$\int$ Calc III : Review Session, Exam 1:

Topics:

1) 3D - coordinates

2) Vectors

3) Dot Product & Cross Product

4) Lines & Planes

5) Vector functions & Space Curves

6) Derivatives & Integrals (of the above)

7) Arc Length

8) TNB Frame

9) Motion in Space

10) Functions of several variables

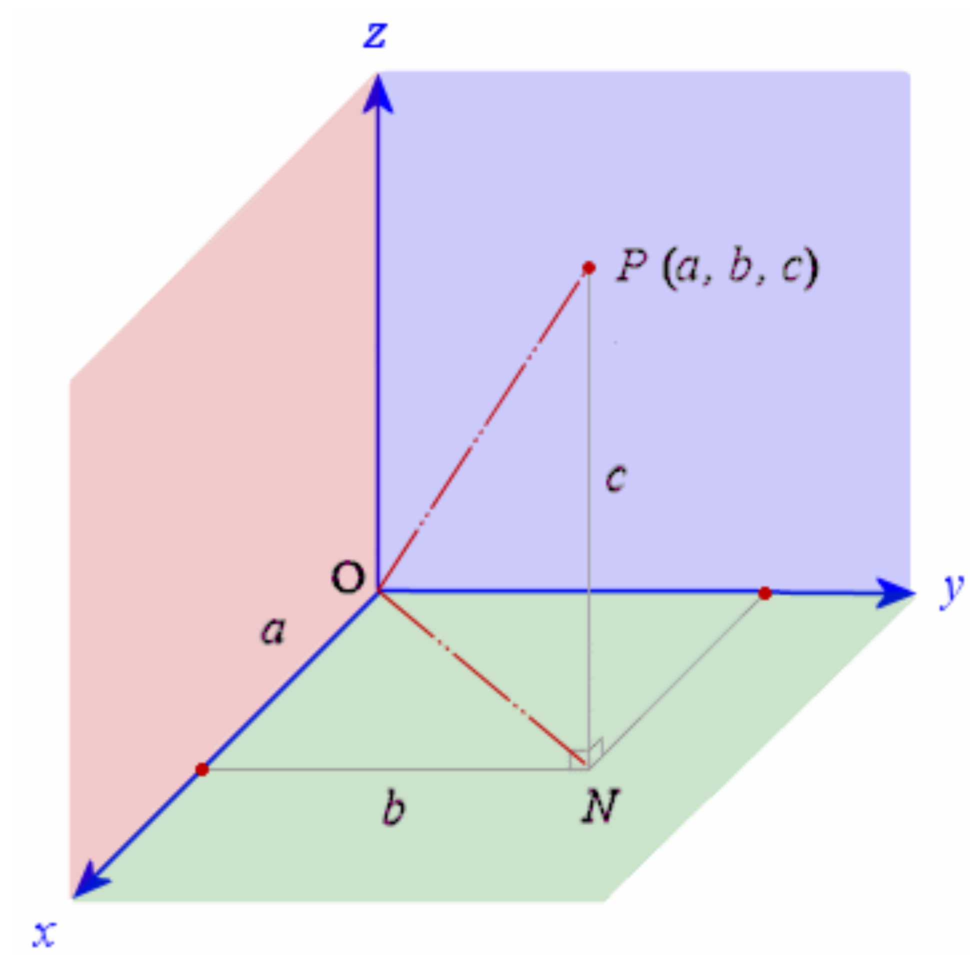
11) Limits & Continuity

1. 3D - Coordinates:

Definition: As a set:

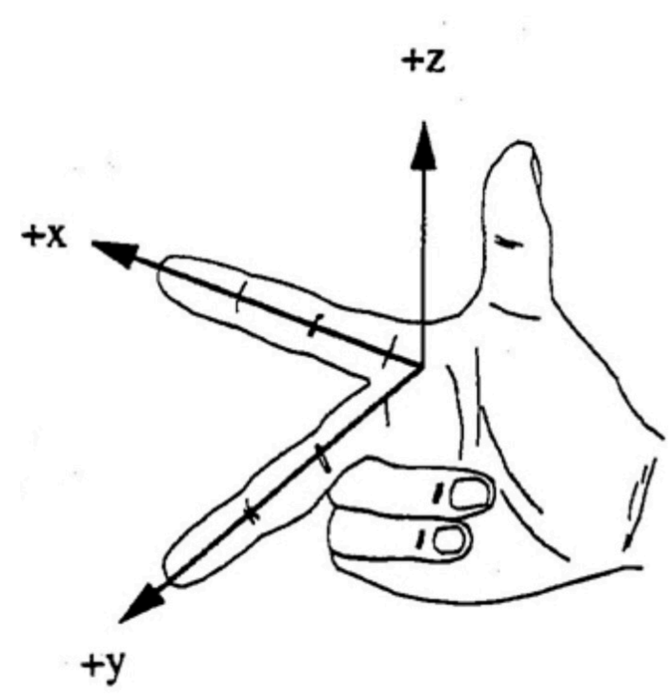
$$\mathbb{R}^3 = \{ (x,y,z) \mid x,y,z \in \mathbb{R} \}$$

Visually:

"$\mathbb{R}^3$ =" 

Remark: Think of $\mathbb{R}^3$ as the "inside" of an infinitely large "box". Similarly, think of $\mathbb{R}^2$ (the "xy-plane") as an infinitely large "sheet".

Right-Hand Rule:



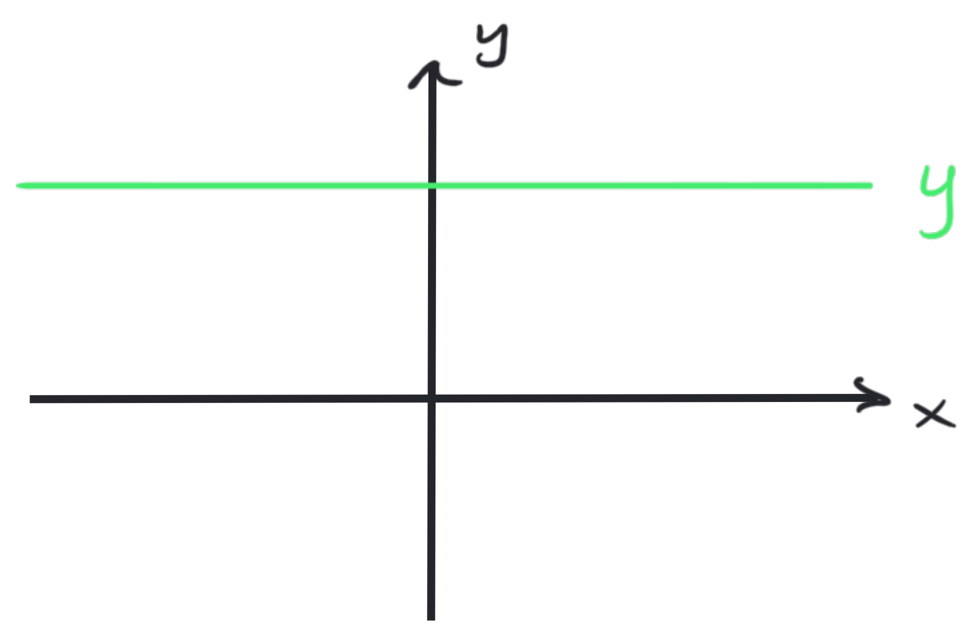
Question: If you were asked to draw $y=3$, what would that look like?

Answer: It depends.

↳ In $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$, this is all pairs (x,y) such that $y=3$.

i.e. $(1,3)$, $(\pi,3)$, $(671,3)$, ...

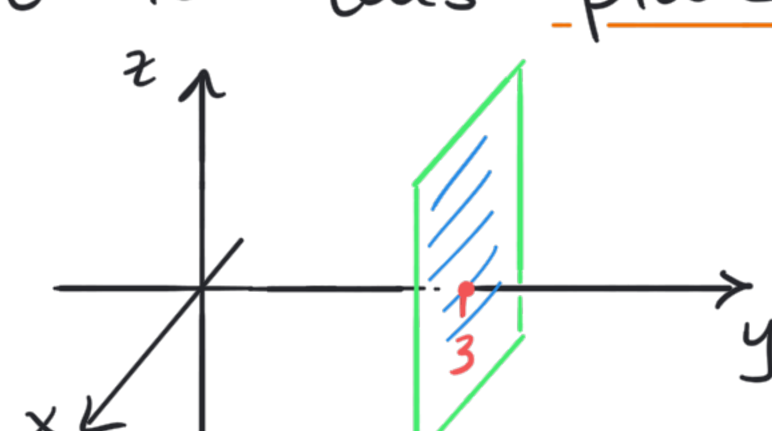
As a picture, it's this green line:



↳ In $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$, this is all (x,y,z), such that $y=3$.

i.e. $(1,3,1)$, $(\pi,3,9)$, ...

As a picture, it is this plane:



Remark: Equations in $\mathbb{R}^2$ usually lead to curves.

Equations in $\mathbb{R}^3$ usually lead to surfaces.

Distance Formula: For $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ in $\mathbb{R}^3$:

$$\boxed{d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}}$$

Exercises: 1) Draw all points in $\mathbb{R}^2$ that are 1 unit from the origin. Find an equation describing this set.

