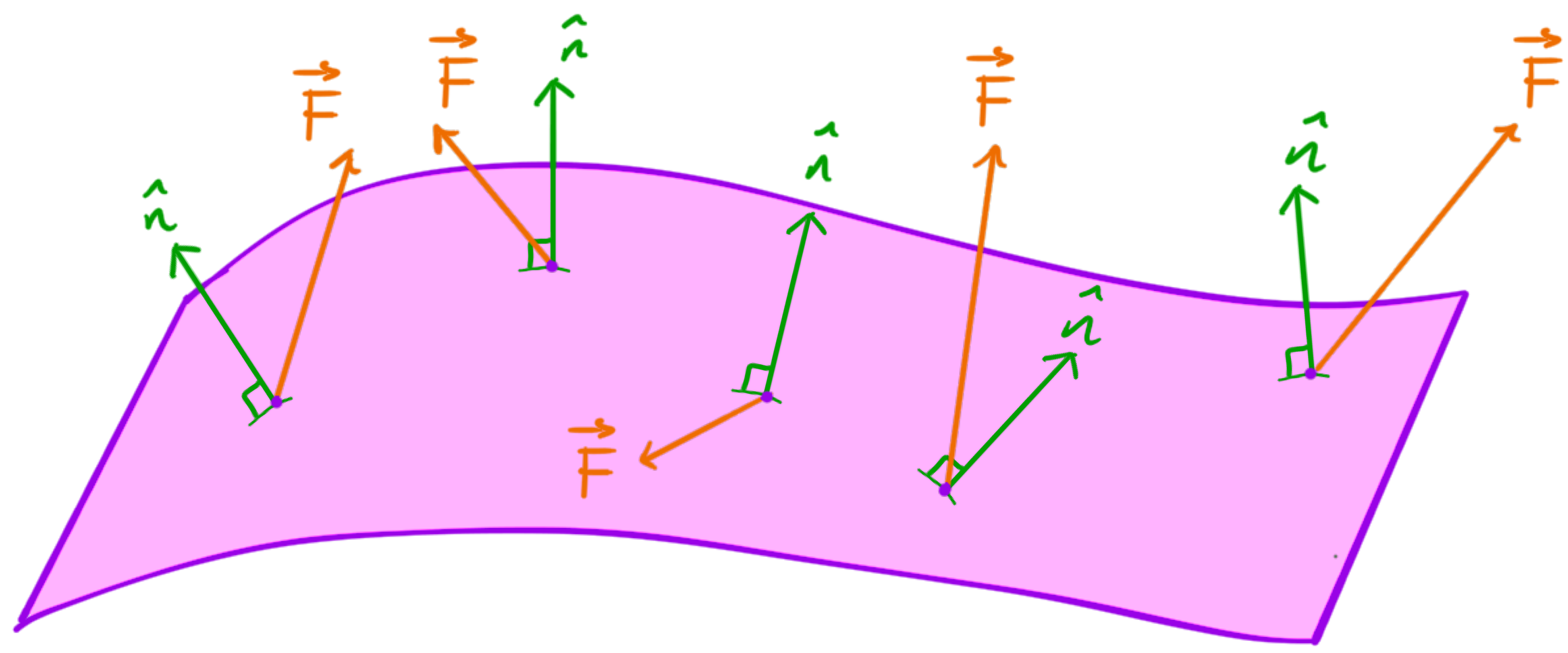
Example: The outward pointing unit normal to a sphere of radius R is $\hat{n} = \frac{1}{R} \langle x, y, z \rangle$.

Surface Integrals of Vector Fields:

Suppose we have an oriented surface S with unit normal $\hat{n}$.

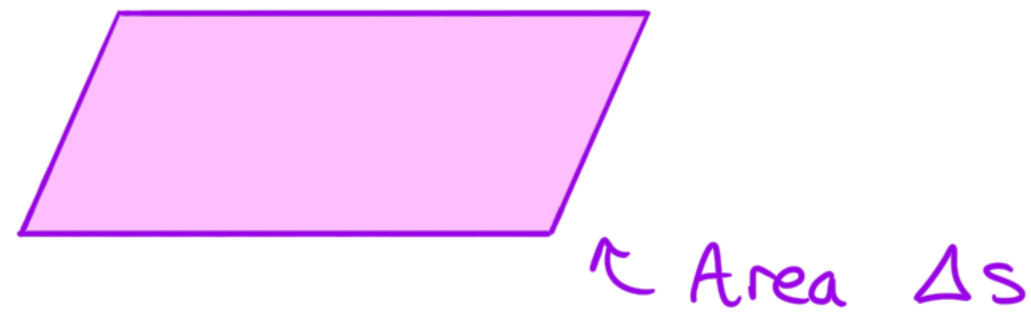


Let's say we have a vector field $\vec{F}$ on S:

