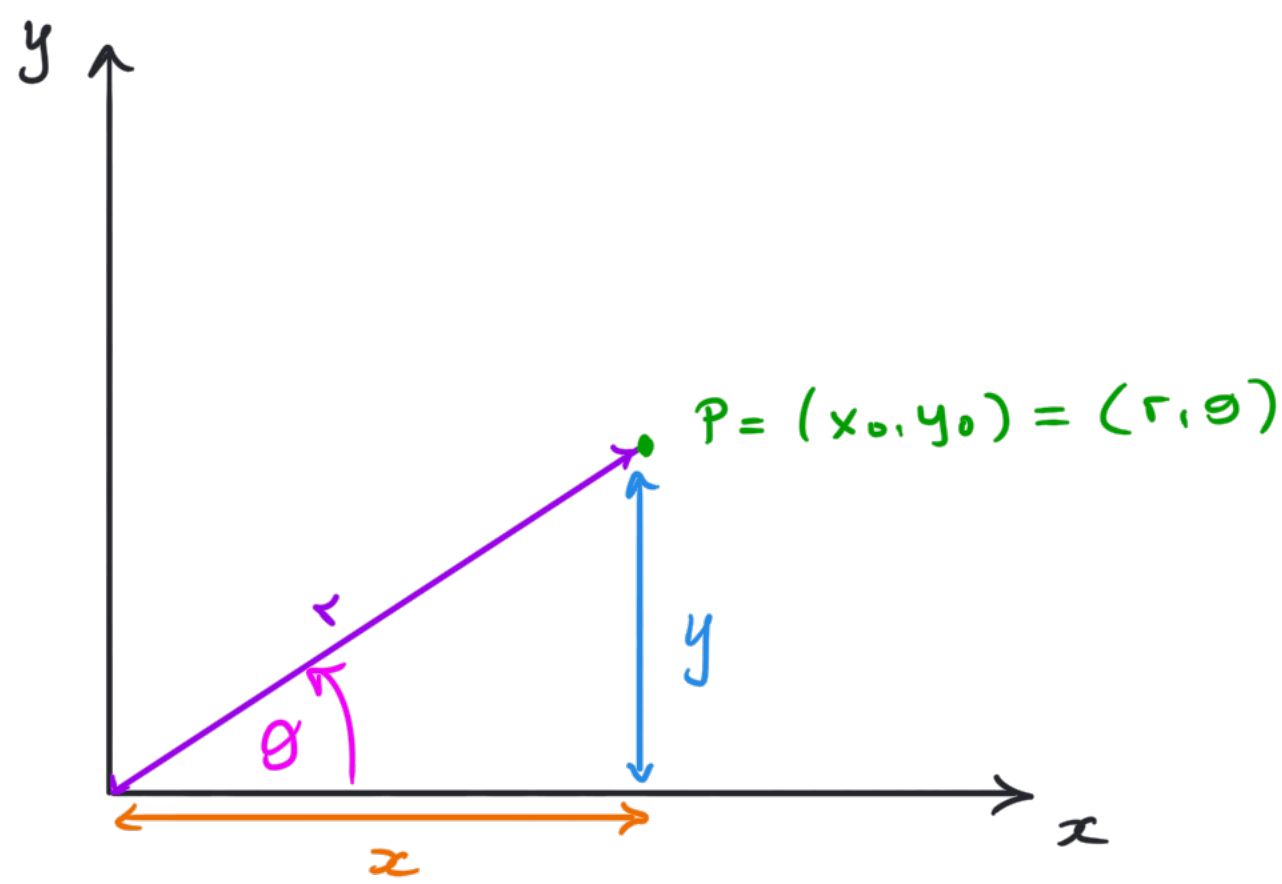
Calculus III : Exam 3 Notes :

Polar Coordinates :



We can represent the point p in either cartesian : (x, y),

or polar : (r, θ) coordinates.

We can see from the picture that we should have :

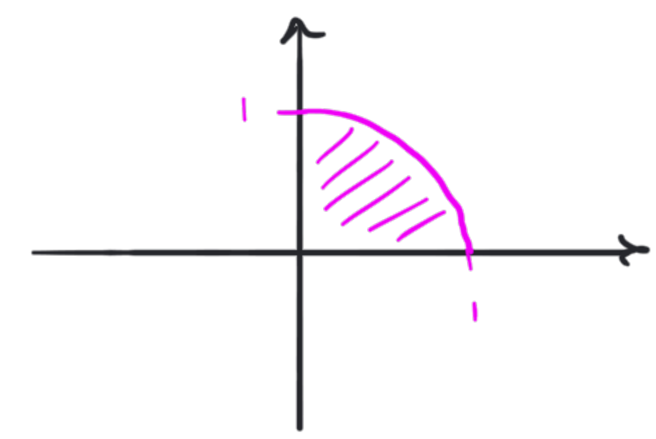
$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

We also have :

$$dA = r\,dr\,d\theta$$

Remark : Some regions in $\mathbb{R}^2$ are easier to describe in polar coordinates.

Example: Describe this region:



Remark: Hence it is sometimes useful to integrate over regions in $\mathbb{R}^2$ using polar coordinates.