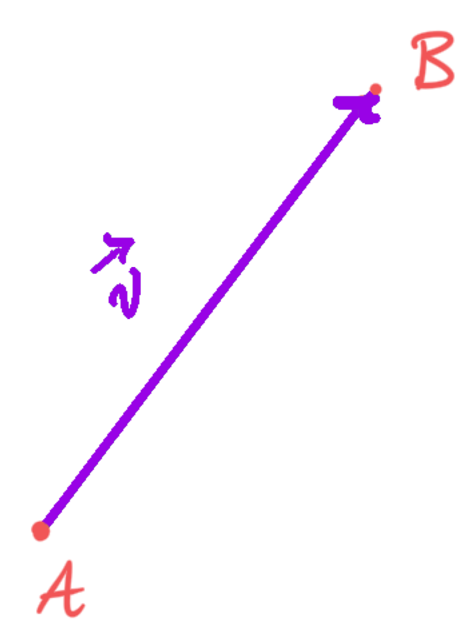
2) Do the same in $\mathbb{R}^3$.

2. Vectors :

Please see "Lecture 2" on my website for more details : www.pheslin.com/calculusiii

Main points :

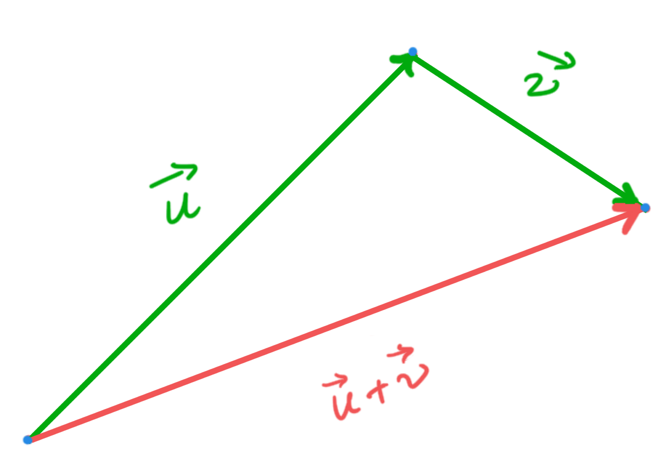
- A vector is something with a magnitude & direction.
- If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are points in $\mathbb{R}^3$, then the vector connecting A to B is denoted by $\overrightarrow{AB}$:



and $\vec{v}$ has coordinates :

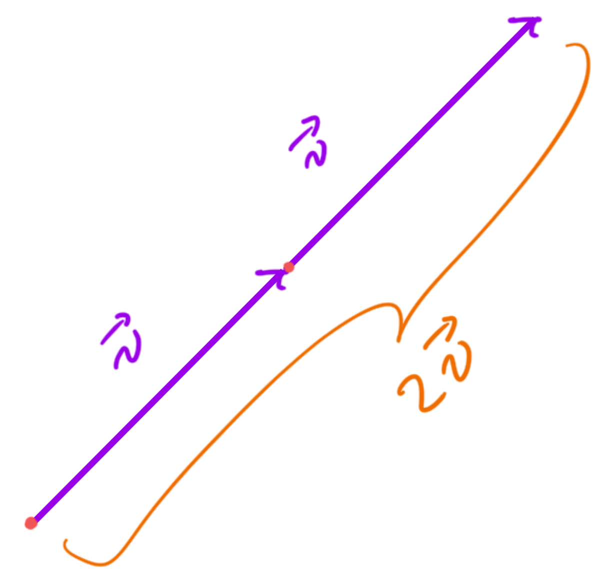
$$\boxed{\vec{v} = \overrightarrow{OB} - \overrightarrow{OA} = (b_1 - a_1, \; b_2 - a_2, \; b_3 - a_3)}$$

- How to visually add vectors :



- How to visually scale vectors:

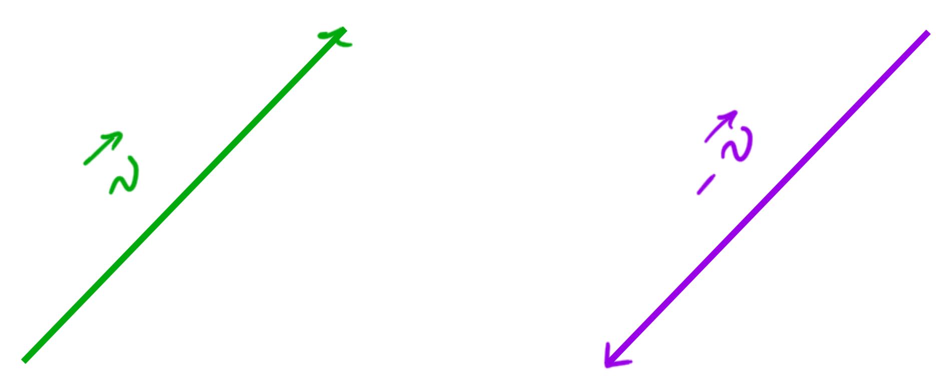
Fig. 5:



Direction? No change.

Length? Twice as long.

Fig. 6:



Direction? Opposite.

Length? Same.

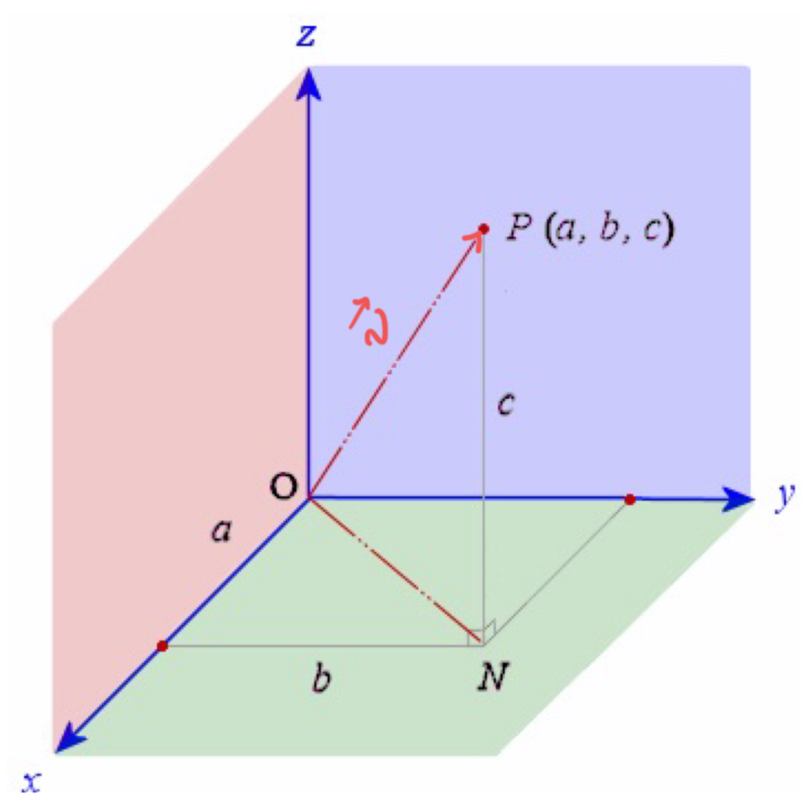
- Writing vectors in components:

Main Point: $\vec{v}$ represented in components is:

$$\vec{v} = \begin{pmatrix} \text{displacement } \vec{v} \text{ causes in } x\text{-direction}, & \text{displacement } \vec{v} \text{ causes in } y\text{-direction}, & \text{displacement } \vec{v} \text{ causes in } z\text{-direction} \end{pmatrix}$$

- How to compute the length of a vector:

Hence, we compute the length of $\vec{v}$, which we denote by $\|\vec{v}\|$, by using its representation in components:



Using pythagoras, we can see:

$$\boxed{\|\vec{v}\| = \sqrt{a^2+b^2+c^2}}$$

- How to add | scale vectors algebraically:

(i) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

(iii) For $c \in \mathbb{R}$: $c\vec{a} = (ca_1, ca_2, ca_3)$

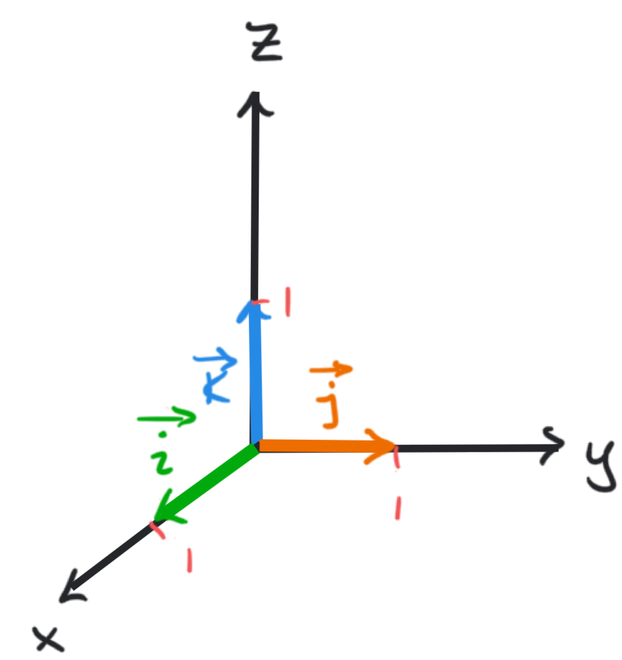
- General Properties of vectors:

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a}=c\vec{a}+d\vec{a}$
7. $(cd)\vec{a}=c(d\vec{a})$
8. $1\vec{a}=\vec{a}$

• Standard Basis Vectors:

Three vectors in $\mathbb{R}^3$ play a special role:

$$\left.\begin{aligned} \vec{i} &= (1,0,0) \\ \vec{j} &= (0,1,0) \\ \vec{k} &= (0,0,1) \end{aligned}\right\} \text{Standard Basis Vectors for } \mathbb{R}^3$$



Why?

