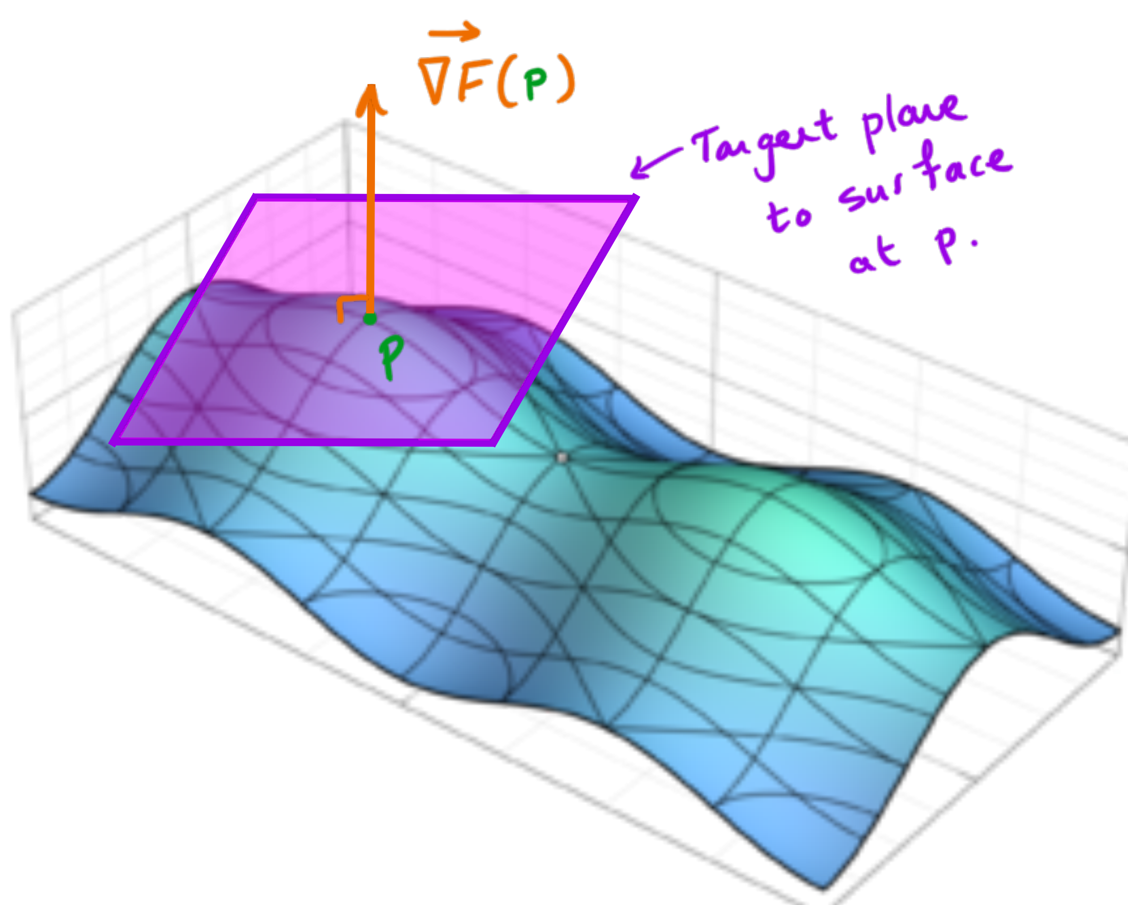
Problem Session 1 :

Definition : $\vec{\nabla} f(x,y,z) := \left(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \right)$

Theory : If a surface S is given by $F(x,y,z) = C$, and p is a point on the surface, then $\vec{\nabla} F(p)$ is normal to the tangent plane to the surface at p.



Special Case : If S is the graph of a function : $z = g(x,y)$, define $F(x,y,z) = z - g(x,y)$. So S is now given by $F(x,y,z) = 0$, and hence for any point p on S $\nabla \vec{F}(p) = (-g_x, -g_y, 1)$ is normal to the tangent plane to the surface at p.

Theory : $\vec{\nabla} F(p)$ points in the direction which will cause the greatest rate of change in the outputs of F "near" p.