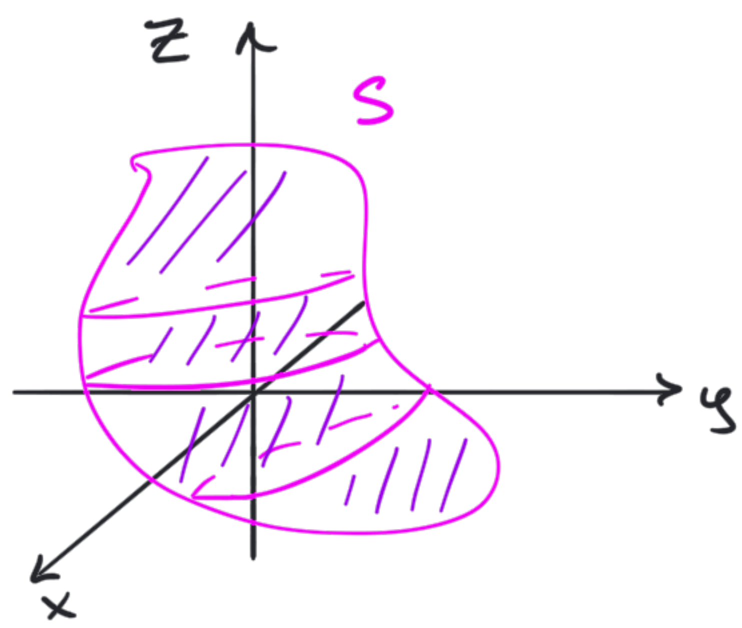
Here's how to switch from cartesian to polar:

$$\boxed{\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = \iint_{R_{(r,\theta)}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Where $R_{(x,y)}$ means the region expressed in cartersian coordinates, and $R_{(r,\theta)}$ means the region expressed in polar coordinates.

Triple Integrals :

Consider a solid $S \subset \mathbb{R}^3$:



$$\boxed{\text{Volume}(S) = \iiint_S dV}$$

To integrate a function f over a solid S :

$$\boxed{\iiint_S f \, dV}$$

In Cartesian coordinates : $\boxed{dV = dx\,dy\,dz}$ and

$$\boxed{\text{Vol}(S) = \iiint_{S_{(x,y,z)}} dx\,dy\,dz}$$

$$\boxed{\iiint_S f \, dV = \iiint_{S_{(x,y,z)}} f(x,y,z) \, dx\,dy\,dz}$$

Applications of Double/Triple Integrals:

2D: For a lamina $D \subset \mathbb{R}^2$ with density function δ:

- $$\boxed{\text{Mass}(D) = \iint_D \delta \, dA = \iint_{D_{(x,y)}} \delta(x,y)\,dx\,dy =: m}$$

- Moment of lamina around:

  x-axis: $$\boxed{M_x = \iint_{D_{(x,y)}} y\,\delta(x,y)\,dx\,dy}$$

  y-axis: $$\boxed{M_y = \iint_{D_{(x,y)}} x\,\delta(x,y)\,dx\,dy}$$

- Center of Mass of D: $(\bar{x}, \bar{y})$ where:

  $$\boxed{\bar{x} = \frac{M_y}{m}} \quad \text{and} \quad \boxed{\bar{y} = \frac{M_x}{m}}$$

- Moment of Inertia around:

  x-axis: $$\boxed{I_x = \iint_{D_{(x,y)}} y^2\,\delta(x,y)\,dx\,dy}$$ y-axis: $$\boxed{I_y = \iint_{D_{(x,y)}} x^2\,\delta(x,y)\,dx\,dy}$$

  origin: $$\boxed{I_0 = \iint_{D_{(x,y)}} (x^2+y^2)\,\delta(x,y)\,dx\,dy}$$

3D: For a solid $S \subset \mathbb{R}^3$ with density function δ:

- $$\boxed{\text{Mass}(S) = \iiint_S \delta \, dV = \iiint_{S_{(x,y,z)}} \delta(x,y,z) \, dV_{(x,y,z)} =: m}$$

- Moments around each coordinate plane:

$$\boxed{\begin{aligned} M_{yz} &= \iiint_{S_{(x,y,z)}} x\,\delta(x,y,z)\, dV_{(x,y,z)} \\ M_{xz} &= \iiint_{S_{(x,y,z)}} y\,\delta(x,y,z)\, dV_{(x,y,z)} \\ M_{xy} &= \iiint_{S_{(x,y,z)}} z\,\delta(x,y,z)\, dV_{(x,y,z)} \end{aligned}}$$

