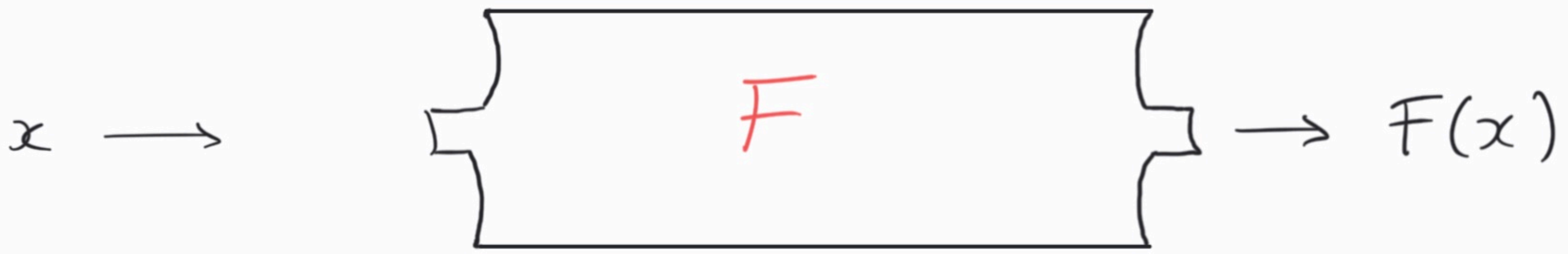
## §14. Chain Rule:

### Goals for Today:

1) Introduce the Chain Rule for a function of several variables.

2) Use this to expand our previous knowledge of Implicit Differentiation.

**Recall:** In Calc. I we see how to break complicated relationships between a single input $x$ and a single output $F(x)$:




By breaking it into a chain of simpler processes:

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x))$$

such that $F(x) = f(g(x))$

i.e. $F = f \circ g$

We arrived at the one dimensional chain rule:

$$\boxed{(f \circ g)'(x) = f'(g(x))\, g'(x)}$$

So, let's proceed with our objectives for this lecture:

Example: Say we want to design a new car.

What will determine / influence it's top speed?

1) Weight $(w)$

2) Aero dynamic efficiency $(s)$

3) Engine Power $(p)$

4) Tire grip $(g)$

5) Transmission efficiency $(t)$

Say $F$ is our top speed function.

Algebraically it might look something like:

$$F(w, s, p, g, t) = \frac{(s+p+t)g}{w}$$

So $\frac{\partial F}{\partial w}$ would be ____________________.

$\frac{\partial F}{\partial s}$ would be ____________________.

$\vdots$

But if we think a layer deeper, each variable $w, s, p, g, t$ could depend on other variables:

1) Weight :
- ↳ Density of material ($\rho$)
- ↳ Surface Area of Car ($A$)

2) Aero-dynamic efficiency :
- ↳ Surface Area of Car ($A$)
- ↳ "Narrowness" of Car ($N$)

3) Engine power :
- ↳ Number of cylinders ($c$)
- ↳ Size of engine ($z$)

4) Tire grip :
- ↳ Thread depth ($d$)
- ↳ Tire size ($\tau$)
- ↳ Material density ($m$)

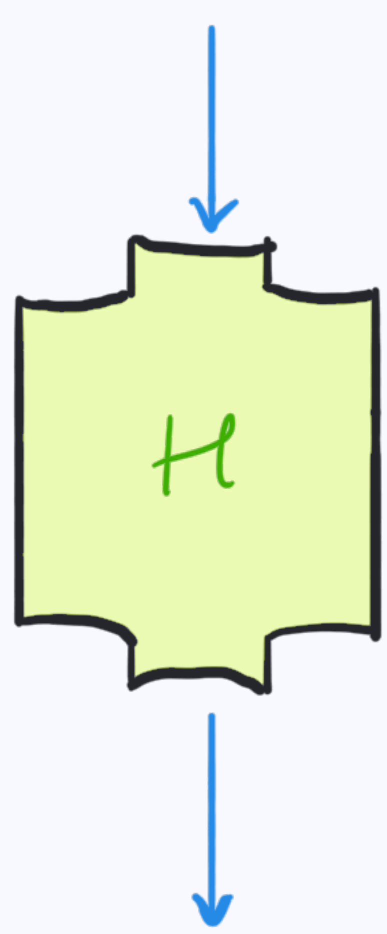
5) Trasmission :
- ↳ Number of cogs ($\sigma$)
- ↳ Friction between cogs ($\mu$).