Formula: If $x = g(t)$, $y = h(t)$ and $z(t) = f(g(t), h(t))$:

$$\boxed{\frac{dz}{dt}(t) = f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)}$$

Formula: If $x = g(s,t)$, $y = h(s,t)$ and $z(s,t) = f(g(s,t), h(s,t))$:

$$\frac{\partial z}{\partial t}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t))\, \frac{\partial h}{\partial t}(s,t)$$

&

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}(g(s,t), h(s,t)) \cdot \frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}(g(s,t), h(s,t))\, \frac{\partial h}{\partial s}(s,t)$$

Formula:

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla_{\vec{u}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \vec{\nabla} f(x,y,z) \cdot \hat{u}}$$

Method: If two surfaces $S_1 : F(x,y,z) = C_1$ & $S_2 : G(x,y,z) = C_2$ intersect at a point p, a direction for the tangent line to the curve of intersection at p can be given by $\boxed{\vec{v} = \vec{\nabla} F(p) \times \vec{\nabla} G(p)}$.

Formulas: If a surface is given by $F(x,y,z) = C$:

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}} \quad \& \quad \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

Problem Session 2 :

Theory: If a function has a local maximum or a local minimum at a point p, then $\vec{\nabla} f(p) = 0$.

Definition: A point p is called a critical point of f if $\vec{\nabla} f(p) = \vec{0}$.

NB: Critical points need not be local max. or mins.

Method: Suppose that f is a "nice" function, and that p is a critical point of f: $\vec{\nabla} f(p) = \vec{0}$.

Define

$$\boxed{D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2}$$

, or equivalently:

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{vmatrix}$$

Then:

(i) If $D(P) > 0$ and $f_{xx}(P) > 0$, f has a local min. at p.

(ii) If $D(P) > 0$ and $f_{xx}(P) < 0$, f has a local max. at p.

(iii) If $D(P) < 0$, f has a saddle point at p.

Method: To find the absolute max./min. of a function f on a closed set whose boundary is made up of **straight lines** (e.g. a triangle or square):

Step 1: Find all critical points of f.

Step 2: Evaluate f at these points, ignoring ones that are outside of the region.

Step 3: Evaluate f at each "corner".

Step 4: Pick a side of the boundary and write it as $y = mx + c$, for x values in some interval (if the side is vertical, write it as $x = k$).

Step 5: Define $h(x) = f(x, mx+c)$ (or $g(y) = f(k, y)$).

Step 6: Find all critical points of h: $h'(x) = 0$ in the interval for x (or $\frac{dg}{dy} = 0$).

Step 7: Evaluate h at each critical point (or g).

Step 8: Do this for each side.

Step 9: Pick out the absolute max. and absolute min.

Method: To find the maximum and minimum values of a function f, subject to a constraint $g(x,y,z) = K$:

Step 1: Find all (x,y,z) such that there is a $\lambda \in \mathbb{R}$;

$$\boxed{\begin{aligned} \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \\ \text{and} \quad g(x,y,z) &= K \end{aligned}}$$

Step 2: Evaluate f at these points and pick out the maximum and minimum values.

Method: To find the maximum and minimum values of a function f, subject to two constraints: $g(x,y,z) = K$ and $h(x,y,z) = \ell$:

Step 1: Find all (x,y,z) such that there is a λ & $\mu \in \mathbb{R}$;

$$\boxed{\begin{aligned} &\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) + \mu \vec{\nabla} h(x,y,z) \\ &\& \\ &g(x,y,z) = K \quad \text{and} \quad h(x,y,z) = \ell \end{aligned}}$$

Step 2: Evaluate f at these points and pick out the max/min values.

Method: To find the absolute max./min. of a function f on a closed set whose boundary is given by a curve $g(x,y,z) = k$ (e.g. an ellipse or a disc):

Step 1: Find all critical points of f.