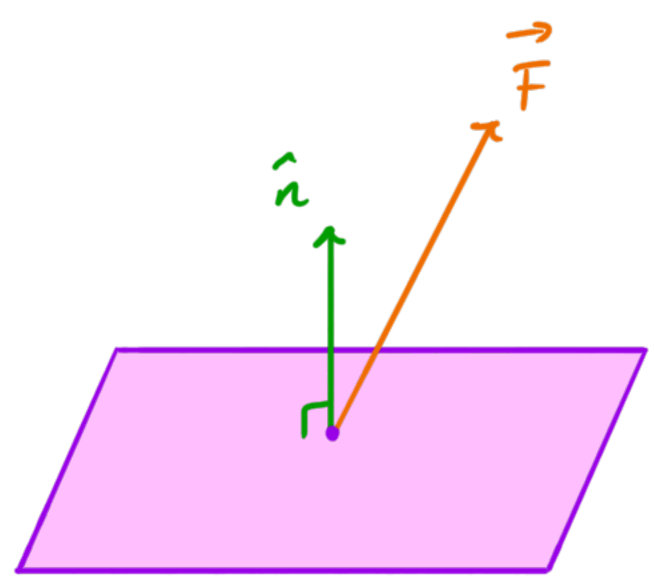


Recall: Our intuition told us that Flux should capture how much $\vec{F}$ is "flowing through" S.

Let's zoom in on a small patch of area ΔS:

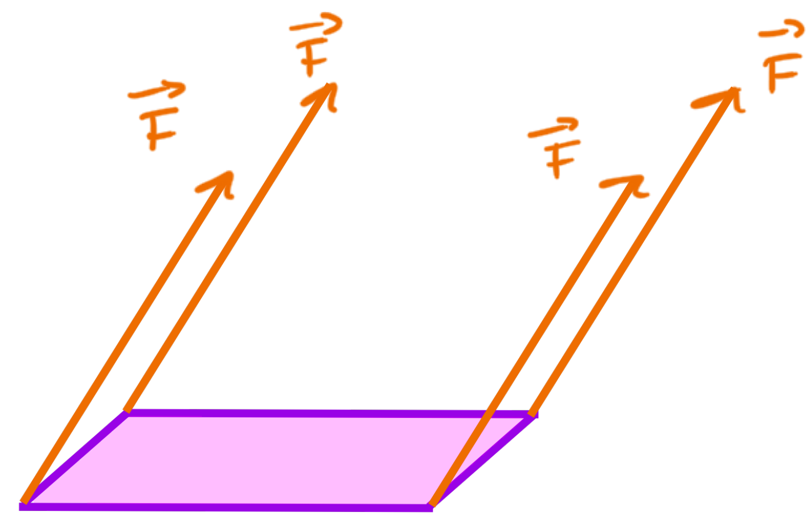


On this small patch, as our vector fields $\vec{F}$ and $\hat{n}$ are "well behaved", they should look pretty much constant on this small patch:

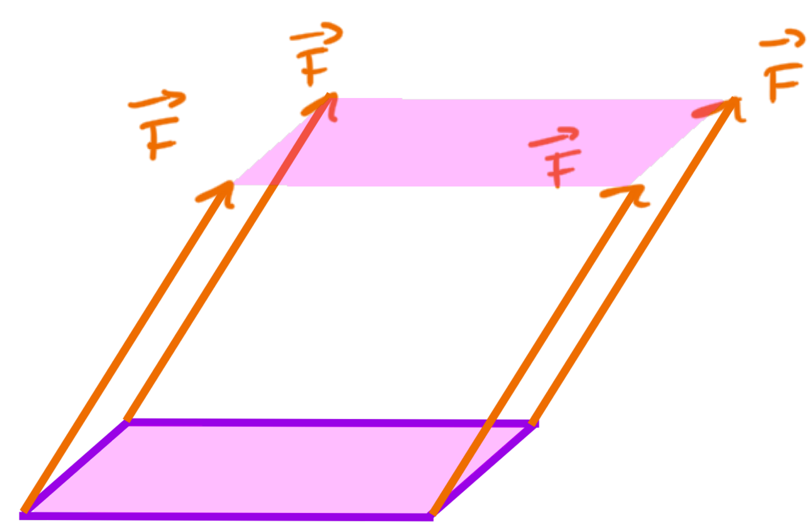


If we think of $\vec{F}$ as the rate at which water is flowing across the points in this patch, what volume of water will flow through per unit time?