We can represent these dependencies by writing

$$w(\rho, A),\ s(A, N),\ p(c, z),\ g(d, \tau, m),\ t(\sigma, \mu)$$

So now we can think of the machine that takes in values for $(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$ and spits out the corresponding top speed:

$$(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$$

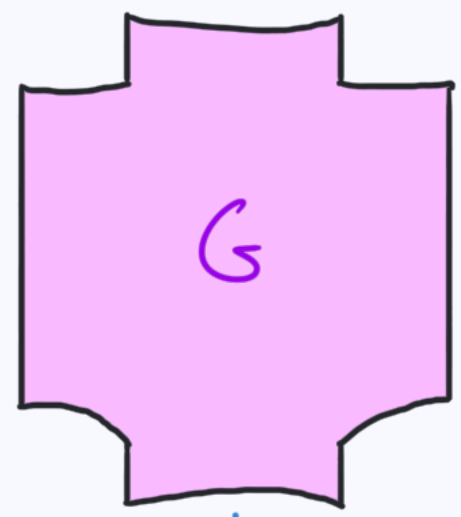



$$H(\rho, A, N, c, z, d, \tau, \mu, \sigma, \lambda)$$

This machine would have very complicated relationships with all of the variables it depends on.

So we try to break it into a chain of simpler machines:

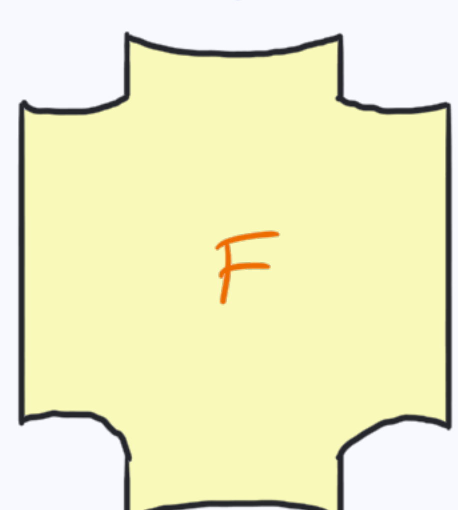
$$(\rho, A, N, c, z, d, \tau, M, \sigma, \lambda)$$




$$G(\rho, A, N, c, z, d, \tau, M, \sigma, \lambda)$$

$$\parallel$$

$$(w(\rho, A), s(A, N), p(c, z), g(d, \tau, M), t(\sigma, \lambda))$$




$$F(w(\rho, A), s(A, N), p(c, z), g(d, \tau, M), t(\sigma, \lambda))$$

Such that $H = F \circ G$.

Then the multivariable chain rule says:

$$\frac{\partial H}{\partial e} = \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial e} + \frac{\partial F}{\partial s} \cdot \frac{\partial s}{\partial e} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial e} + \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial e} + \frac{\partial F}{\partial t} \cdot \frac{\partial t}{\partial e}$$

$$\frac{\partial H}{\partial A} = \cdots$$

$$\vdots$$

In general: <u>The Chain Rule</u>

If $u$ is a differentiable function of $n$ variables: $x_1, x_2, \dots, x_n$

and each $x_j$ is a differentiable function of $m$ variables: $t_1, t_2, \dots, t_m$, then $u$ is a function of $t_1, t_2, \dots, t_m$

and:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

for any $i = 1, 2, \dots, m$.

<u>Remark:</u> Don't worry too much about this formula. It is very general, and we are mainly concerned with the following two versions, which correspond to the cases:

↳ $m = 1$, $n = 2$

↳ $m = 2$, $n = 2$

## "Just $t$":

If we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ : $f(x,y)$, and we are given $x = g(t)$, $y = h(t)$, i.e. $x$ and $y$ are now considered to be functions of $t$, we can write $z(t) = f(g(t), h(t))$. i.e. we have the process:

$$t \longrightarrow \boxed{(g,h)} \longrightarrow (g(t), h(t)) \longrightarrow \boxed{f} \longrightarrow f(g(t), h(t))$$

Combined to:

$$t \longrightarrow \boxed{z} \longrightarrow z(t) \quad (= f(g(t), h(t)))$$

And we want to see how sensitive $z(t)$ is to a change in $t$:

$$\boxed{\frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(g(t), h(t)) \cdot \frac{dg}{dt}(t) + \frac{\partial f}{\partial y}(g(t), h(t)) \cdot \frac{dh}{dt}(t)}$$