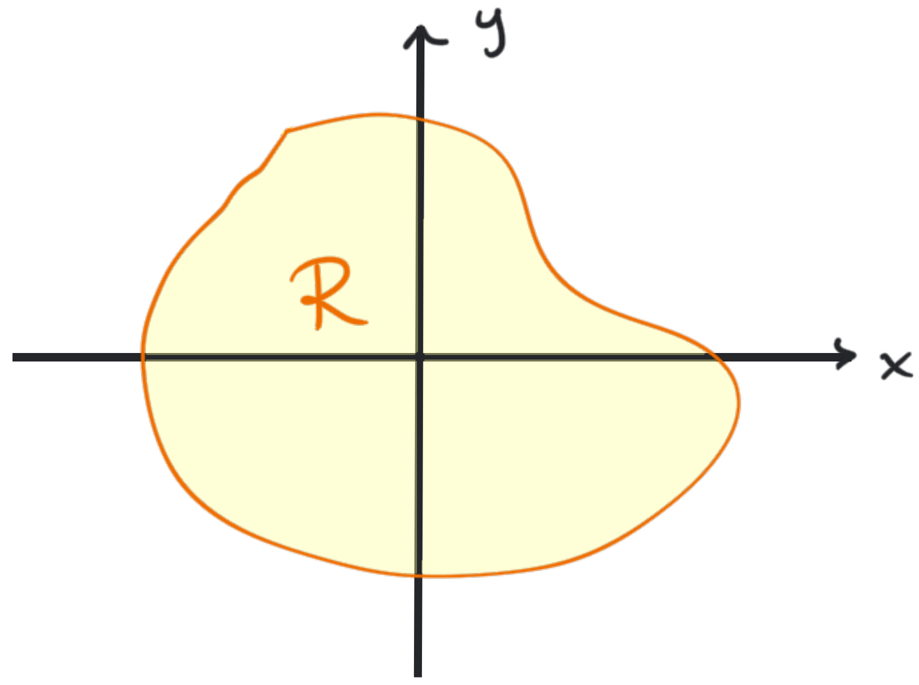
Step 2: Evaluate f at these points, ignoring ones that are outside of the region.

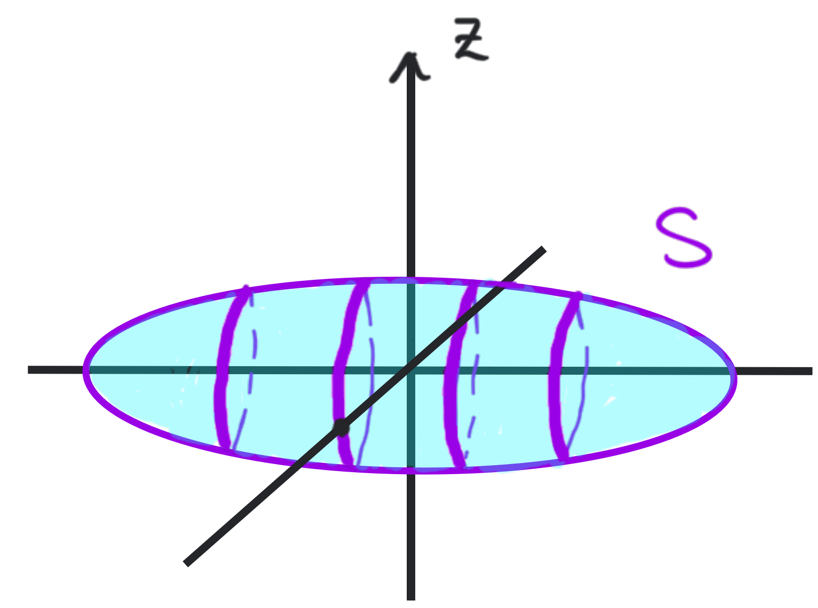
Step 3: Apply the method of Lagrange Multipliers to find the maximum/minimum of f on the boundary $g(x,y,z) = k$ i.e. subject to the constraint $g(x,y,z) = k$.

Step 4: Pick out the maximum and minimum values.

Problem Session 3 :

Theory: If R is a region in $\mathbb{R}^2$ (the "xy-plane"), and S is some solid sitting in $\mathbb{R}^3$:





$$\text{Area}(R) = \iint_R dA \qquad\qquad \text{Volume}(S) = \iiint_S dV$$

- In Cartesian / Retangular coordinates :

$$\boxed{dA = dx\,dy} \qquad\qquad \boxed{dV = dx\,dy\,dz}$$

- In Polar / Cylindrical Coordinates :

$$\boxed{dA = r\,dr\,d\theta} \qquad\qquad \boxed{dV = r\,dz\,dr\,d\theta}$$

Method: It is almost always useful to sketch the region of integration.