Because any vector $\vec{v}$ can be represented, algebraically as a linear combination of $\vec{i}, \vec{j}, \vec{k}$:

$$\vec{v} = (a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$$

$$= a\vec{i} + b\vec{j} + c\vec{k}$$

Definition: A unit vector is a vector with length 1.

Example: $\vec{i}$, $\vec{j}$ and $\vec{k}$ are all unit vectors.

Remark: For any non-zero vector $\vec{u}$, there is a unit vector pointing in the same direction as $\vec{u}$. This vector is usually denoted as $\hat{u}$, and is given algebraically by:

$$\boxed{\hat{u} = \frac{1}{\|\vec{u}\|} \cdot \vec{u}}$$

This process $(\vec{u} \rightsquigarrow \hat{u})$ is called normalizing $\vec{u}$.

Remark: Unit vectors are sometimes referred to as directions.

Exercise: Normalize $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$.

Trick: If you need to normalize a vector, and you can pull out a positive constant from the vector, just normalise the vector you get if you drop the constant.

Example: Normalize $\vec{v} = (8, 16, 24)$.

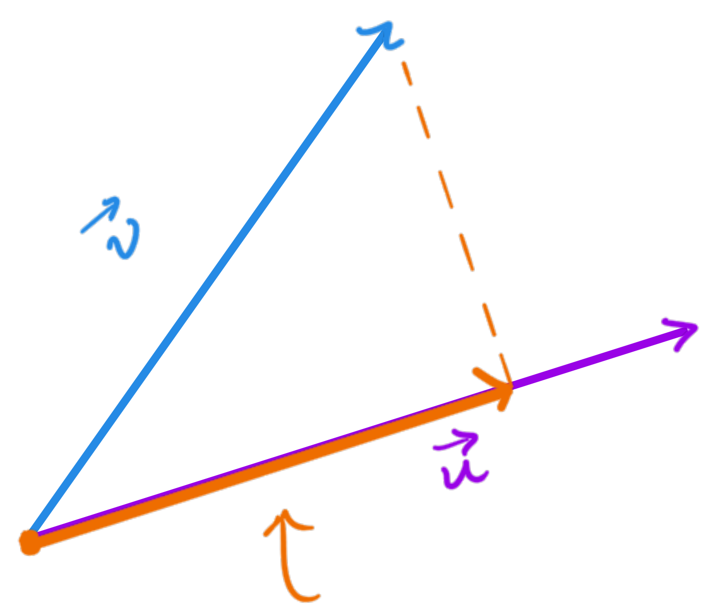
Soln: $\vec{v} = 8(1, 2, 3)$

$$\Rightarrow \hat{v} = \frac{(1, 2, 3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}(1, 2, 3)$$

3. Dot Product & Cross Product :

Motivation : Say I have two vectors $\vec{u}$ and $\vec{v}$, and I would like to know :

How much of $\vec{v}$ points in the direction of $\vec{u}$?

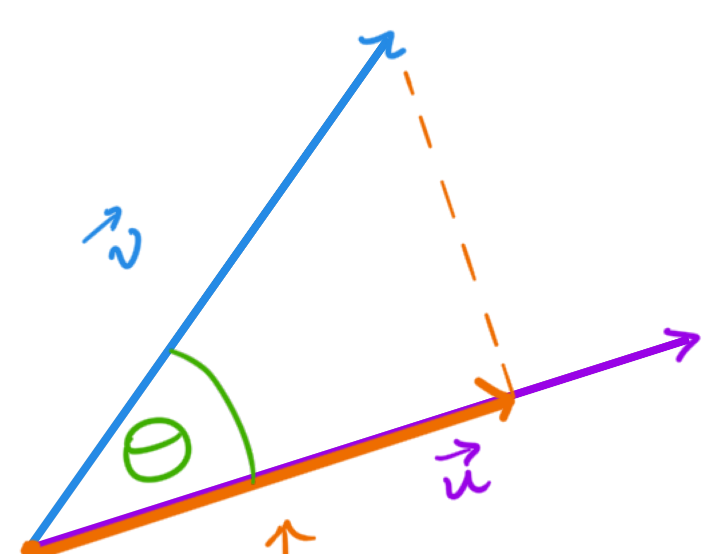


Let's call this : $\text{proj}_{\vec{u}}(\vec{v})$

But how can we find a formula for $\text{proj}_{\vec{u}}(\vec{v})$?

Well, if we knew θ, we could find it's length :

(Assume θ is acute - if not, use $-\vec{v}$)



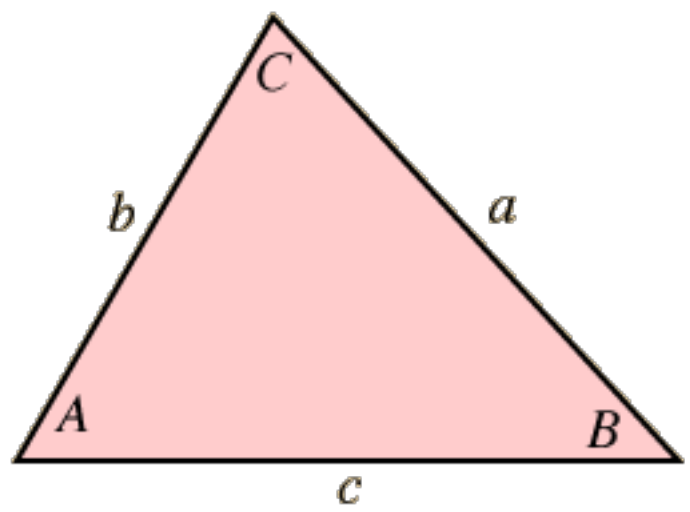
$\|\text{proj}_{\vec{u}}(\vec{v})\| = |\vec{v}| \cos\theta$

Let's call this length : $\boxed{\text{comp}_{\vec{u}}(\vec{v}) := \|\text{proj}_{\vec{u}}(\vec{v})\|}$

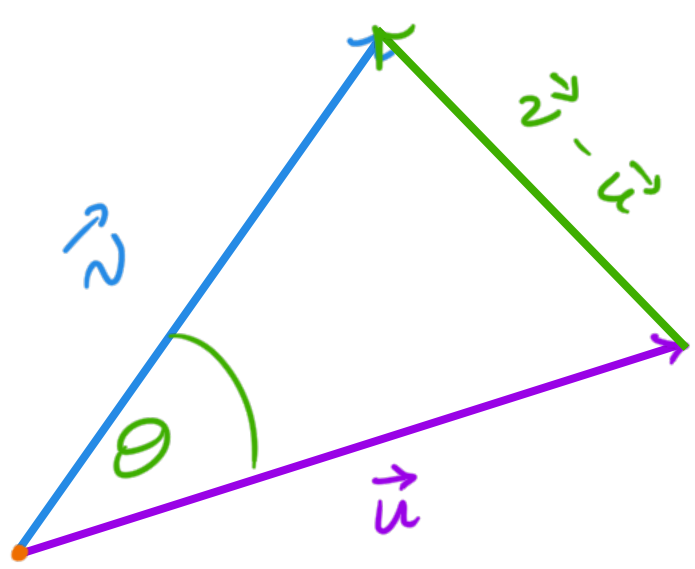
So, how can we find θ ?

Recall : Cosine Rule :

$$c^2 = a^2 + b^2 - 2ab\cos C$$



So :



$$\Rightarrow \quad \|\vec{v} - \vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 = v_1^2 + v_2^2 + v_3^2 + u_1^2 + u_2^2 + u_3^2 - 2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad -2v_1u_1 - 2v_2u_2 - 2v_3u_3 = -2\|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad v_1u_1 + v_2u_2 + v_3u_3 = \|\vec{v}\|\|\vec{u}\|\cos\theta$$

$$\Rightarrow \quad \boxed{\cos\theta = \frac{v_1u_1 + v_2u_2 + v_3u_3}{\|\vec{v}\|\|\vec{u}\|}}$$

← Assuming $\vec{v}$ and $\vec{u}$ are non-zero.