If we think of $\vec{F}$ as a velocity field for some fluid, and assume that on this patch it's moving at v meters per second:



What is the volume of this "box"?



Definition: If $\vec{F}$ is a continuous vector field on an oriented surface S with unit normal $\hat{n}$, then the surface integral of $\vec{F}$ over S or the Flux of $\vec{F}$ across S is given by:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS}$$

Example:

Let S be the unit sphere : $x^2 + y^2 + z^2 = 1$.

Let $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$

Solution:

If S is given by $\vec{r}(u,v)$ then :

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS$$

$$= \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v|\, dA$$

Hence we have :

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv}$$

Example :

20.(7 pts.) Find the flux of the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$$

over a surface with **downward** orientation, whose parametric equation is given by

$$\mathbf{r}(u, v) = 2u\mathbf{i} + 2v\mathbf{j} + (5 - u^2 - v^2)\mathbf{k}$$

with $u^2 + v^2 \leq 1$.

(a) $-\dfrac{56\pi}{3}$ (b) $\dfrac{112\pi}{3}$ (c) -18π (d) -36π (e) 9π

Special Case :

If S is the graph of a function : $z = g(x,y)$:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

Why?

Example: Let S be the surface given by $z = x^2 + y^2$ above the region D: $x^2 + y^2 \leq 1$.

Let $F(x, y, z) = \langle -x, -y, x^2 + y^2 \rangle$.

Compute $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$

Solution:

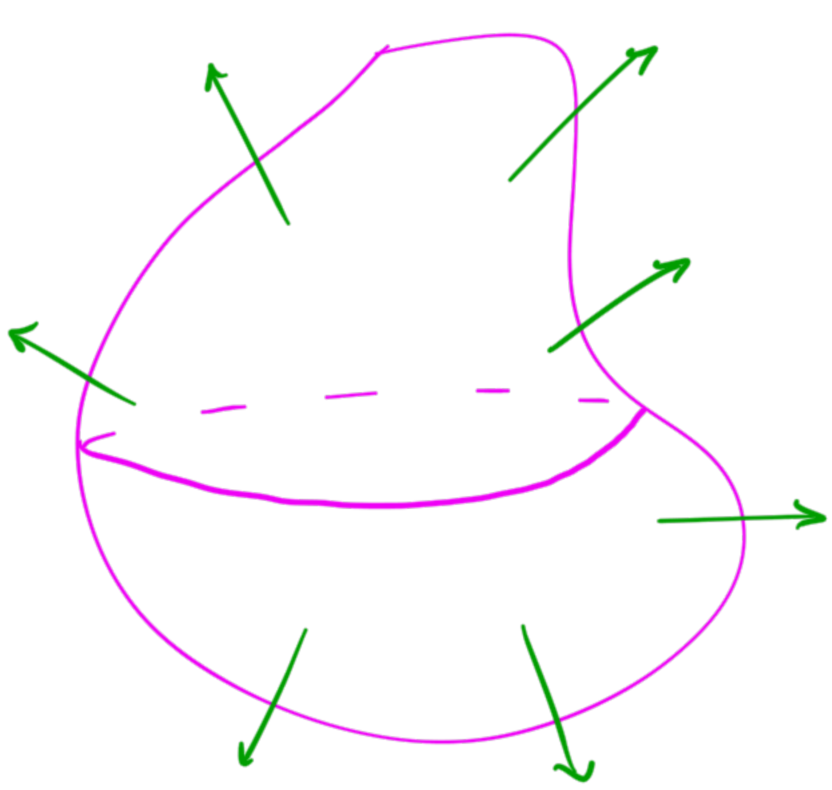
$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA}$$

The Divergence Theorem:

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation.

Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\, dV}$$



S = Shell (Hollow)

E = Solid (Filled In)

Intuition:

Example:

4.(7 pts.) Use the Divergence theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is calculate the flux of $\mathbf{F}$ across S.

$$\mathbf{F} = \langle e^y, zy, xy^2 \rangle,$$

S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2$ and $z = 4$ with outward orientation.