More compactly:

$$\boxed{\frac{dz}{dt}(t) = f_x(g(t), h(t))\, g'(t) + f_y(g(t), h(t))\, h'(t)}$$

## Examples:

1) **3.**(6 pts) If $z = f(x, y)$, where $f$ is differentiable, and $x = g(t), y = h(t)$, $g(1) = 3$, $h(1) = 4$, $g'(1) = -2$, $h'(1) = 5$, $f_x(3, 4) = 7$ and $f_y(3, 4) = 6$. Find $dz/dt$ when $t = 1$.

(a) 13 (b) 44 (c) 32 (d) 23 (e) 16

**Sol$\underline{n}$:** ("Just t")

So $x = g(t)$, $y = h(t)$ and hence $z(t) = f(g(t), h(t))$.

By the formula:

$$\frac{dz}{dt}(1) = \frac{\partial f}{\partial x}(g(1), h(1)) \cdot g'(1) + \frac{\partial f}{\partial y}(g(1), h(1))\, h'(1)$$

$$= f_x(3, 4)(-2) + f_y(3, 4)(5)$$

$$= 7(-2) + 6(5)$$

$$= 16$$

## "s and t":

If we have a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y)$ and we are given $x = g(s,t)$, $y = h(s,t)$, i.e $x$ & $y$ are considered to be functions of $s$ and $t$, we can write $z(s,t) = f\big(g(s,t), h(s,t)\big)$ i.e. we have the process:

$$(s,t) \longrightarrow \boxed{(g,h)} \longrightarrow \big(g(s,t),h(s,t)\big) \longrightarrow \boxed{f} \longrightarrow f\big(g(s,t),h(s,t)\big)$$

Combined to:

$$(s,t) \longrightarrow \boxed{z} \longrightarrow z(s,t) \quad \Big(= f\big(g(s,t),h(s,t)\big)\Big)$$

And we want to see how sensitive $z(s,t)$ is to a change in $s$ or $t$:

$$\frac{\partial z}{\partial t}(s,t) = \frac{\partial f}{\partial x}\big(g(s,t),h(s,t)\big) \cdot \frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}\big(g(s,t),h(s,t)\big)\frac{\partial h}{\partial t}(s,t)$$

&

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}\big(g(s,t),h(s,t)\big) \cdot \frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(g(s,t),h(s,t)\big)\frac{\partial h}{\partial s}(s,t)$$

2) **5.**(6 pts) Let $f(x,y)$ be a function of $x(s,t)=st$ and $y(s,t)=2s+t$. If you know that $f_x(1,3)=2$ and $f_y(1,3)=-3$ then what is $\partial f/\partial s$ at when $s=1$ and $t=1$?

(a) $-1$

(b) not enough information to determine the value

(c) 3

(d) $-4$

(e) 0

Sol$^{\underline{n}}$: ( "s and t" )

Here we are given $x = x(s,t) = st$ and $y = y(s,t) = 2s+t$.

Hence $z(s,t) = f\left(x(s,t), y(s,t)\right)$.

We are asked for $\frac{\partial z}{\partial s}(1,1)$.

By the formula:

$$\frac{\partial z}{\partial s}(1,1) = f_x\left(x(1,1), y(1,1)\right)\frac{\partial x}{\partial s}(1,1) + f_y\left(x(1,1), y(1,1)\right)\frac{\partial y}{\partial s}(1,1)$$

• $x(s,t) = st \Rightarrow \frac{\partial x}{\partial s}(s,t) = t$

Hence $x(1,1) = 1$ & $\frac{\partial x}{\partial s}(1,1) = 1$

• $y(s,t) = 2s+t \Rightarrow \frac{\partial y}{\partial s}(s,t) = 2$

Hence $y(1,1) = 3$ & $\frac{\partial y}{\partial s}(1,1) = 2$

$$\Rightarrow \frac{\partial z}{\partial s}(1,1) = f_x(1,3)(1) + f_y(1,3)(2) = (2)(1)+(-3)(2)$$