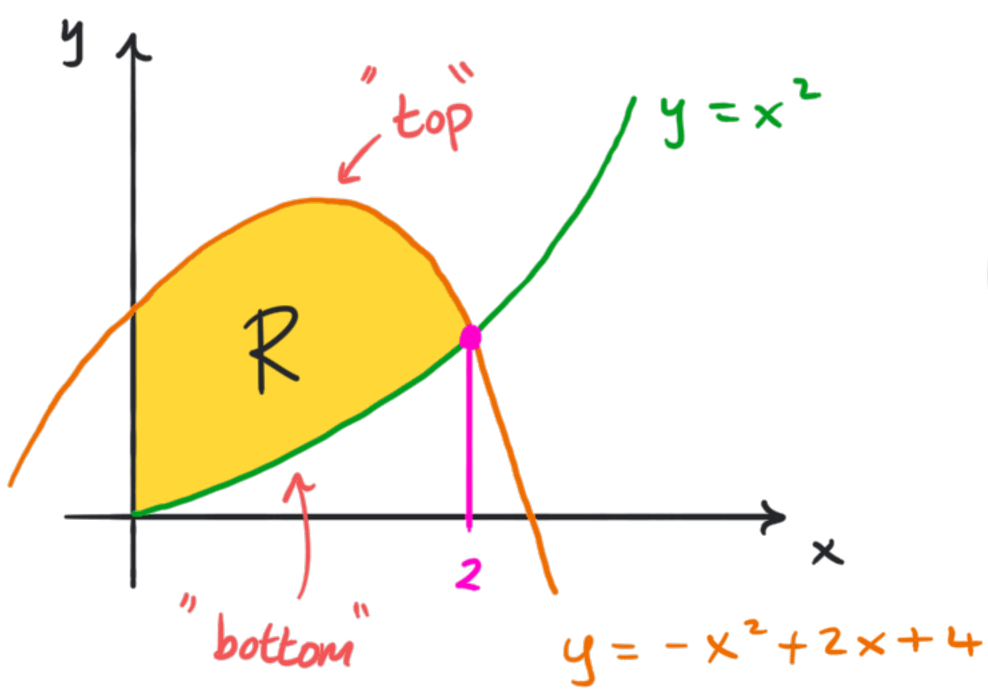
Types of problems:

1) Compute the area of a region $R \subset \mathbb{R}^2$:

Example: Find the area of the region given by:

$$R = \{(x,y) \; ; \; 0 \leq x \leq 2 \; , \; x^2 \leq y \leq x^2 - 2x + 4\}.$$

Solution:



$$\left(\text{Aside: } \begin{aligned} & x^2 = -x^2 + 2x + 4 \\ \Rightarrow \; & 2x^2 - 2x - 4 = 0 \\ \Rightarrow \; & x^2 - x - 2 = 0 \\ \Rightarrow \; & x = 2 \text{ or } x = -1 \end{aligned} \right)$$

"top"

$$\text{Area}(R) = \iint_R dA = \int_0^2 \int_{x^2}^{-x^2+2x+4} dy\,dx = \int_0^2 y \Big|_{x^2}^{-x^2+2x+4} dx$$

"bottom"

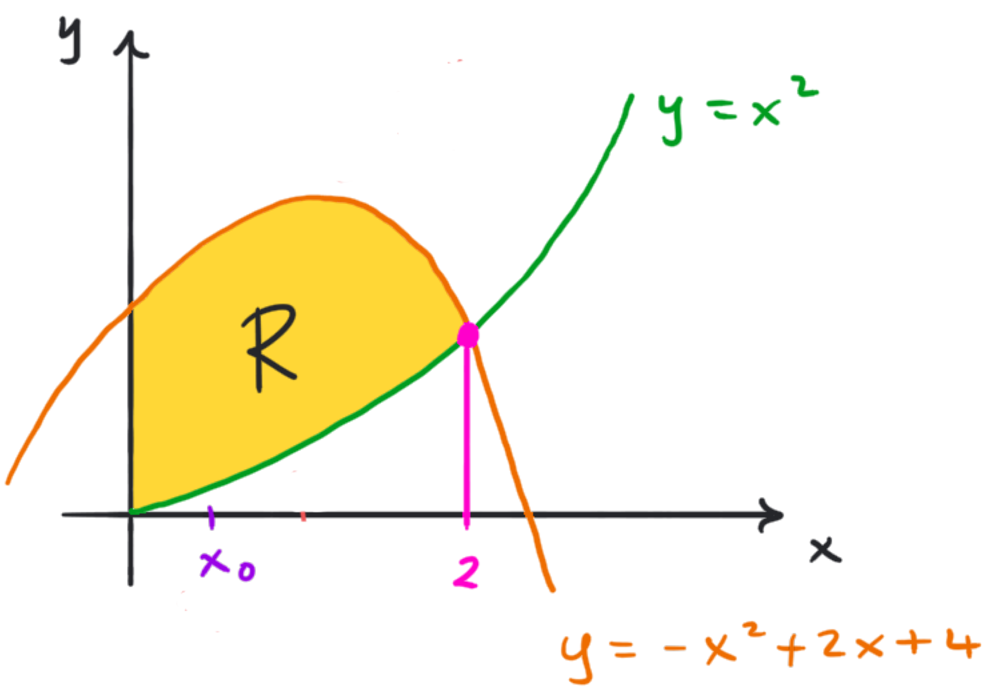
$$= \int_0^2 \left(-x^2 + 2x + 4 - x^2\right) dx = \int_0^2 -2x^2 + 2x + 4 \, dx$$