This motivates :

Definition: The <u>Dot Product</u> of two vectors $\vec{v}$ and $\vec{u}$ is :

$$\boxed{\vec{v}\cdot\vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3}$$

Remark: Hence , we see :

$$\boxed{\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\,\|\vec{u}\|}}$$

OR :

$$\boxed{\vec{v}\cdot\vec{u} = \|\vec{v}\|\,\|\vec{u}\|\cos\theta}$$

Finally :

$$\boxed{\operatorname{comp}_{\vec{u}}(\vec{v}) = \|\vec{v}\|\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}}$$

Recall that by definition, $\operatorname{proj}_{\vec{u}}(\vec{v})$ points in the same direction as $\vec{u}$. Hence :

$$\boxed{\operatorname{proj}_{\vec{u}}(\vec{v}) = \operatorname{comp}_{\vec{u}}(\vec{v})\cdot\hat{u} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}\right)\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u}}$$

Properties of the Dot Product:

1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5) $\mathbf{0} \cdot \mathbf{a} = 0$

NB: $\vec{v}$ and $\vec{u}$ are orthogonal if and only if:

$$\boxed{\vec{v} \cdot \vec{u} = 0}$$

Exercise: Why?

Motivation: Given two vectors $\vec{u}$ and $\vec{v}$, imagine you wanted a vector which is orthogonal to both $\vec{u}$ and $\vec{v}$. Say we find such a vector: $\vec{w}$.

Then: $\vec{u}\cdot\vec{w} = 0$ & $\vec{v}\cdot\vec{w} = 0$.

↳ If you solve this algebraically, the easiest solution is:

$$w_1 = u_2v_3 - v_2u_3 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}$$

$$w_2 = -(u_1v_3 - v_1u_3) = -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$$

$$w_3 = u_1v_2 - v_1u_2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

So $\vec{w} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

This motivates the following definition:

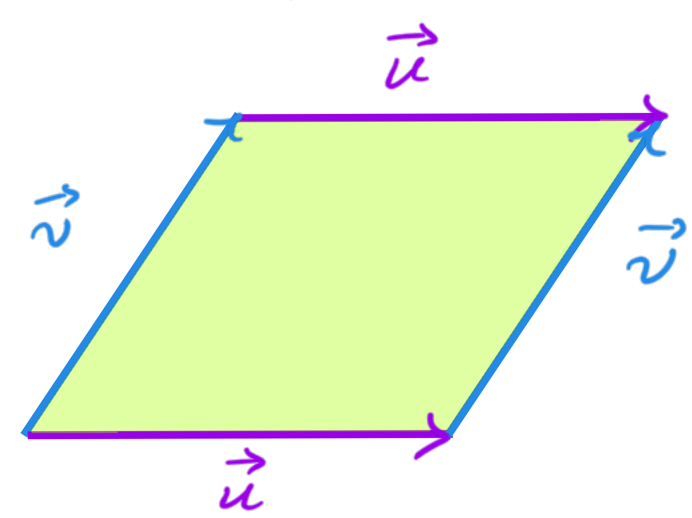
Definition: The Cross Product of two vectors $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \times \vec{v} := \begin{vmatrix} i & j & K \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Key Properties of $\vec{u} \times \vec{v}$:

1) $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{v}$

2) For $\vec{u}$ and $\vec{v}$:



$\|\vec{u} \times \vec{v}\|$ = Area of shaded parallelogram

3) The direction of $\vec{u} \times \vec{v}$ follows the Right Hand Rule: If $\vec{u}$ is the direction of your index finger, and $\vec{v}$ is the direction of your middle finger, then $\vec{u} \times \vec{v}$ points in the direction of your thumb.

Other properties: If $\vec{a}$ and $\vec{b}$ are vectors & c is a scalar:

i) $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

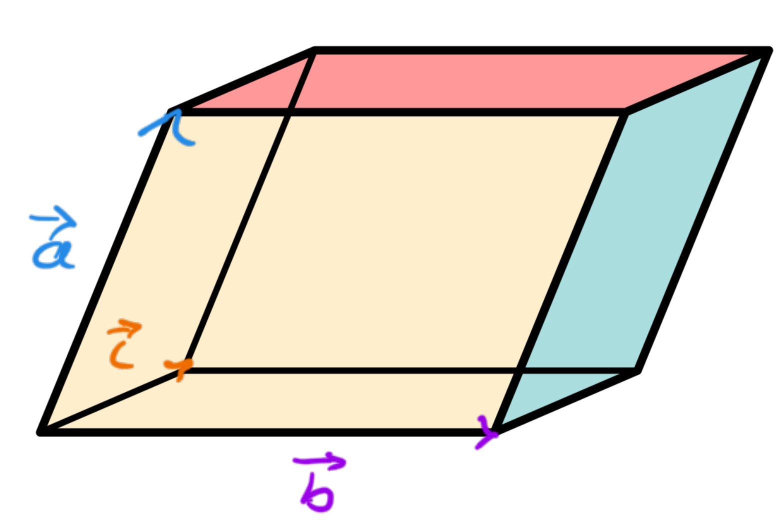
ii) $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

iii)
1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Definition: The vector triple product is defined as: $\vec{a} \cdot (\vec{b} \times \vec{c})$

Remark: We can compute the volume of a parallelopiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$ using the vector triple product:

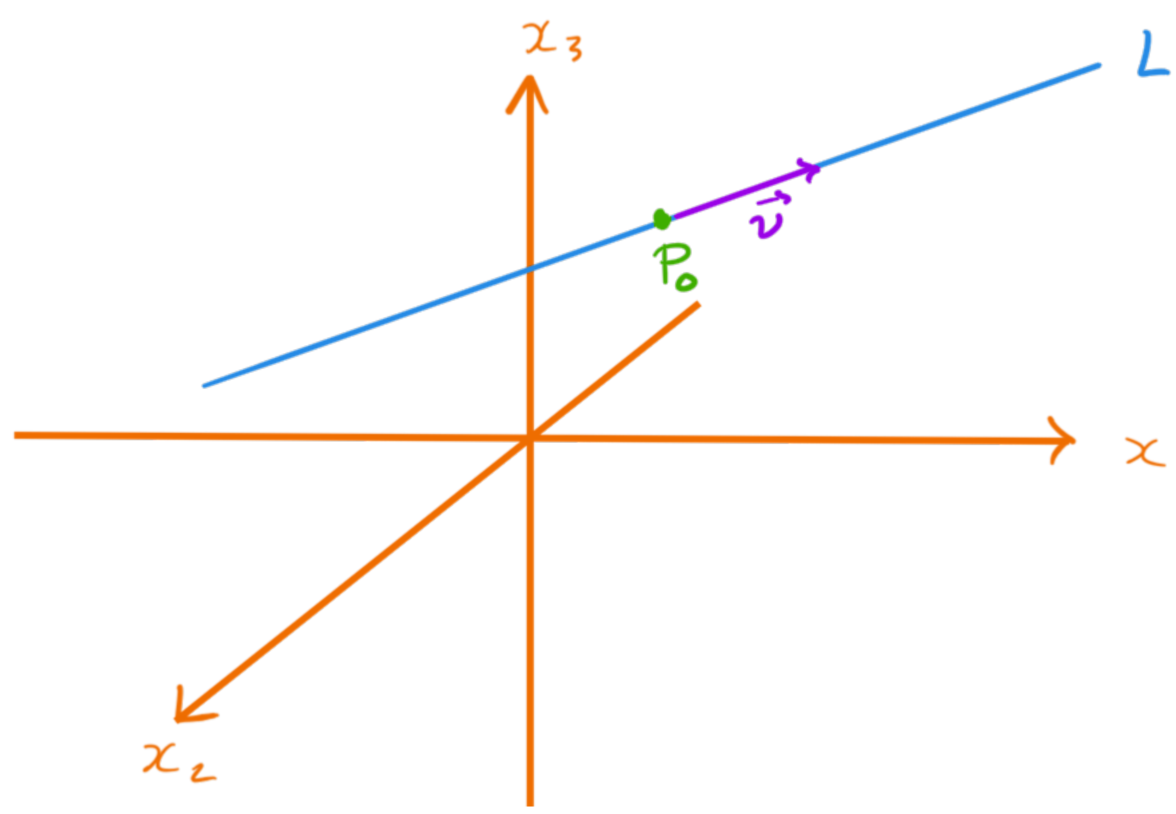
$$\text{Volume}(\vec{a}, \vec{b}, \vec{c}) = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$



4. Lines and Planes:

- We can see geometrically that a line is uniquely defined by a point on the line and the direction of the line (or, alternatively by two points on the line).

$(\vec{v} \neq \vec{0})$



Equation of L: $\boxed{L(t) = \vec{P_0} + t\vec{v}}$ (vector form)

Remarks:

1) We can think of this as a criteria for a point to be on the line. i.e., a point $P=(x,y,z)$ is on the line L $\iff$ there is a t_* such that $\vec{P} = \vec{P_0} + t_*\vec{v}$.

2) We can also think of this as a machine:

You give it a value of t and it gives you a point on the line L.

e.g. $L(1) = \vec{P}_0 + (1)\vec{v} = \vec{P}_0 + \vec{v}$ is on L

$L(\pi) = \vec{P}_0 + \pi\vec{v}$ is on L

$L(-3) = \vec{P}_0 - 3\vec{v}$ is on L

$\vdots$

- If $\vec{P}_0 = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$, then:

$$L(t) = (x(t), y(t), z(t)) = (x_0 + at, y_0 + bt, z_0 + ct)$$

Where we now think of $x(t)$, $y(t)$ and $z(t)$ as "component machines".