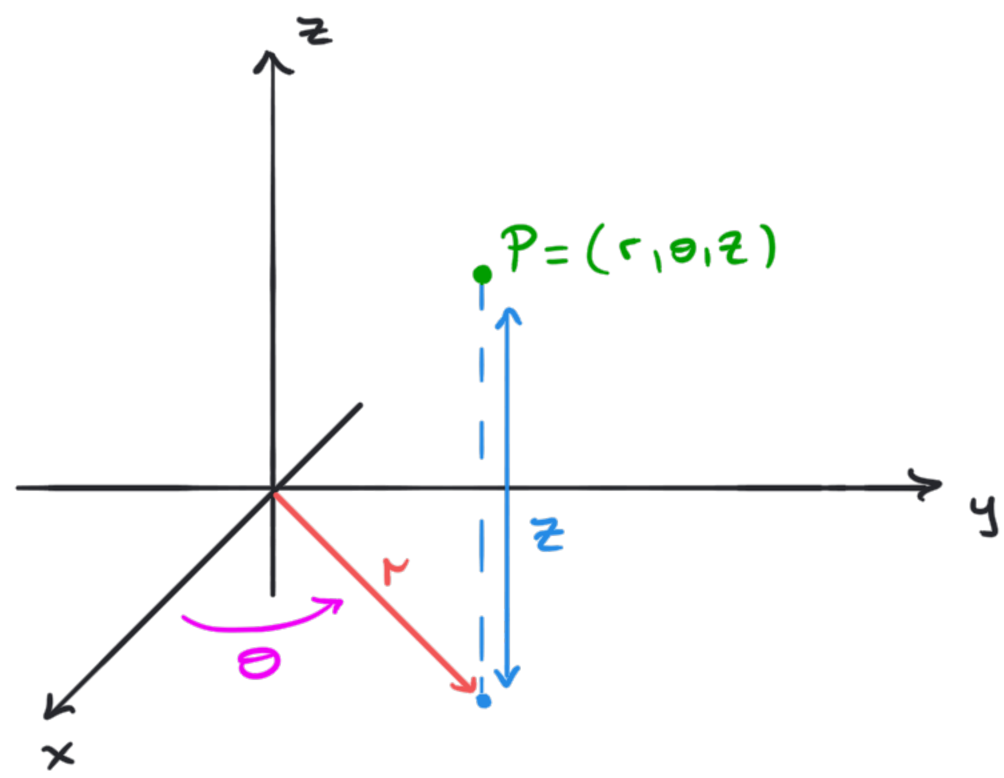
- Center of Mass of S: $(\bar{x}, \bar{y}, \bar{z})$ where:

$$\boxed{\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}}$$

Moments of Inertia around each coordinate axis:

$$\boxed{\begin{aligned} I_x &= \iiint_{S_{(x,y,z)}} (y^2+z^2)\,\delta(x,y,z)\, dV_{(x,y,z)} \\ I_y &= \iiint_{S_{(x,y,z)}} (x^2+z^2)\,\delta(x,y,z)\, dV_{(x,y,z)} \\ I_z &= \iiint_{S_{(x,y,z)}} (x^2+y^2)\,\delta(x,y,z)\, dV_{(x,y,z)} \end{aligned}}$$

Triple Integrals in Cylindrical Coordinates :



$$0 \le r \quad , \quad 0 \le \theta \le 2\pi \quad , \quad -\infty < z < \infty$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

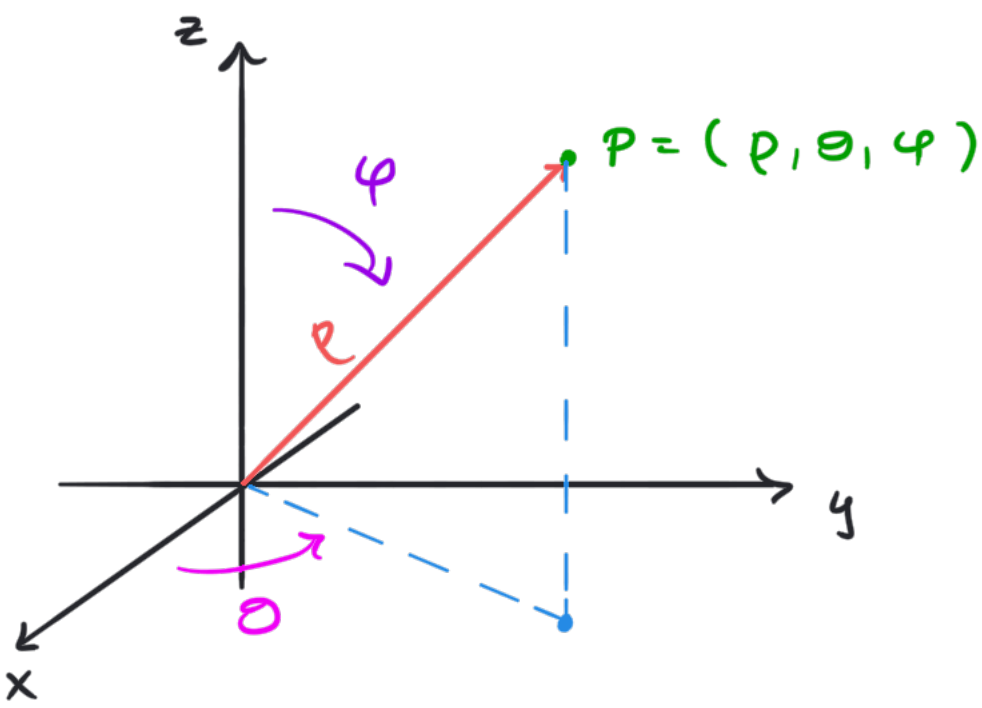
$$z = z$$

$$dV_{(r,\theta,z)} = r\,dz\,dr\,d\theta$$

$$\iiint_{S_{(x,y,z)}} f(x,y,z)\,dV_{(x,y,z)} = \iiint_{S_{(r,\theta,z)}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Example : (Old Exam Q1)