$$= -\frac{2}{3}x^3 + x^2 + 4x \Big|_0^2 = -\frac{16}{3} + 4 + 8 = \frac{20}{3}$$

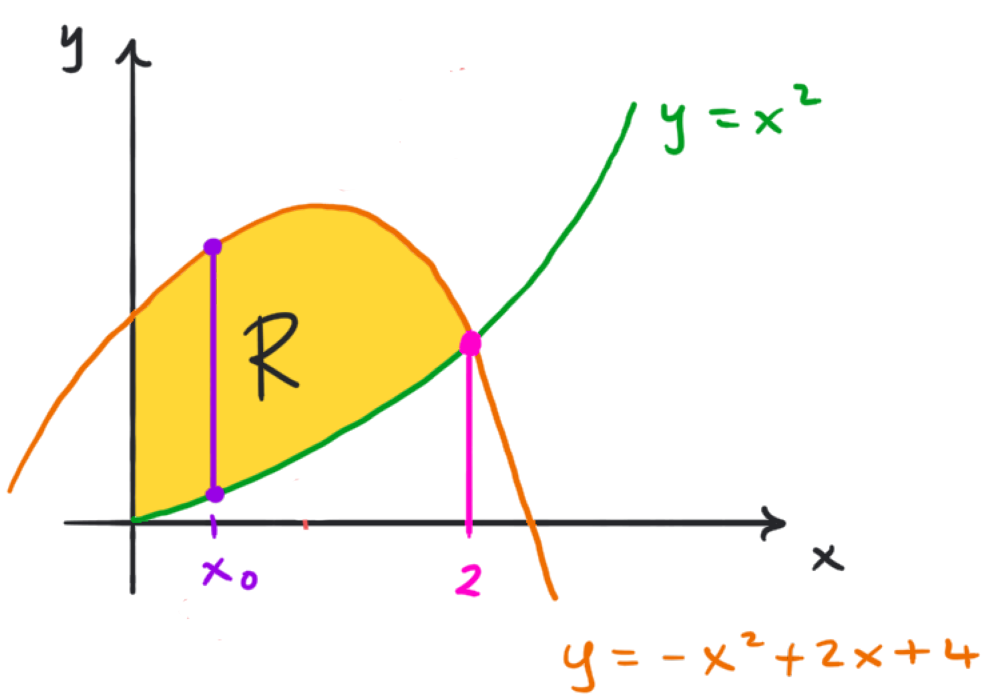
Remark: At this stage : $\int_0^2 \int_{x^2}^{-x^2+2x+4} dy\,dx$, you should read your variables of integration "from the outside in".

i.e. you are handed an $x_0 \in [0,2]$: $\int_{\underline{0}}^{2} \int_{x^2}^{-x^2+2x+4} dy\,d\underline{x}$

outside



and you compute the length of this cord :



which is : $\int_{x_0^2}^{-x_0^2+2x_0+4} dy = y \Big|_{x_0^2}^{-x_0^2+2x_0+4}$

$= (-x_0^2 + 2x_0 + 4) - x_0^2$

$= -2x_0^2 + 2x_0 + 4$

You do this for all $x \in [0,2]$: Lenght of "x-cord" $= -2x^2+2x+4$,

and integrate x over $[0,2]$ to "shade in" the yellow area:

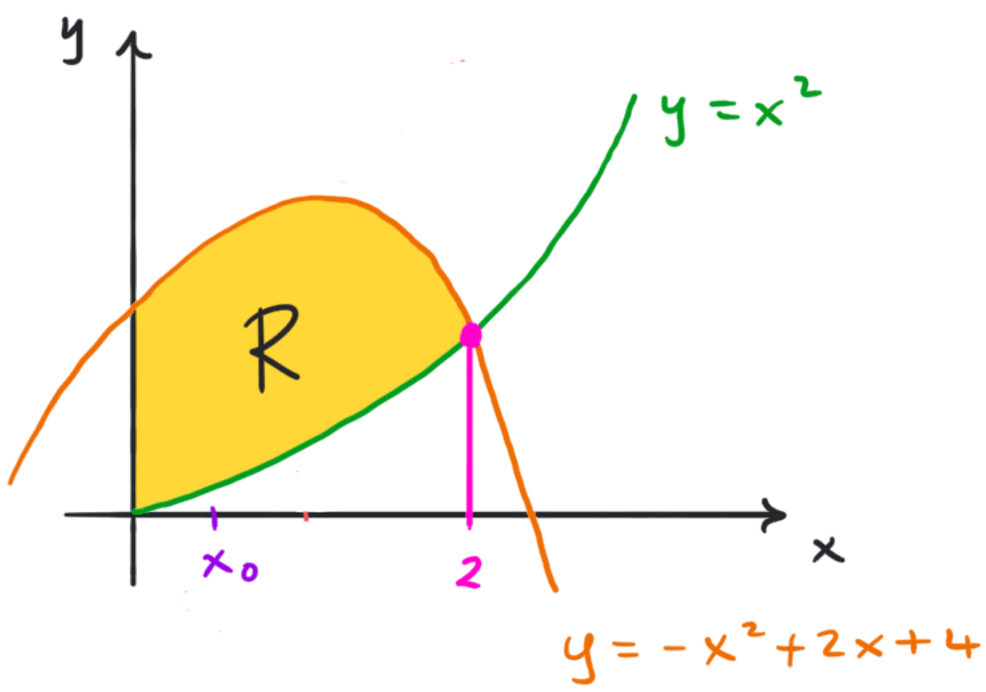
$$\text{Area}(R) = \int_0^2 (-2x^2 + 2x + 4)\,dx = \frac{20}{3}.$$

This way of thinking about it can help you decide your bounds for each of your variables.

2) Integrate a function over a region $R \subset \mathbb{R}^2$:

Example: Let's use the same region as before

for simplicity: $R = \{(x,y) \; ; \; 0 \leq x \leq 2 \; , \; x^2 \leq y \leq -x^2+2x+4\}$



Compute $\iint_R f(x,y)\,dA$

where $f(x,y) = x$

Sol$^{\underline{n}}$: Write bounds as before:

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{-x^2+2x+4} f(x,y)\,dy\,dx = \int_0^2 \int_{x^2}^{-x^2+2x+4} (x)\,dy\,dx$$

$$= \int_0^2 xy \Big|_{x^2}^{-x^2+2x+4} dx$$

$$= \int_0^2 x(-x^2+2x+4) - x(x^2)\,dx$$

$$= \int_0^2 -2x^3 + 2x^2 + 4x\,dx$$

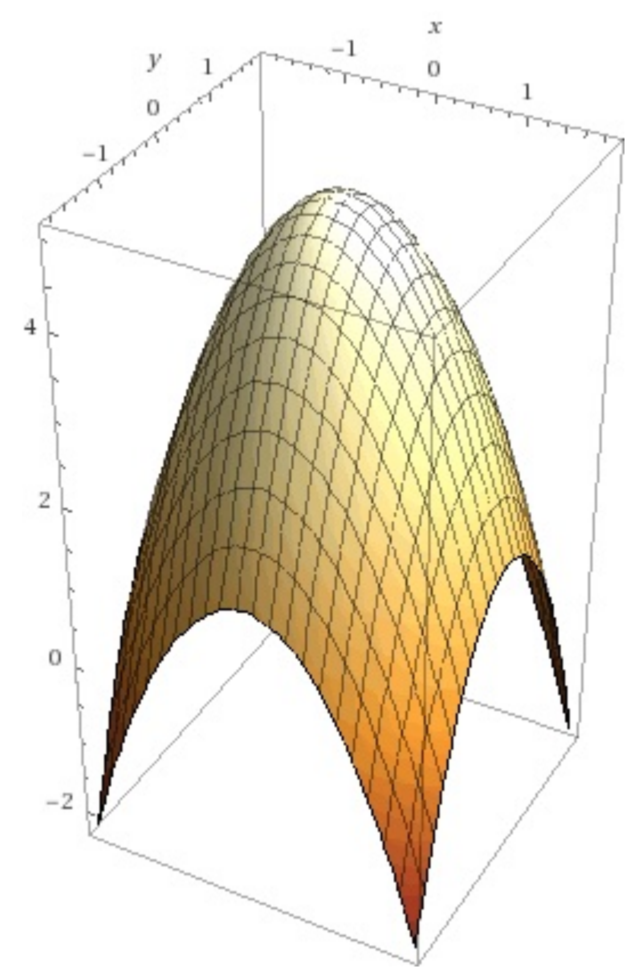
$$= -\frac{1}{2}x^4 + \frac{2}{3}x^3 + 2x^2 \Big|_0^2 = \underline{\underline{\frac{16}{3}}}$$