- Say a point (x, y, z) is on L. Then there must be some time, say t_*, where

$$\left.\begin{aligned} x &= x_0 + a t_* \\ y &= y_0 + b t_* \\ z &= z_0 + c t_* \end{aligned}\right\} \text{Parametric equations of } L$$

Isolating t_* :

$$t_* = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

We can also see if an x, y, z satisfy the last two equalities, they will be the components of $L(t)$ for some time. i.e.

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

is an equivalent description of a line through $P_0 = (x_0, y_0, z_0)$ with direction $\vec{v} = (a, b, c)$.

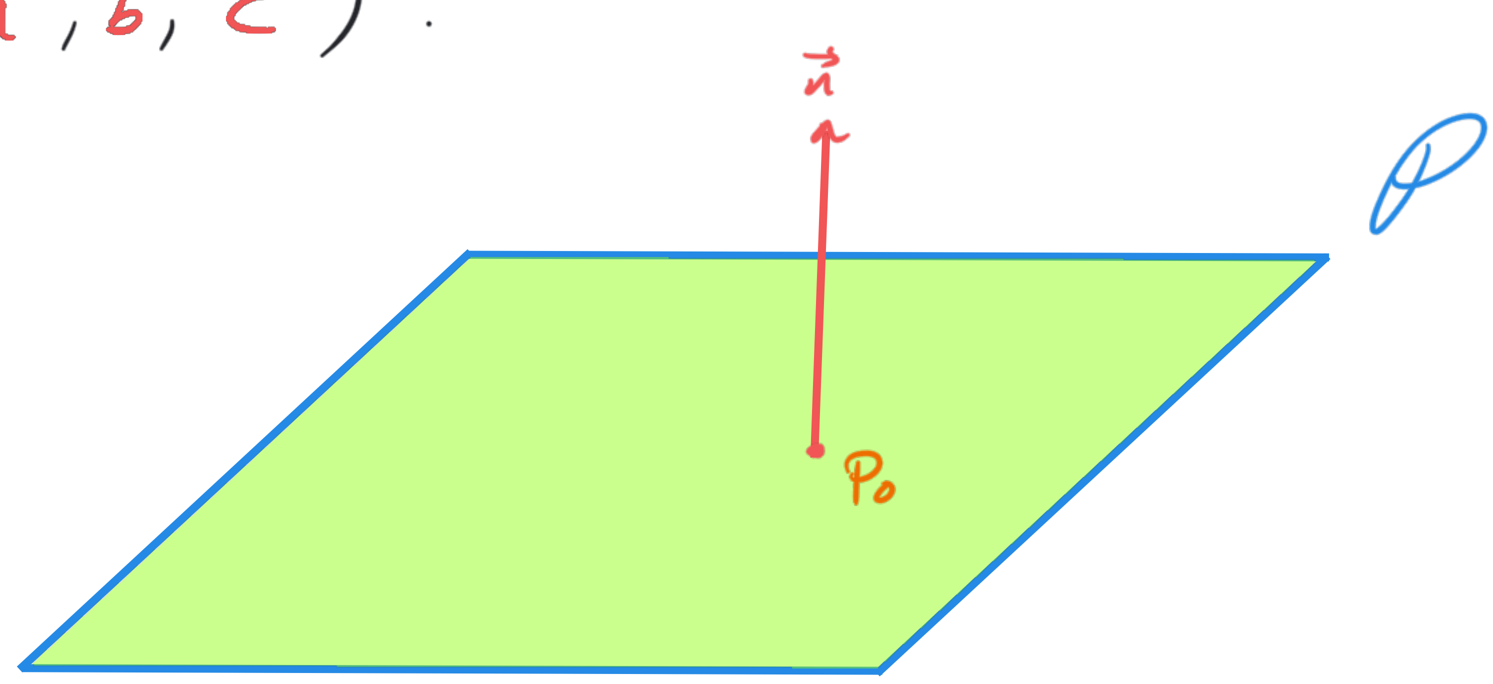
These are called the symmetric equations of L.

- We can geometrically see that a Plane $\mathcal{P}$ is described uniquely by a point it contains, and two vectors that lie in $\mathcal{P}$.

Alternatively, and more usefully, $\mathcal{P}$ can be described by a point it contains and a vector that is orthogonal to the plane.

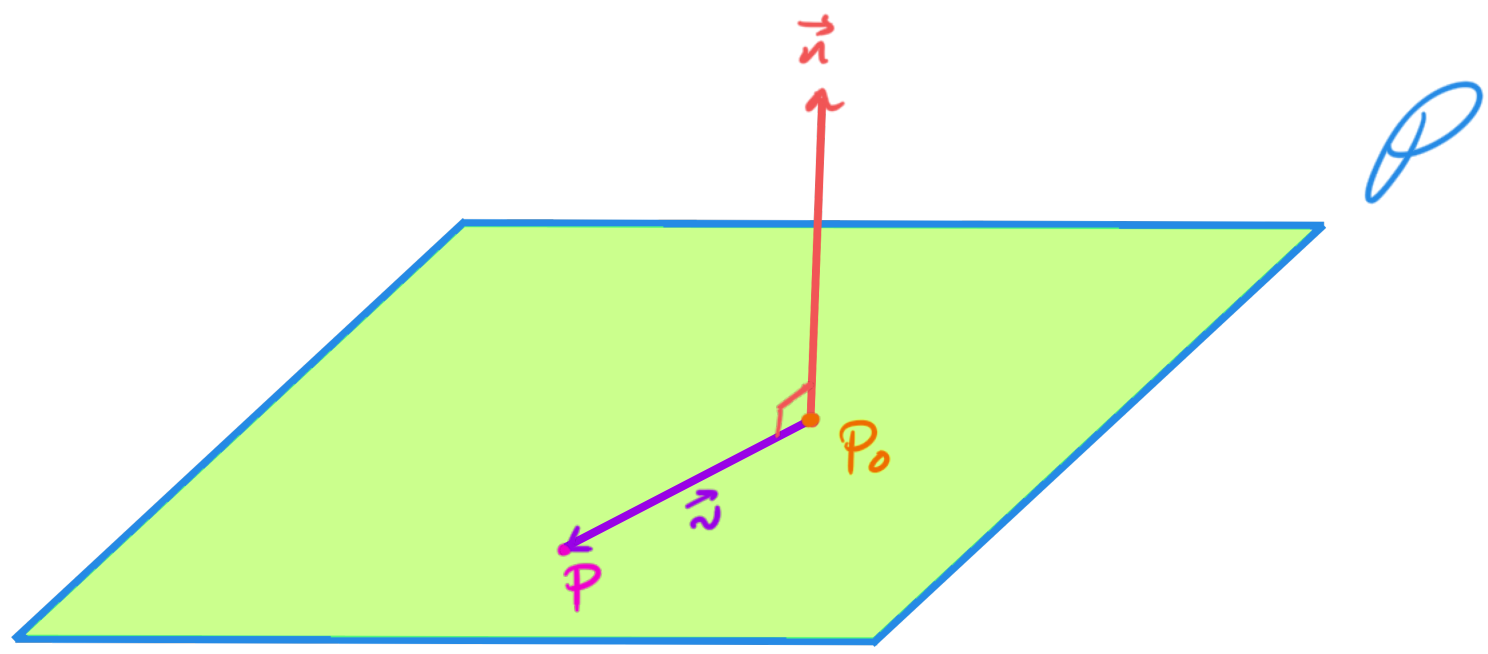
- This orthogonal vector is referred to as being <u>normal to $\mathcal{P}$</u>, and is usually denoted by $\vec{n}$.

- Consider the plane $\mathcal{P}$ below, which contains the point $P_0 = (x_0, y_0, z_0)$ and has normal vector $\vec{n} = (a, b, c)$:



Say $P = (x, y, z)$ is on $\mathcal{P}$.

Then : $\vec{v} = \overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ lies in $\mathcal{P}$:



So we must have $\vec{v} \perp \vec{n}$.

$$\iff \quad \vec{v} \cdot \vec{n} = 0$$

$$\iff \quad (x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

$$\iff \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\iff \quad \boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$

The above uniquely describes $\mathcal{P}$ (criteria).

Remark : We are usually given planes by equations of the form : $ax + by + cz = d$.

Here d is just $ax_0 + by_0 + cz_0$ (simplified).

NB: We can read off $n = (a, b, c)$ from this equation.

Example: A normal vector to the plane given by

$P: \; 2x + y - z = 10$ is $\vec{n} = (2, 1, -1)$.

Another is $\vec{N} = (4, 2, -2)$. Why?

Exercise: Given 3 points, how would you find the plane containing them?