Triple Integrals in Spherical Coordinates:



$0 \le \rho \quad , \quad 0 \le \theta \le 2\pi \quad , \quad 0 \le \varphi \le \pi$

$x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$

$z = \rho \cos\varphi$

$dV_{(\rho,\theta,\varphi)} = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

$$\iiint_{S(x,y,z)} f(x,y,z) \, dV_{(x,y,z)}$$

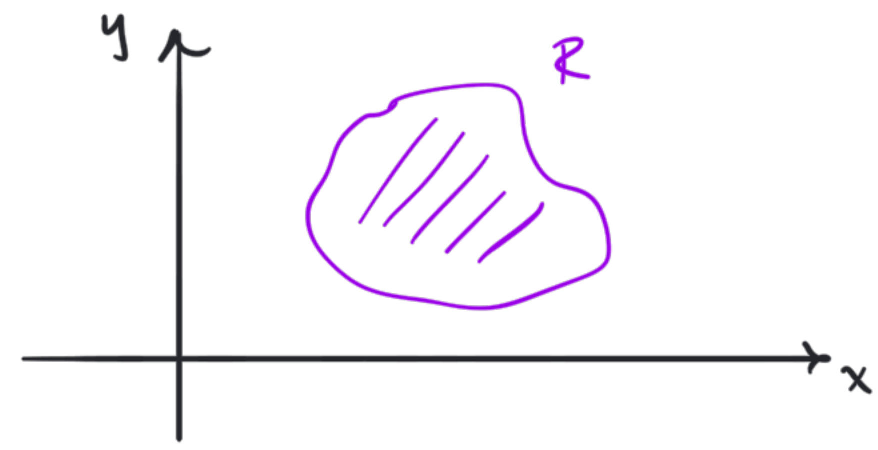
$$\|$$

$$\iiint_{S_{(\rho,\theta,\varphi)}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Example: (Old Exam Q4)

Change of Variables in Multivariate Integrals:

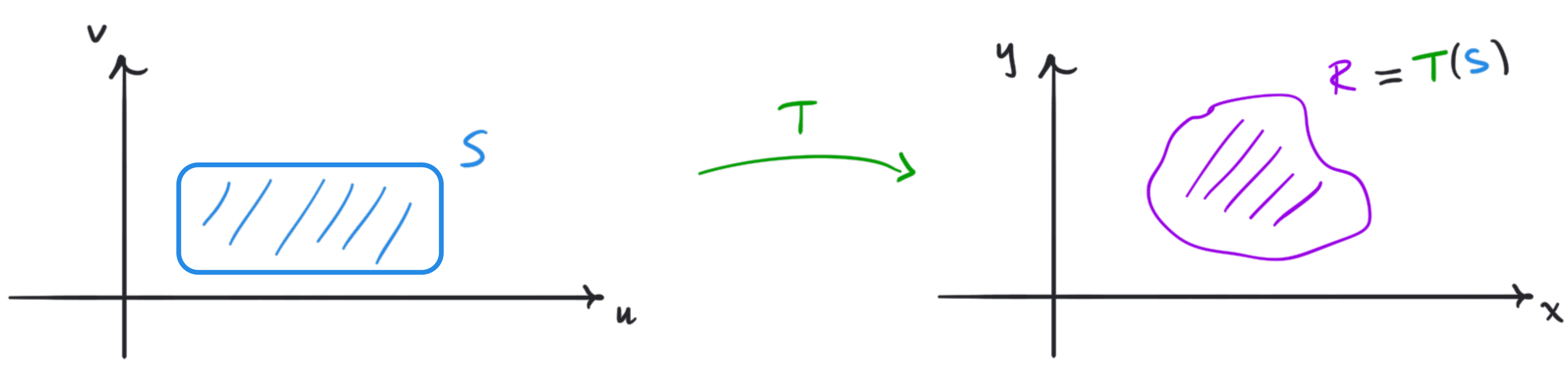
Let's say you're tasked with integrating a function f over a region $R \subset \mathbb{R}^2$:



$$\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = ?$$

R may be very difficult to describe in Cartesian coordinates: $(R_{(x,y)} = ?)$.

But imagine we have a simpler set $S \subset \mathbb{R}^2$, and a "nice" map T such that $T(S) = R$:



We can use a change of variables $T(u,v) = (x,y) = \big(x(u,v), y(u,v)\big)$ to integrate over the simpler set S:

$$\boxed{\iint_{R_{(x,y)}} f(x,y)\,dx\,dy = \iint_{S_{(u,v)}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

where :

$$\boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|}$$

↳ This is referred to as the Jacobian of the transformation.

Example : $R_{(x,y)} = \{(x,y) \,;\, 0 \leq x \leq 2 \,,\, 0 \leq y \leq 2\}$, $x = 2u$, $y = 2v$.

Find Area(R).