3) Find the volume of a solid :

Find the volume of the solid bounded by the planes :

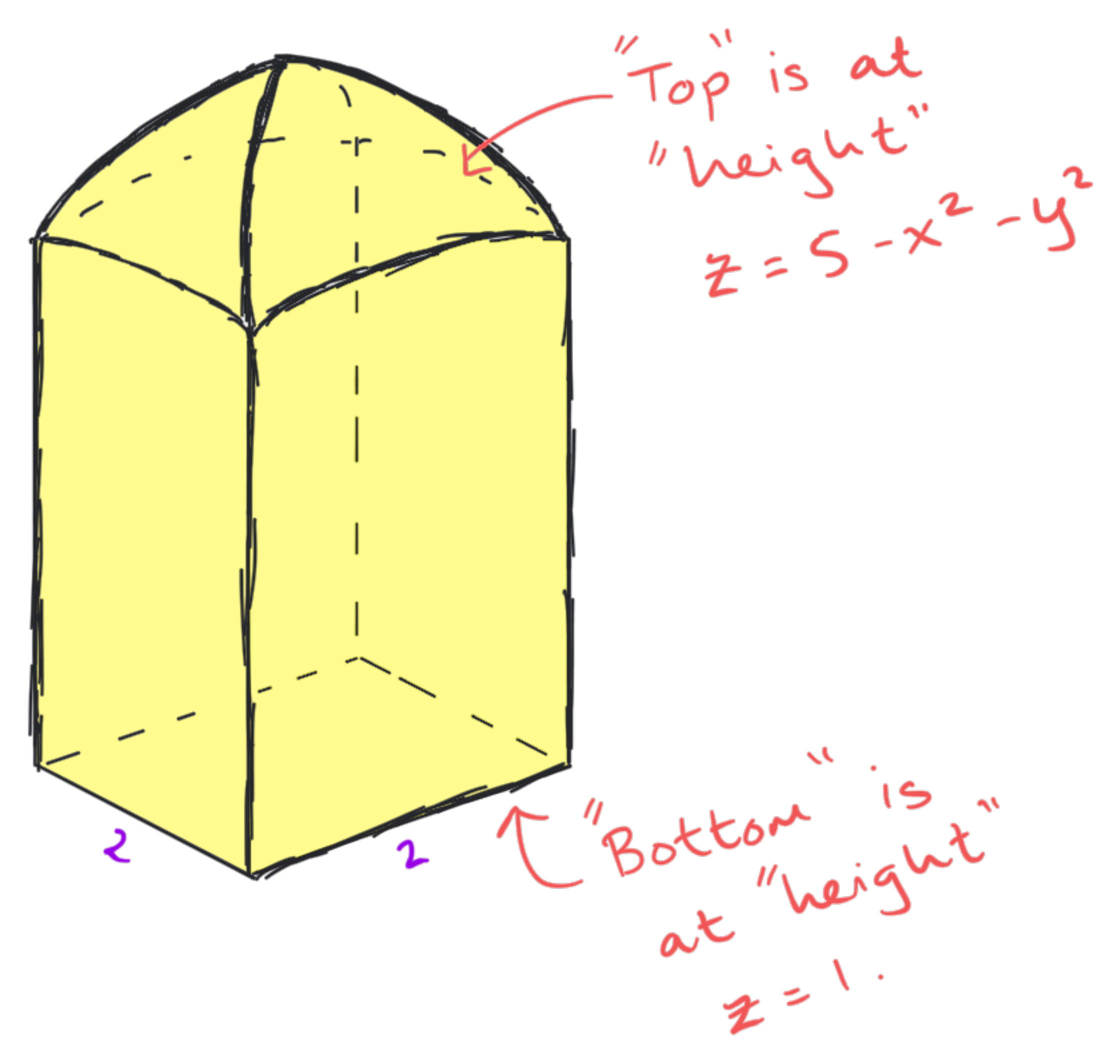
$x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1$