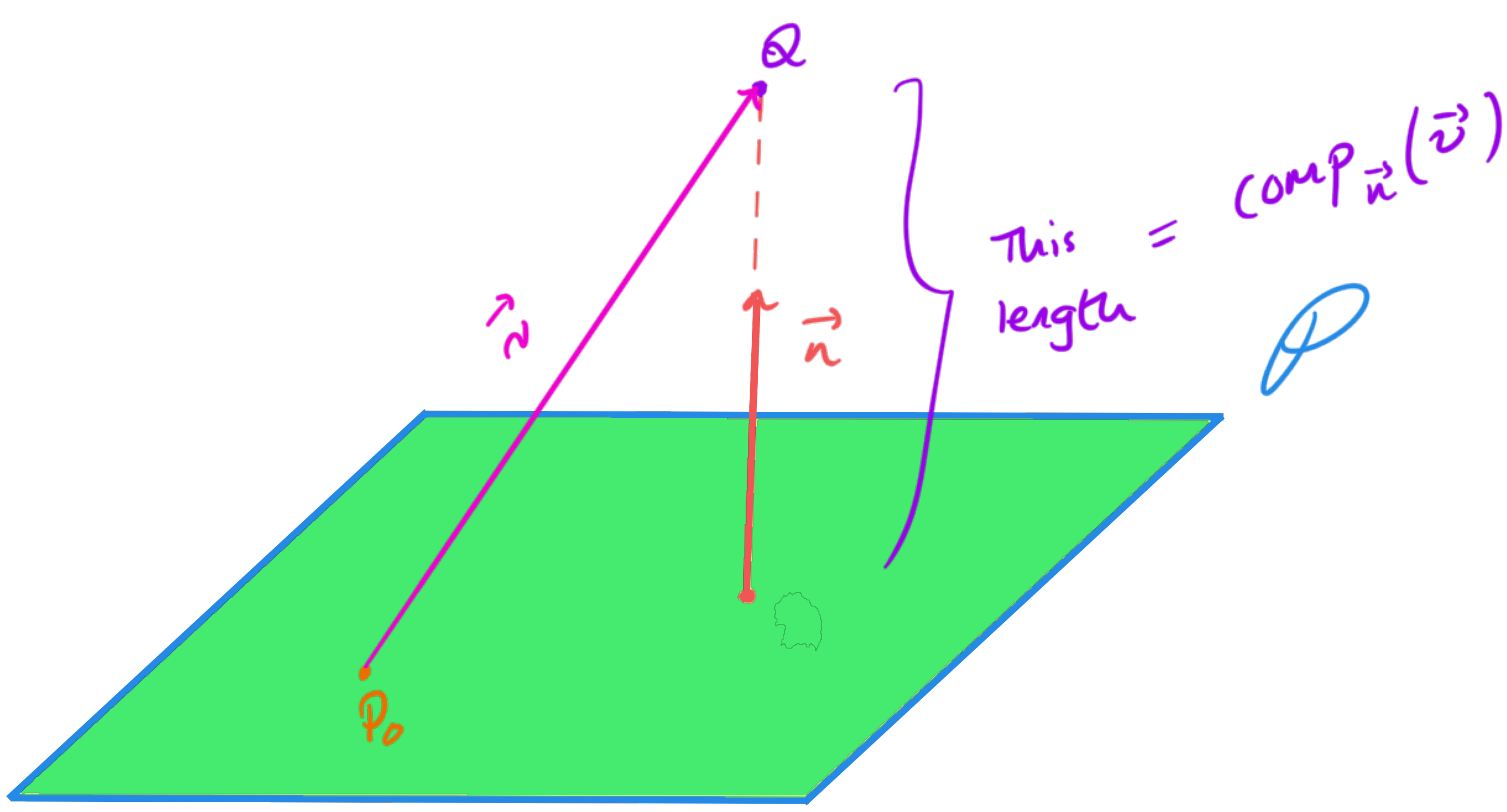
Remark: The angle between two planes can be found by finding the angle between their respective normal vectors.

- If two planes intersect to give a line, you can find the line's direction vector by taking the cross product of the normal vectors of the planes:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2$$

- Alternatively, you can try find the symmetric eq$^{\underline{n}}$s of the line by manipulating the equations of both planes (tutorial problem).

- The formula for the distance of a point to a plane is:



$$\boxed{\text{Dist}(Q, \mathcal{P}) = \frac{|\vec{v} \cdot \vec{n}|}{|\vec{n}|}}$$

5. Vector Functions and Space Curves :

For us, these are two different ways of thinking about the same thing (a function versus it's graph) :

$$r(t) = (x(t), y(t), z(t))$$

You can think about $r(t)$ as being the position of a particle at time t (allow "negative time").

Example : $r(t) = (\cos(t), \sin(t), -t)$.

Draw this space curve / vector valued function / particle trajectory.

Sol^n :

6. Derivatives of Space Curves:

- If $r(t) = (x(t), y(t), z(t))$, then

(i) $r'(t) = \left(x'(t), y'(t), z'(t) \right)$

(ii) $r''(t) = \left(x''(t), y''(t), z''(t) \right)$

(iii) $\int r(t)\,dt = \left(\int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right)$

Remark: If we consider $r(t)$ to be the position of a particle at time t, then $r'(t) = v(t)$ is its velocity and $r''(t) = a(t)$ is its acceleration.

- If $\vec{u}(t)$ and $\vec{v}(t)$ are vector valued functions and $f : \mathbb{R} \xrightarrow{C^1} \mathbb{R}$, then:

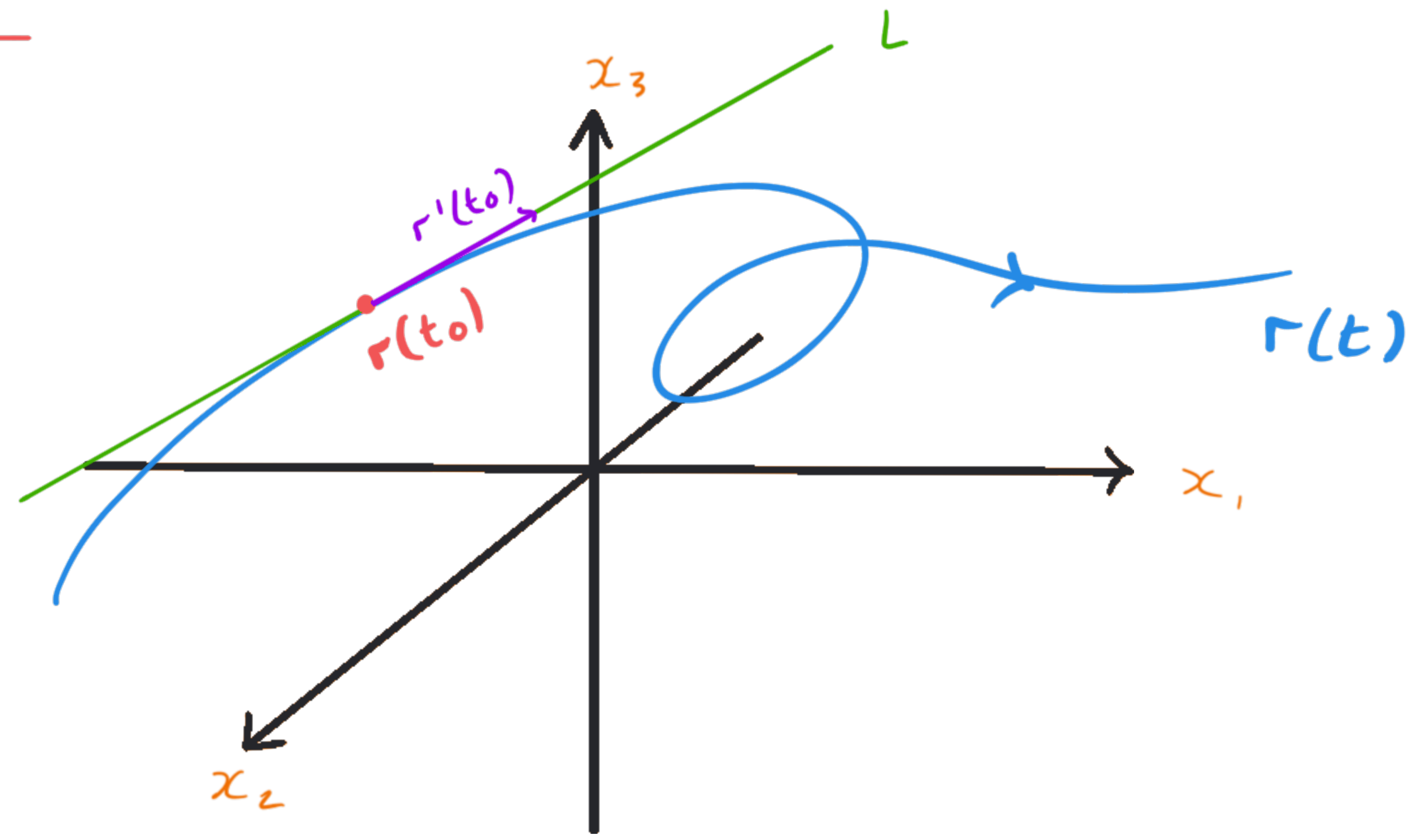
(i) $(\vec{u}(t) \cdot \vec{v}(t))' = u'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

(ii) $(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

(iii) $(f(t)\vec{u}(t))' = f'(t)\,\vec{u}(t) + f(t)\,u'(t)$

(iv) $(\vec{u}(f(t)))' = f'(t)\,\vec{u}\left(f(t)\right)$

Definition:

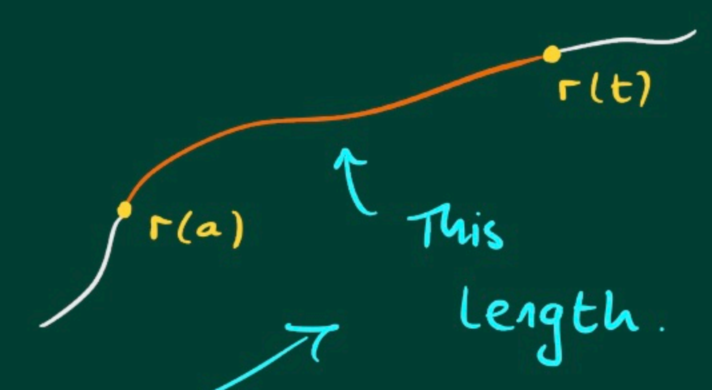


The tangent line to r at $P_0 = r(t_0)$ is the line containing P_0 with direction vector $r'(t_0)$:

$$\boxed{L_{P_0}(s) = P_0 + s\,r'(t_0)}\quad , \quad s \in \mathbb{R}$$

7. Arc Length

·) Arc Length : $\boxed{s(t) = \int_a^t |r'(\tau)| \, d\tau}$ $\leftarrow$

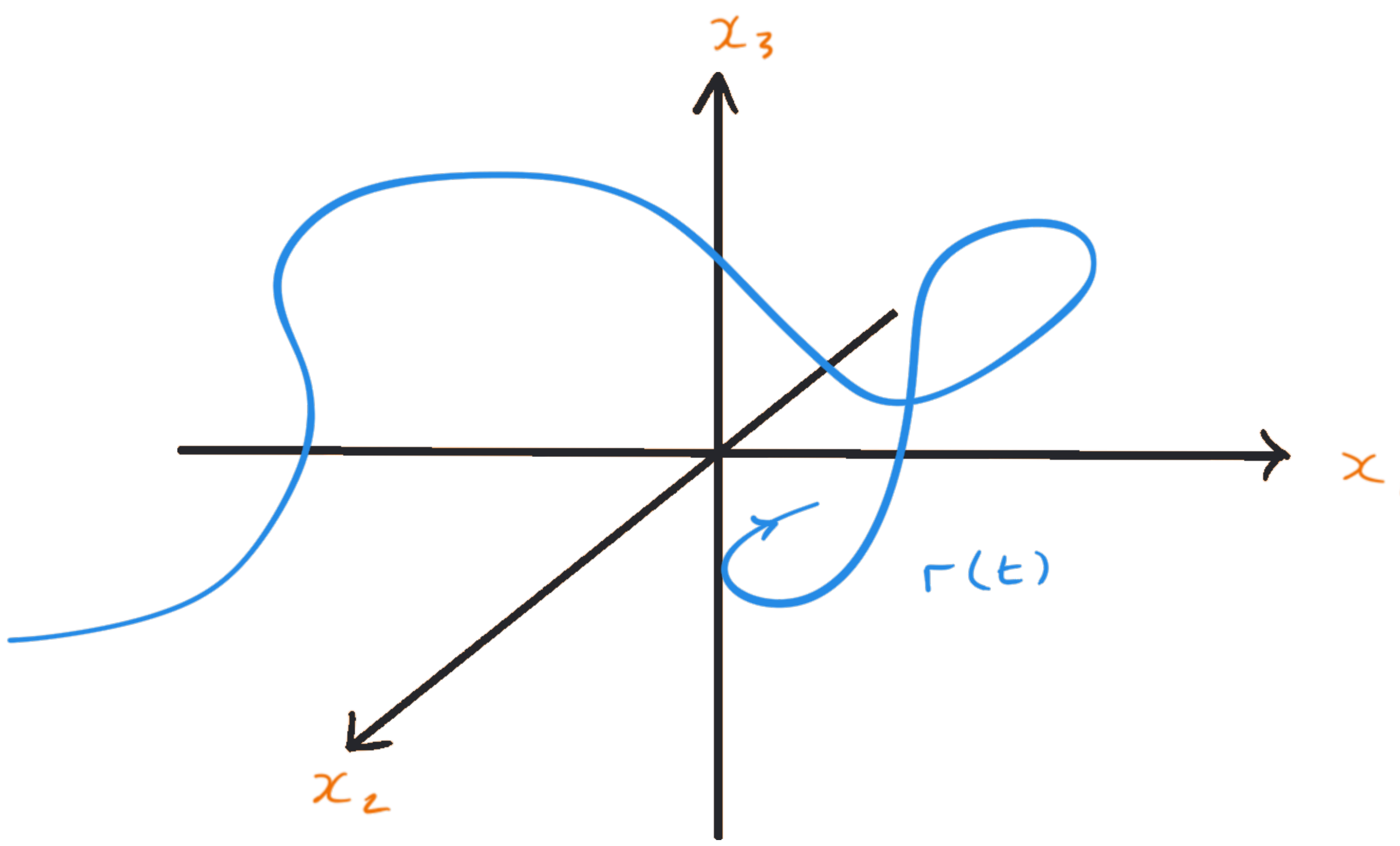


Example: Find the length of the arc of the circular helix $r(t) = (\cos(t), \sin(t), t)$ from the point $(1,0,0)$ to $(1,0,2\pi)$ and from $(1,0,0)$ to $(1,0,4\pi)$.

Sol$\underline{n}$:

8. TNB Frame :

Consider the space curve $r(t)$:



Goal : To describe the "shape of the curve".

Remark : The curve is shaped differently at each point $r(t)$, so whatever description we come up with, if it's "good" will vary from point to point i.e. vary with time.

Idea : One piece of information that tells us something about the shape of the curve is where it's "pointing" / "the direction it's heading" at at each point.

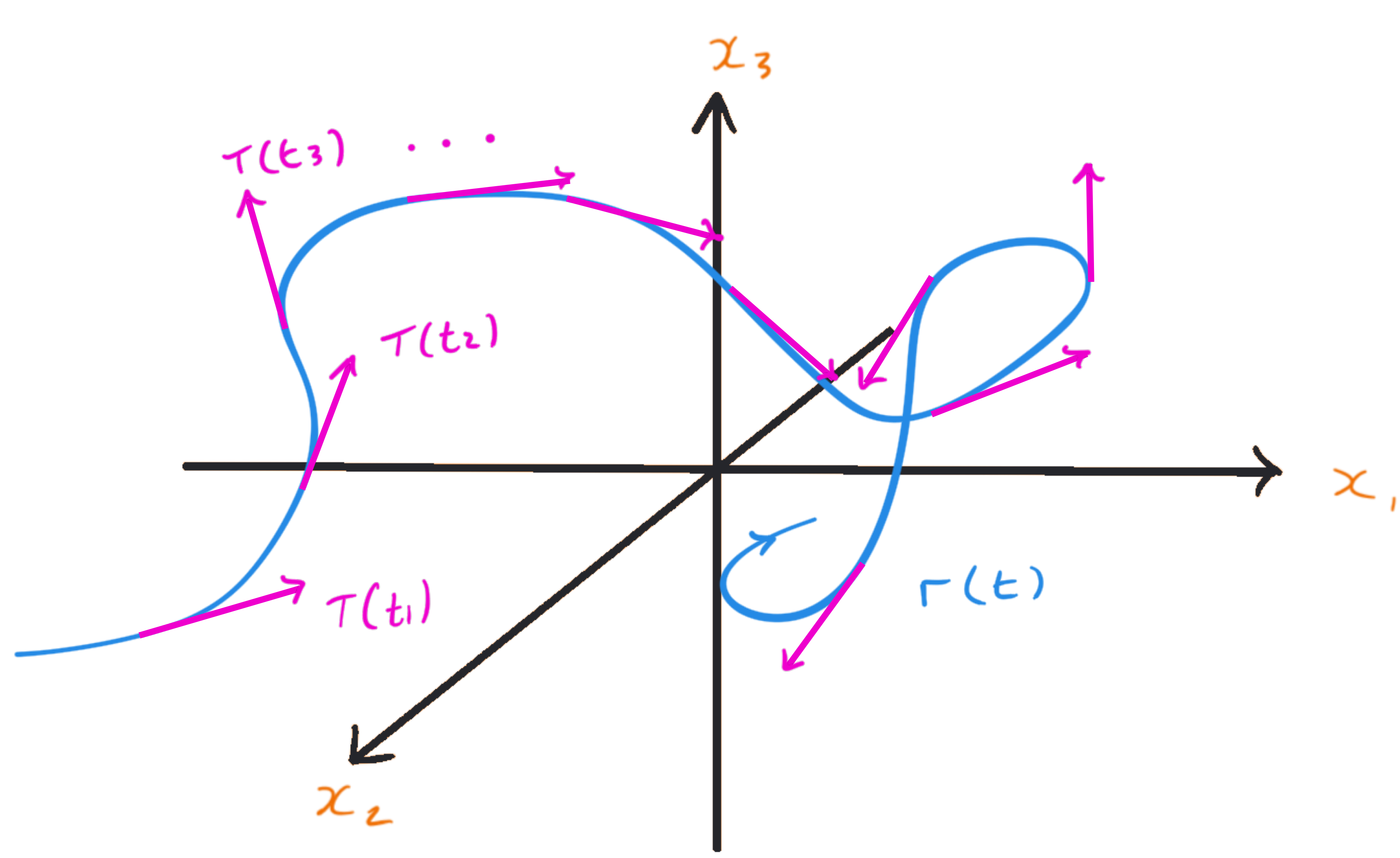
i.e. we should consider $r'(t)$.