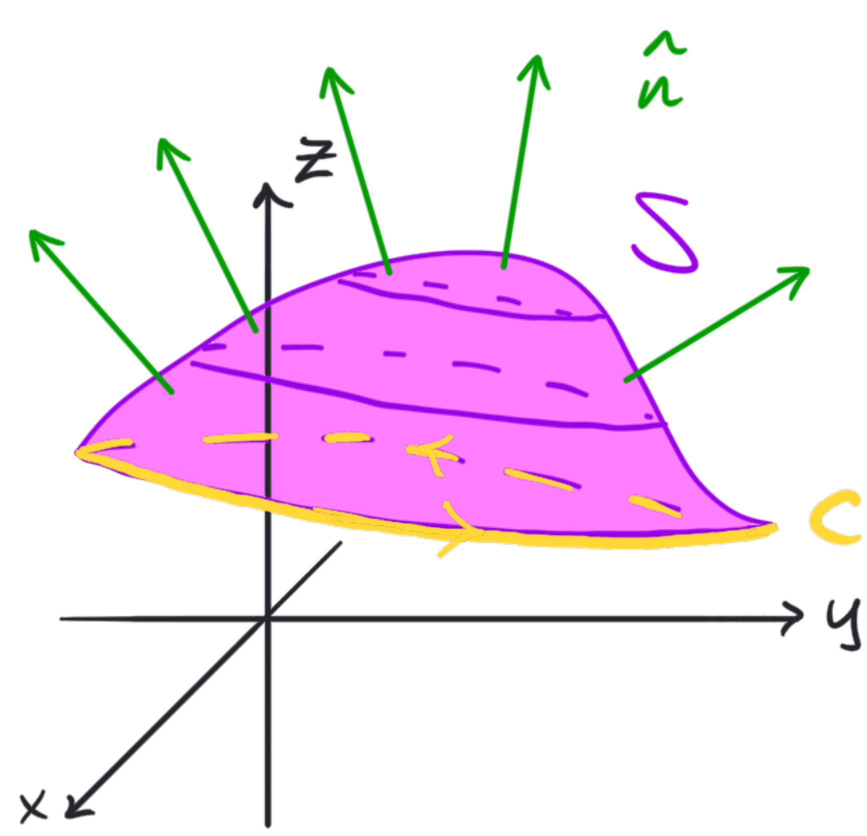
(a) $\dfrac{3\pi}{2}$ (b) 6π (c) 4π (d) 2π (e) π

Stokes' Theorem:

Let S be an oriented piecewise smooth surface that is bounded by a simple closed piecewise smooth curve C with positive orientation.

Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on open region in $\mathbb{R}^3$ that contains S. Then:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F})\, d\vec{S}}$$



Intuition:

Remark: If S is a closed surface (no boundary curve), what does Stoke's Theorem say?

Question: Can we relate Stoke's Theorem to Green's Theorem?

Example:

8.(7 pts.) Let C be the rectangle in the $z = 1$ plane with verticies $(0, 0, 1)$, $(1, 0, 1)$,$(1, 3, 1)$, and $(0, 3, 1)$ oriented counterclockwise when viewed from above. Use Stokes' Theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + \frac{x}{3}\mathbf{j} + xy\mathbf{k}$.

(a) 1 (b) 9/2 (c) 0 (d) 6 (e) $-3/2$

Name: Harder Problems Date: /12/2019

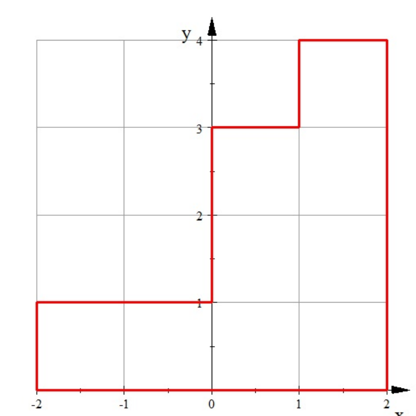
Main Questions

4. Let S be the portion of the graph $z = 4 - 2x^2 - 3y^2$ that lies over the region in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = 1$. Write the integral that computes $\iint_S (x^2 + y^2 + z)\, \mathrm{d}S$.

5. Compute $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by
$$x = 2u, \quad y = 2v, \quad z = 5 - u^2 - v^2,$$
where $u^2 + v^2 \leq 1$. S has downward orientation.

6. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.

7. Evaluate $\int_C (x^4 e^{5y} - 3y)\mathrm{d}x + (4x + x^5 e^{5y})\mathrm{d}y$ where C is the curve below and C is oriented in clockwise direction.



8. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

9. Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, z, 1 \rangle$ across the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with orientation away from the origin.

10. Let S be the boundary surface of the region bounded by $z = \sqrt{36 - x^2 - y^2}$ and $z = 0$, with outward orientation. Find $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$.

11. Let C be the boundary curve of the part of the plane $x + y + 2z = 2$ in the first octant. C has counterclockwise orientation when viewing from above. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{\sin x^2}, z, 3y \rangle$.

12. Evaluate
$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\, dz$$
where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$ with counterclockwise direction when viewed from above. (*Hint*: the curve C lies on the surface $z = 2xy$.)