and the graph of $z = 5 - x^2 - y^2$:



So, the shape looks like :

S :



So $S = \{ (x,y,z) \, ; \, -1 \leq x \leq 1, \, -1 \leq y \leq 1 \, , \, 1 \leq z \leq 5 - x^2 - y^2 \}$

So : $\mathrm{Vol}(S) = \iiint\limits_S dV = \int_{-1}^{1} \int_{-1}^{1} \int_{1}^{5-x^2-y^2} dz\,dy\,dx$

$$= \int_{-1}^{1}\int_{-1}^{1} z \Big|_{1}^{5-x^2-y^2} dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} (5-x^2-y^2)-1 \; dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 4-x^2-y^2 \; dy\,dx$$

$$= \int_{-1}^{1} \left(4y - x^2y - \frac{y^3}{3}\right)\Big|_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left\{4 - x^2 - 1/3\right\} - \left\{-4 + x^2 + 1/3\right\} dx$$

$$= \int_{-1}^{1} \left(\frac{22}{3} - 2x^2\right) dx$$

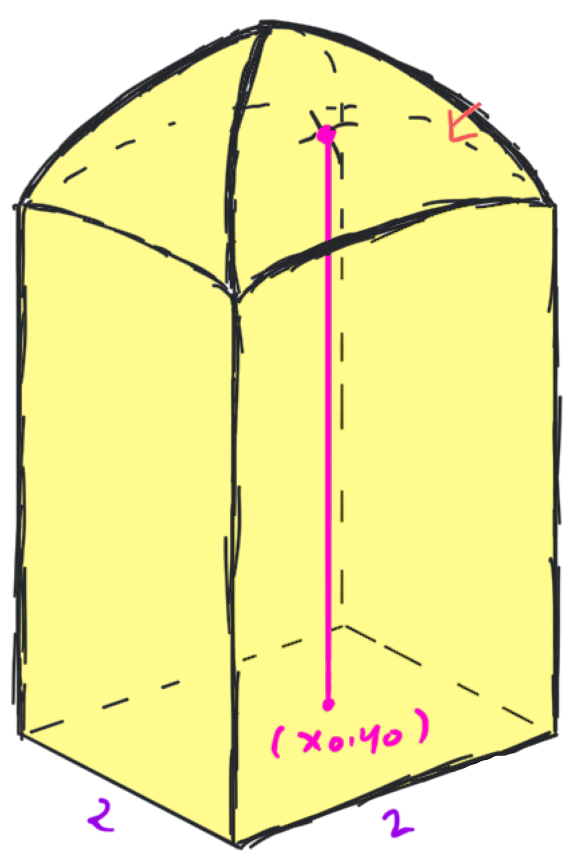
$$= \frac{22}{3}x - \frac{2x^3}{3} \Big|_{-1}^{1}$$

$$= \left(\frac{22}{3} - \frac{2}{3}\right) - \left(-\frac{22}{3} + \frac{2}{3}\right)$$

$$= \frac{40}{3}$$

Remark: Similar to the area problem, we can think of this volume being constructed by: Picking (x_0, y_0) and computing the length of the "cord" above (x_0, y_0):

S :



Then integrate over all $(x,y) \in [-1,1] \times [-1,1]$ to "fill in" S with cords.

Bonus Questions: 1) $\frac{40}{3} = 8 + \frac{16}{3}$

Explain from the picture why $\int_{-1}^{1}\int_{-1}^{1}(2 - x^2 - y^2)\,dy\,dx = \frac{16}{3}$.

Hint: Think of S as "a cap sitting on a box".

2) Assume $f(x,y) \geq 0$ on a region R. Why does $\iint_R f(x,y)\,dA$ = the Volume "trapped" under the graph of f over R?