$$= -4$$

## Implicit Differentiation:

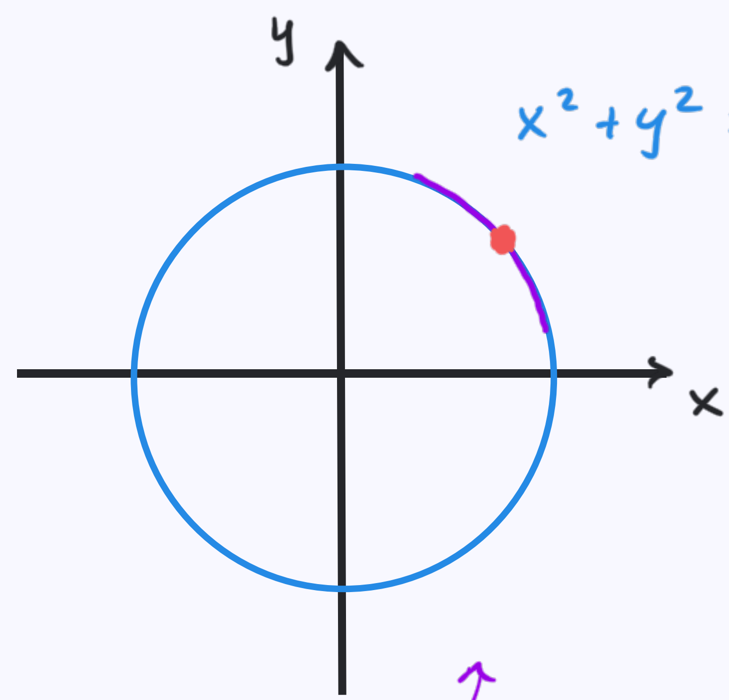
Recall in Calc. I we learned how to find tangents to curves defined by equations. e.g. $x^2 + xy = 2y^2$

By treating $y$ (locally) as a function of $x$ we were able to find $\frac{dy}{dx}$:

e.g.

This is not the graph of a function $y = f(x)$, but we can write $y$ locally as a function of $x$:

$y = \sqrt{1-x^2}$ around red point




$x^2+y^2=1$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$, for $y \neq 0$.

↑ depends on $y$. Why?

- Let's apply what we've learned about chain rule.

Let's think of $y$ as a function of $x$ (locally).

So we can think of the machine:

$$x \longrightarrow \boxed{\phantom{xx}} \longrightarrow (x, y(x))$$

Let's go back to our example curve:

**Example:** Find the slope of the tangent line to the curve $x^2 + xy = 2y^2$ at the point $(1,1)$.

**Sol$^{n}$:** We will try to outline a general method here.

**Step 1:** Bring everything to the LHS.

$$x^2 + xy - 2y^2 = 0 \qquad (*)$$

**Step 2:** Let $F(x,y) =$ LHS.

$$F(x,y) = x^2 + xy - 2y^2$$

**Idea:** If $y$ is a function of $x$, we can think of the LHS as $F(x, y(x))$. Writing this as a chain:

$$x \longrightarrow \boxed{\phantom{xx}} \longrightarrow (x, y(x)) \longrightarrow \boxed{\phantom{xx}} \longrightarrow F(x, y(x))$$

From the "just t" chain rule (here "$t = x$"), if we take the derivative of $F(x, y(x))$ w.r.t. $x$ we get:

$$F_x(x, y(x)) \underbrace{\frac{dx}{dx}}_{1}(x) + F_y(x, y(x)) \frac{dy}{dx}(x)$$

$$= F_x(x, y(x)) + F_y(x, y(x)) \frac{dy}{dx}(x) \qquad (*)$$

But $(*)$ says $F(x, y(x)) = 0$ on this curve.

So that derivative we got, $(*)$, should be zero.

i.e. $F_x(x, y(x)) + F_y(x, y(x)) \frac{dy}{dx}(x) = 0$

Solving for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx}(x) = \frac{-F_x(x, y(x))}{F_y(x, y(x))}}$$

More compactly:

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}} \qquad (*)$$

If you are not concerned with theory, you can just memorize the formula in the purple box and proceed:

Step 3: $F_x(x, y) = 2x + y$

$F_y(x, y) = x - 4y$

$(*) \Rightarrow \frac{dy}{dx} = -\frac{(2x+y)}{x - 4y}$

$$\Rightarrow \frac{dy}{dx}(1,1) = -\frac{3}{-3} = \boxed{1}$$

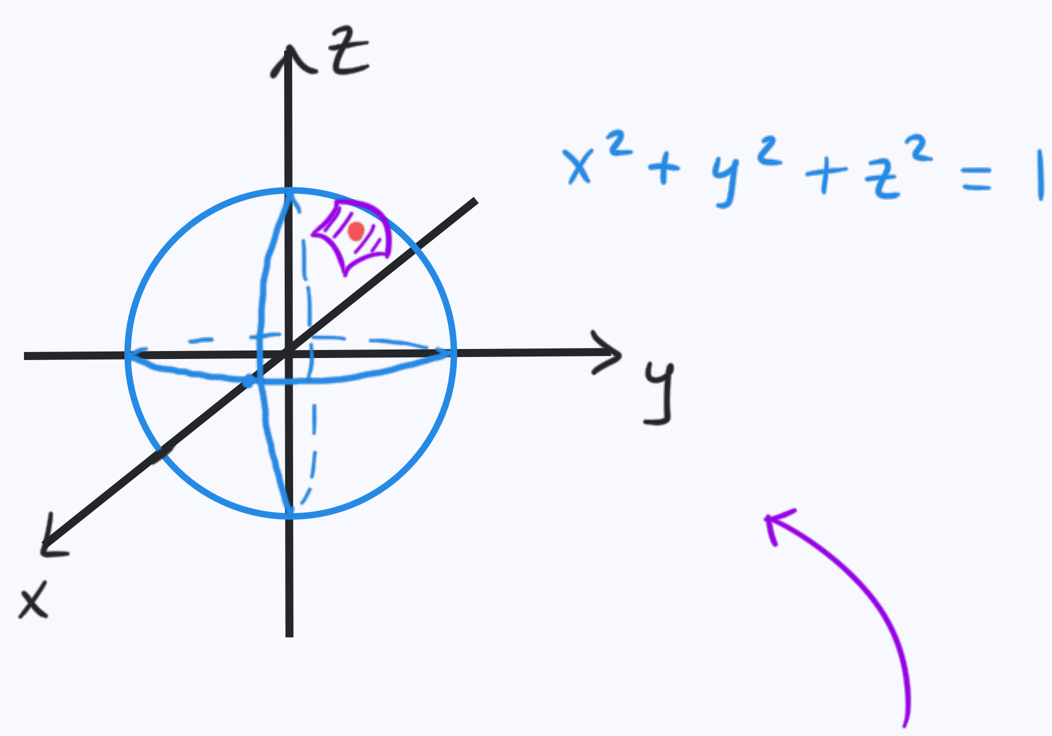
- Again, when answering a question, go straight from step 2 to step 3.

Even more variables: If instead of having a function define a "surface" $z = f(x,y)$, what if we were just given an equation:

e.g. $z^2 + 2xz + 3yz = 2xy$

Under certain circumstances (which will always be satisfied in examples we see in this course) we can think of $z$ as being locally a function of $x$ & $y$.

e.g.




This is not the graph of a function $z = f(x,y)$, but we can write $z$ locally as a function of $x$ & $y$: $z = \sqrt{1 - x^2 - y^2}$ "around" the red point.

- Let's apply what we've learned about chain rule.
  Let's think of $z$ as a function of $x$ and $y$ (locally).

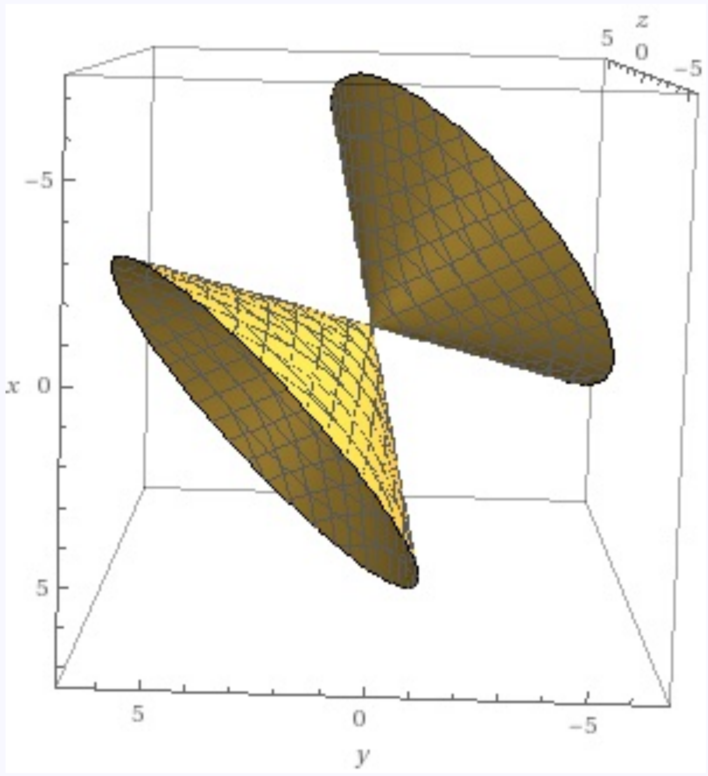
So we can think of the machine:

$$(x,y) \longrightarrow \boxed{\phantom{XX}} \longrightarrow (x, y, z(x,y))$$

Back to our example:

**Example:** Find $\frac{\partial z}{\partial x}$ at $(1,-3,1)$ for the surface described by: $z^2 + 2xz + 3yz = 2xy$

**Aside:**
This is a graph of the "surface":
What does $\frac{\partial z}{\partial x}$ "mean" at $(1,-3,1)$?




**Sol$^{\underline{n}}$:** We will again try to give a general method here, with some motivation/explanation between step 2 and step 3 which you can ignore if you're not interested.

**Step 1:** Gather everything to the LHS.

$$z^2 + 2xz + 3yz - 2xy = 0 \qquad (*)$$

**Step 2:** Let $F(x,y,z) = \text{LHS}$.

$$F(x,y,z) = z^2 + 2xz + 3yz - 2xy$$

Idea: If $z$ is a function of $x$ and $y$, we can think of the LHS as $F(x,y,z(x,y))$. Writing this as a chain:

$$(x,y) \longrightarrow \boxed{\phantom{XX}} \longrightarrow (x,y,z(x,y)) \longrightarrow \boxed{\phantom{XX}} \longrightarrow F(x,y,z(x,y))$$

We can see how sensitive this end output is to a change in $x$ by using chain rule (very similar to "s and t" case):

$$F_x(x,y,z(x,y)) \cdot \underbrace{\frac{\partial x}{\partial x}(x,y)}_{=1} + F_y(x,y,z(x,y)) \underbrace{\frac{\partial y}{\partial x}(x,y)}_{\substack{=0 \\ \text{as } y \text{ is} \\ \text{indep. of} \\ x \text{ here}}} + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y)$$

$$= F_x(x,y,z(x,y)) + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y) \qquad (*)$$

But (*) says $F(x,y,z(x,y)) = 0$ on this surface, so it's partial derivative w.r.t. $x$ should be zero.

i.e. by (*) we have:

$$F_x(x,y,z(x,y)) + F_z(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y) = 0$$

Isolating $\frac{\partial z}{\partial x}(x,y)$ we arrive at:

$$\boxed{\frac{\partial z}{\partial x}(x,y) = \frac{-F_x(x,y,z(x,y))}{F_z(x,y,z(x,y))}}$$

or, more compactly:

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}} \qquad (*)$$

similarly with $y$ instead of $x$:

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}} \qquad (*)$$

Step 3: Apply relevant formula in purple box.

$$F_x(x,y,z) = 2z - 2y$$

$$F_z(x,y,z) = 2z + 2x + 3y$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-(2z-2y)}{2z+2x+3y}$$

So, at $(1,-3,1)$:

$$\frac{\partial z}{\partial x} = -\frac{8}{-5} = \boxed{\frac{8}{5}}$$

## Example :

**2.**(6 pts) Use implicit differentiation to find $\partial z/\partial x$ when $xz + z^2 = y$.

(a) $\dfrac{\partial z}{\partial x} = \dfrac{-z}{x+2z}$

(b) $\dfrac{\partial z}{\partial x} = \dfrac{y}{x+z}$

(c) $\dfrac{\partial z}{\partial x} = \dfrac{-x}{2z}$

(d) $\dfrac{\partial z}{\partial x} = \dfrac{y-z}{x+2z}$

(e) $\dfrac{\partial z}{\partial x} = \dfrac{y-x}{2z}$

## Exam Question Method :

Step 1 : $xz + z^2 - y = 0$

Step 2 : $F(x,y,z) = xz + z^2 - y$

Step 3 : $F_x(x,y,z) = z$

$F_z(x,y,z) = x + 2z$

Step 4 : $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_y} = \dfrac{-z}{x+2z}$