Intuition :

3D: $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$.

$$\boxed{\iiint_{R_{(x,y,z)}} f(x,y,z)\,dx\,dy\,dz = \iiint_{S_{(u,v,w)}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

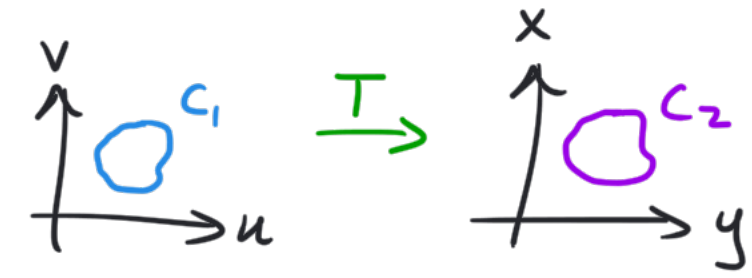
where

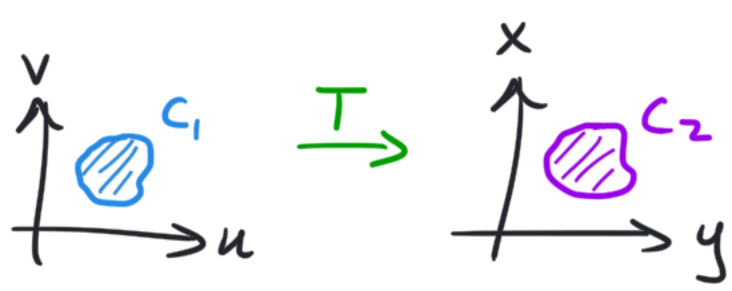
$$\boxed{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}}$$

Examples: (i) Cylindrical coordinates

(ii) Spherical coordinates

Remark: These transformations "map interiors to interiors".

i.e. If T sends a "loop" C_1 in the uv-plane to a "loop" C_2 in the xy plane: 

then it maps the region "inside" C_1 to the region "inside" C_2: 

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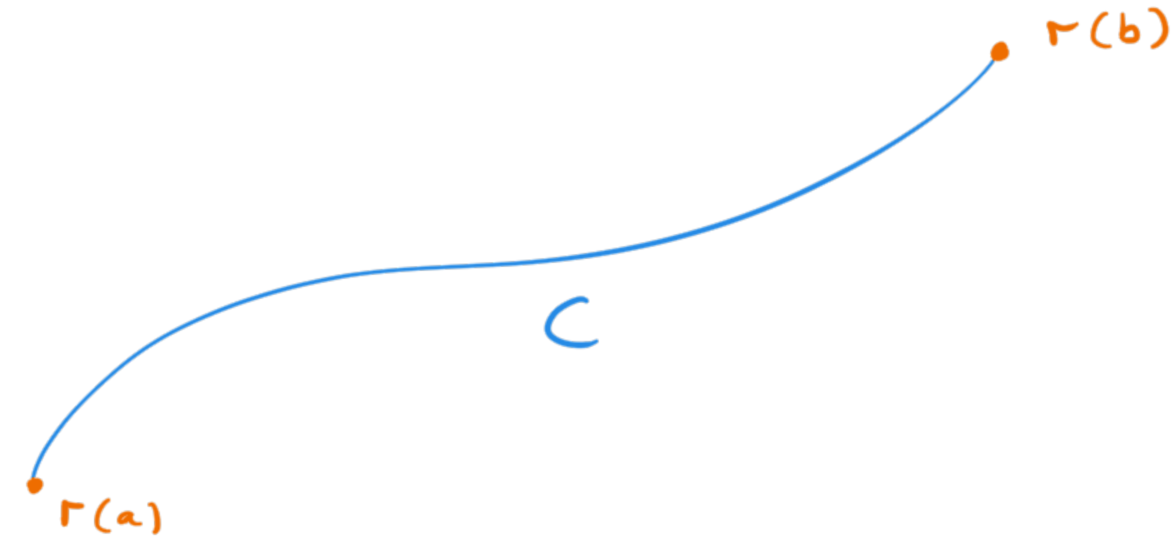
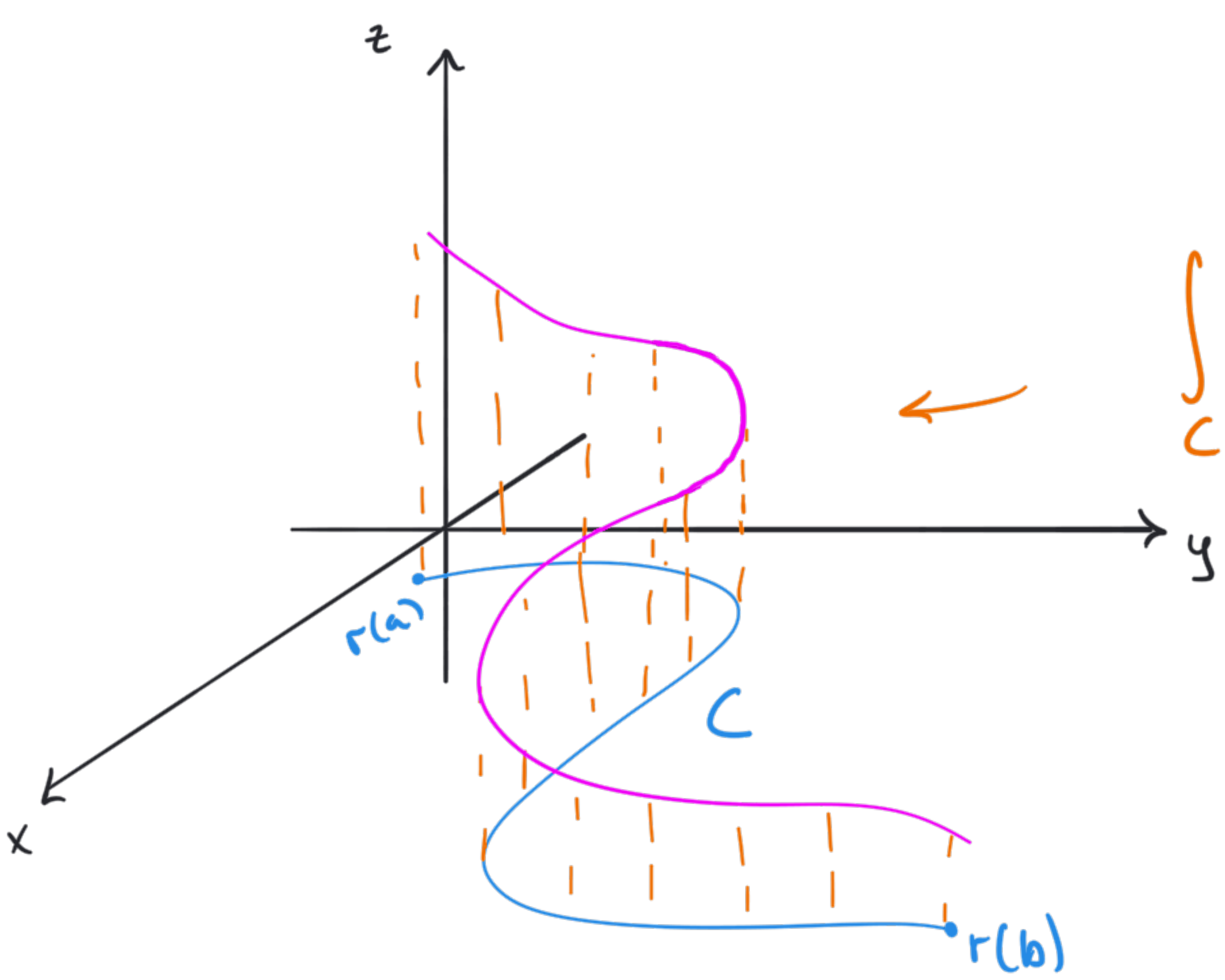
Example: Webassign parallelogram problems.