But $r'(t)$ has a magnitude, which we don't care about. This is why we concern ourselves with the Unit Tangent Vector:

$$\boxed{T(t) = \frac{r'(t)}{|r'(t)|}}$$

Remark: Here we are assuming our particle never "stops moving": i.e. $r(t) \neq 0$ for any t.

- Let's consider how $T(t)$ varies with time:



Remark: We can see, at the places where the curve is most ... curved, that $T(t)$ changes direction quite rapidly.

This gives us an idea of how the curve is "bending" at a point. This motivates the following definition:

Definition: The unit Normal to r at a point $r(t)$ is given by:

$$\boxed{N(t) := \frac{T'(t)}{|T'(t)|}}$$

Remark: $N(t) \perp T(t)$ for all t.

Definition: We define the Unit Binormal to r at $r(t)$ by:

$$\boxed{B(t) = T(t) \times N(t)}$$

Remark: $B(t) \perp T(t)$ and $N(t)$ for all t.

- To summarize \ visualize :

For a parametrised curve $r(t)$:

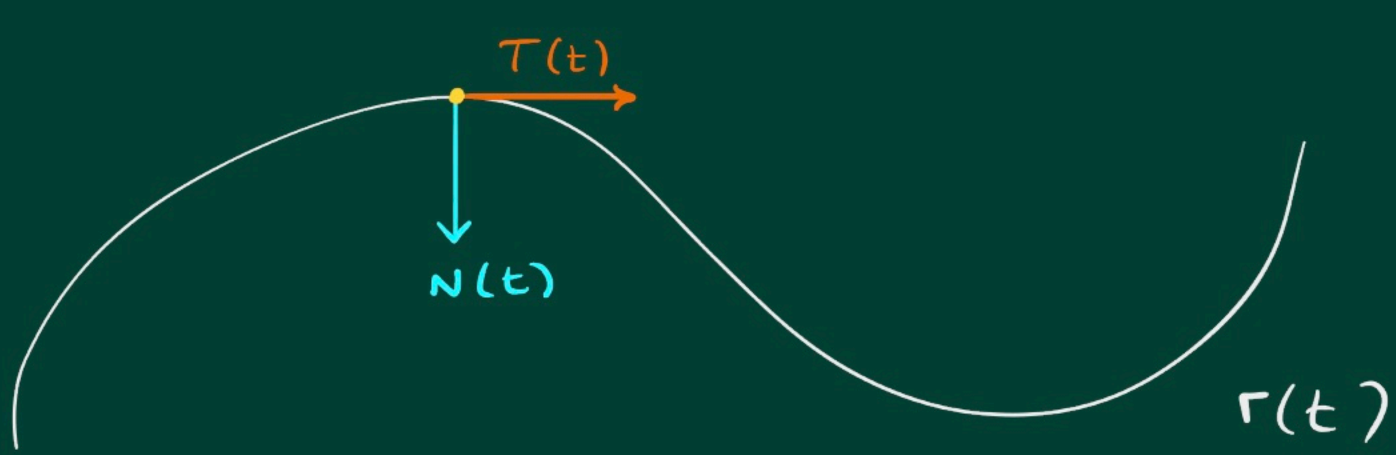
Unit Tangent : $\boxed{T(t) := \dfrac{r'(t)}{|r'(t)|}}$

Unit Normal : $\boxed{N(t) := \dfrac{T'(t)}{|T'(t)|}}$

Unit Binormal : $\boxed{B(t) := T(t) \times N(t)}$

Calc. III is a

Big Terrible Nightmare



$B(t) = T(t) \times N(t)$

Can show: 1) $\boxed{r''(t) =: \vec{a}(t) = \underbrace{a_T(t)}_{\in \mathbb{R}} \vec{T}(t) + \underbrace{a_N(t)}_{\geq 0} \vec{N}(t)}$

2) $\boxed{B(t) = \dfrac{r'(t) \times r''(t)}{|r'(t) \times r''(t)|}}$

3) $\boxed{N(t) = B(t) \times T(t)}$

... but it's

Not Bad Today

NB: • When you are asked to calculate these quantities, the above formulas make things a lot easier.

• If you are asked to evaluate these at a given point, always evaluate $r'(t)$ and $r''(t)$ at the point first and then compute!

·) $$\boxed{a_T(t) = \frac{r'(t) \cdot r''(t)}{|r'(t)|}} \qquad \boxed{a_N(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|}}$$

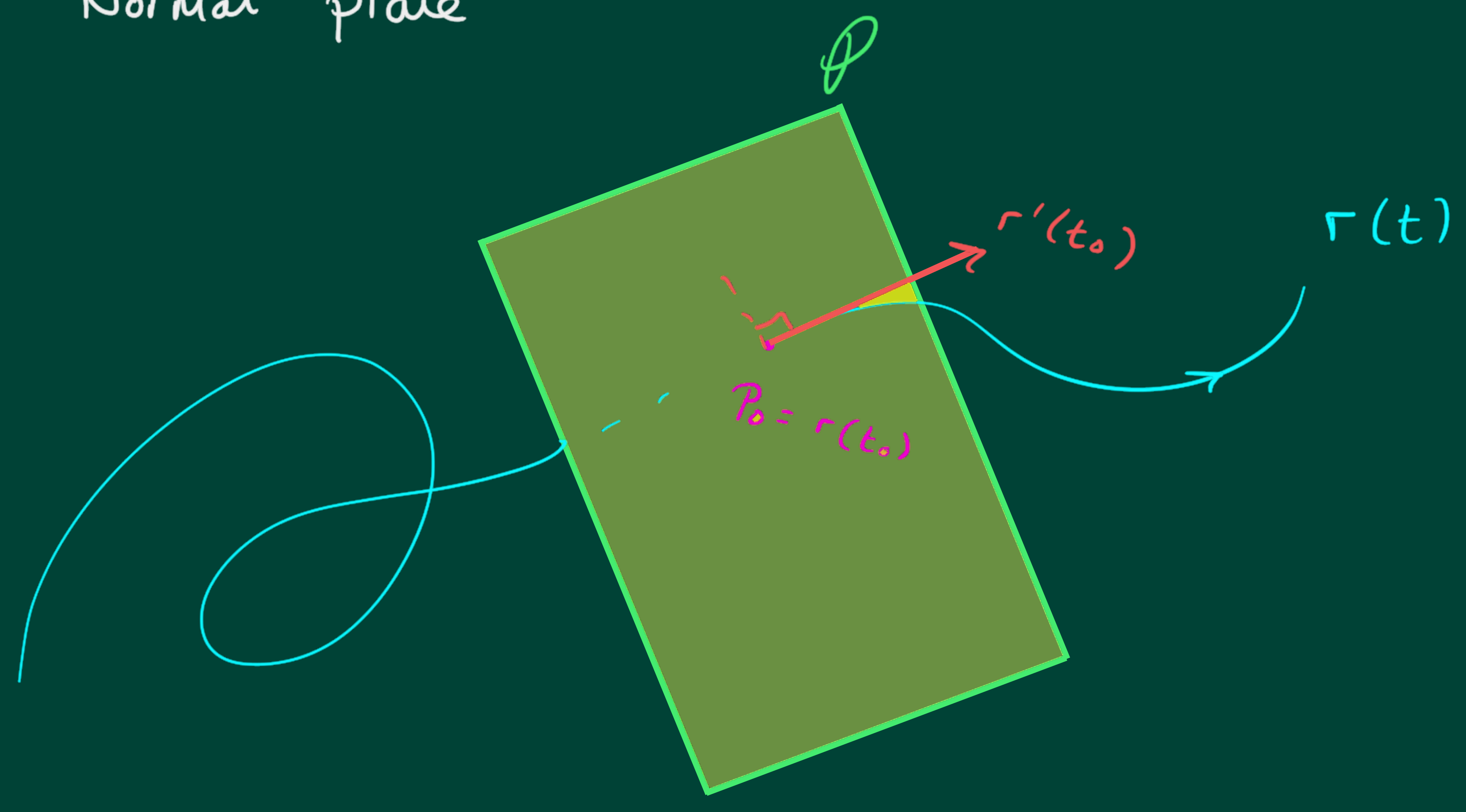
·) Trick : $$\boxed{a_N(t) = \sqrt{|\vec{a}(t)|^2 - a_T(t)^2}}$$

Definition: The plane at a point $r(t_0)$ on a space curve r determined by $N(t_0)$ and $B(t_0)$ is called the Normal Plane of r at $r(t_0)$.

Shortcut: $r'(t_0)$ is normal to this plane. ← NB

Remark: It consists of all lines $\perp$ to the tangent line.

Fig 1 : Normal plane



<u>Definition</u>: The Osculating Plane of a curve C (parametrized by $r(t)$) at a point $P_0 = r(t_0)$ is the plane determined by $T(t_0)$ and $N(t_0)$.

<u>Shortcut</u>: $r'(t_0) \times r''(t_0)$ is normal to this plane.

NB

<u>Remark</u>: Intuitively, the osculating plane is the plane that comes closest to containing part of the curve near P_0.

Fig 2: Osculating Plane