Line Integrals:

2D: Let C be a curve in $\mathbb{R}^2$ parametrized by $r(t) = (x(t), y(t))$ for $a \leq t \leq b$:



Consider a function $f: \mathbb{R}^2 \to \mathbb{R}$ and restrict the graph of f to just those points "above C":



$\int_C f(x,y)\,ds$ = Area of this "curtain".

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

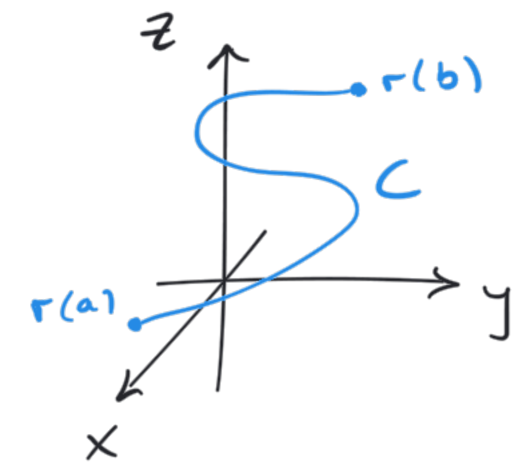
$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Example :

3D: If C is a curve in $\mathbb{R}^3$, parametrised by

$r(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$:

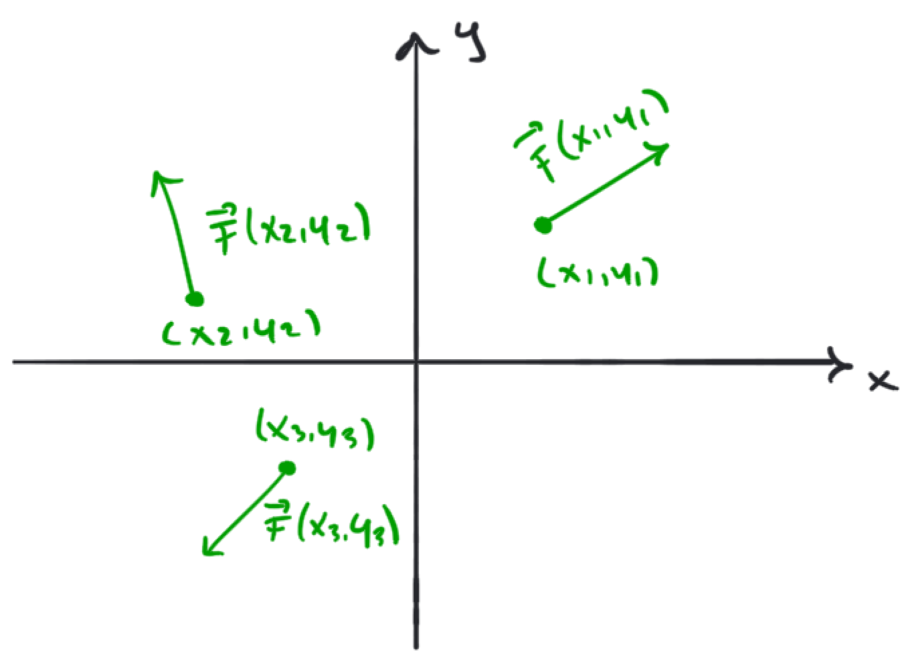


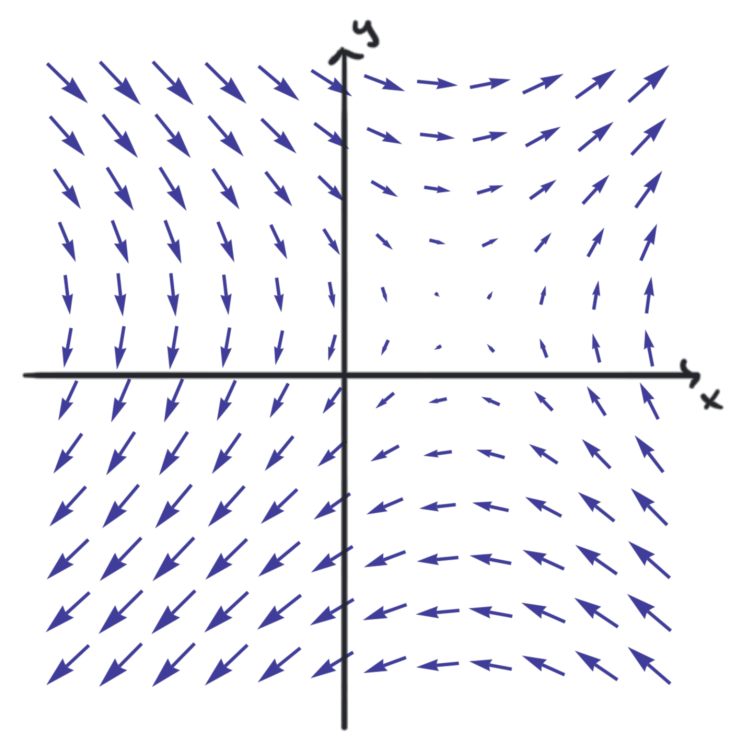
$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,|r'(t)|\,dt \qquad \text{where } |r'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}$$

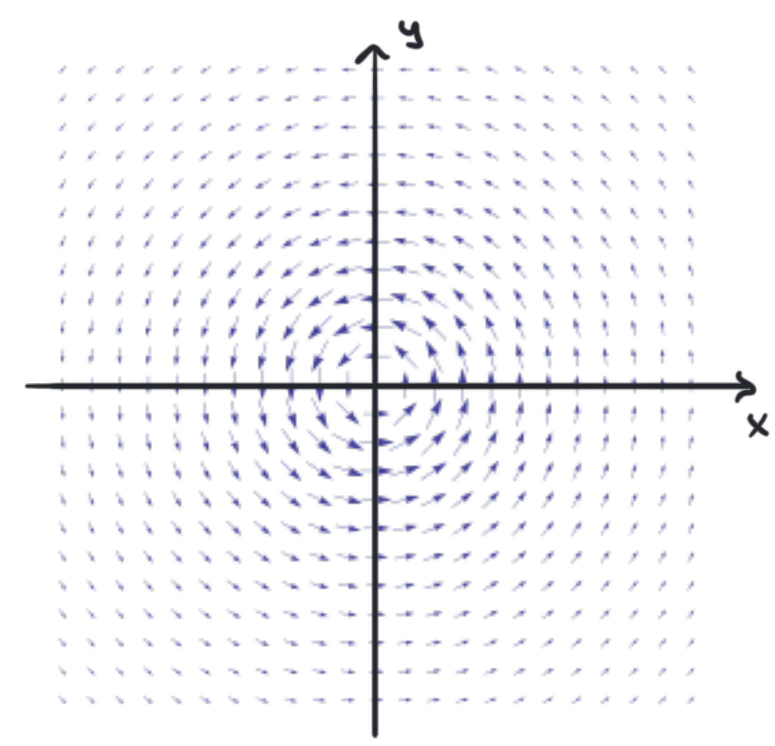
Example:

Vector Fields:

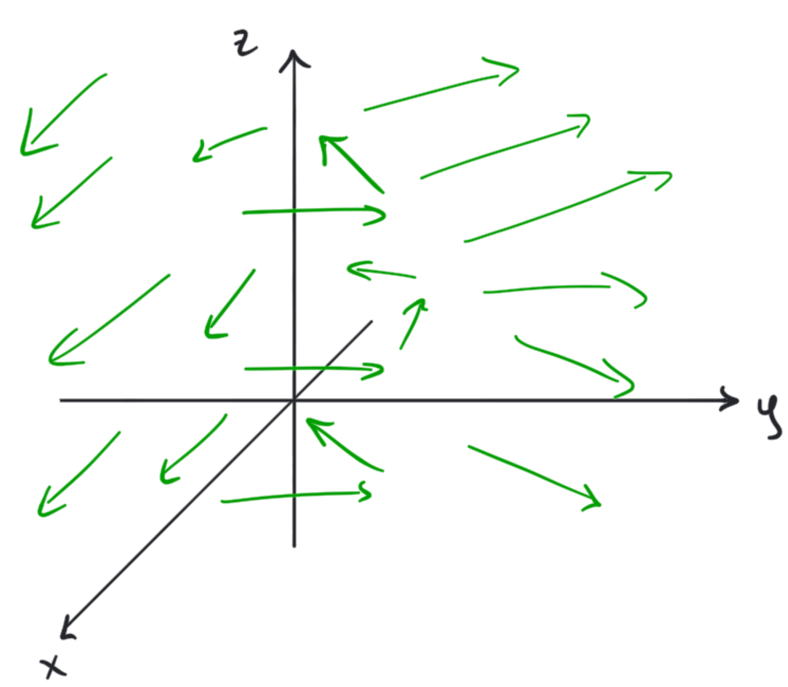
2D: A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that for every point (x,y) in $\mathbb{R}^2$ assigns a vector $\vec{F}(x,y)$ at that point:







3D: A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that for every point (x,y,z) in $\mathbb{R}^3$ assigns a vector $\vec{F}(x,y,z)$ at that point:



Remark The gradient of a function $f: \mathbb{R}^3 \to \mathbb{R}$ (or $\mathbb{R}^2$) : $\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f)$ is a vector field.

Definition: A vector field $\vec{F}$ is called <u>conservative</u> if there is a scalar function f such that $\boxed{\vec{F} = \vec{\nabla} f}$.

Line Integrals of Vector Fields:

Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ and let C be a curve in $\mathbb{R}^3$.

If we consider $\vec{F}$ to be a "force field", we can ask:

What is the work done by $\vec{F}$ in moving a particle along C?

Answer: If $r(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$ parametrizes C, then:

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot r'(t)\, dt = \int_C \vec{F} \cdot \vec{T}\, ds}$$

Why?

- If $\vec{F}$ is given in components by:

$$\vec{F}(x,y,z) = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$$

i.e. $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then:

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz}$$

The Fundamental Theorem for Line Integrals:

- Let C be a smooth curve parametrized by $r(t)$, $a \le t \le b$.

Let f be a differentiable function whose gradient $\vec{\nabla} f$ is continuous on C. Then:

$$\boxed{\int_C \vec{\nabla} f \cdot d\vec{r} = f(r(b)) - f(r(a))}$$

Remark: This implies that if C_1 and C_2 are two distinct smooth curves with the same start and endpoints, then:

$$\boxed{\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = \int_{C_2} \vec{\nabla} f \cdot d\vec{r}}$$

Definition: For an arbitrary vector field $\vec{F}$, continuous on a domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1, C_2 in D with the same start and end points.

Remark: Conservative vector fields are independent of path.

Example :

Theorems :

- $\int_C \vec{F}.d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}.d\vec{r} = 0$ for every closed path C in D.

- Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F}.d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative. i.e. there exists a function f such that $\vec{F} = \vec{\nabla} f$.

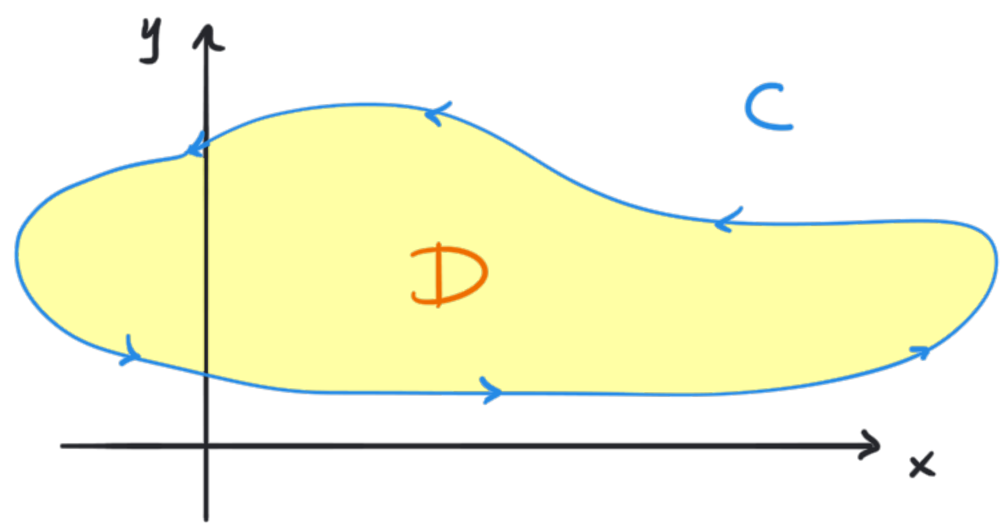
- If $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a conservative vector field, and P and Q have continuous first order partial derivatives on D, then we have: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

- Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

  Then $\vec{F}$ is conservative.

Green's Theorem :

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region enclosed by C :



If P and Q have continuous partial derivatives on an open region containing D, then :

$$\boxed{\int_C P\,dx + \int_C Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Example :