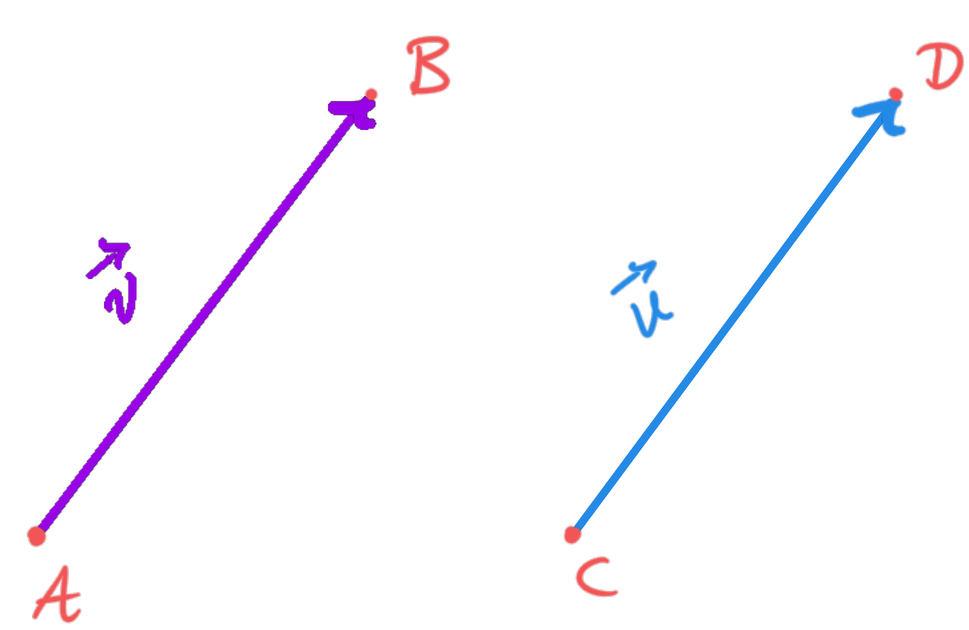
§2. Vectors :

When scientists talk about 'vectors', they are referring to something which has a 'magnitude' (or 'length') and a 'direction'.

Vectors can be denoted in multiple ways :

$$\vec{v}\ ,\ \mathbf{v}\ ,\ \underline{v}\ ,\ \ldots$$

Fig. 1 :



Consider the above vectors.

$\vec{v}$ moves from the point A to the point B.

We express this by writing $\vec{v} = \overrightarrow{AB}$.

Notice $\vec{u} = \overrightarrow{CD}$ has the same length and direction as $\vec{v}$.

We actually identify these vectors and write $\vec{v} = \vec{u}$.

Remark: The zero vector $\vec{0}$ has zero length and is the only vector with no specific direction.

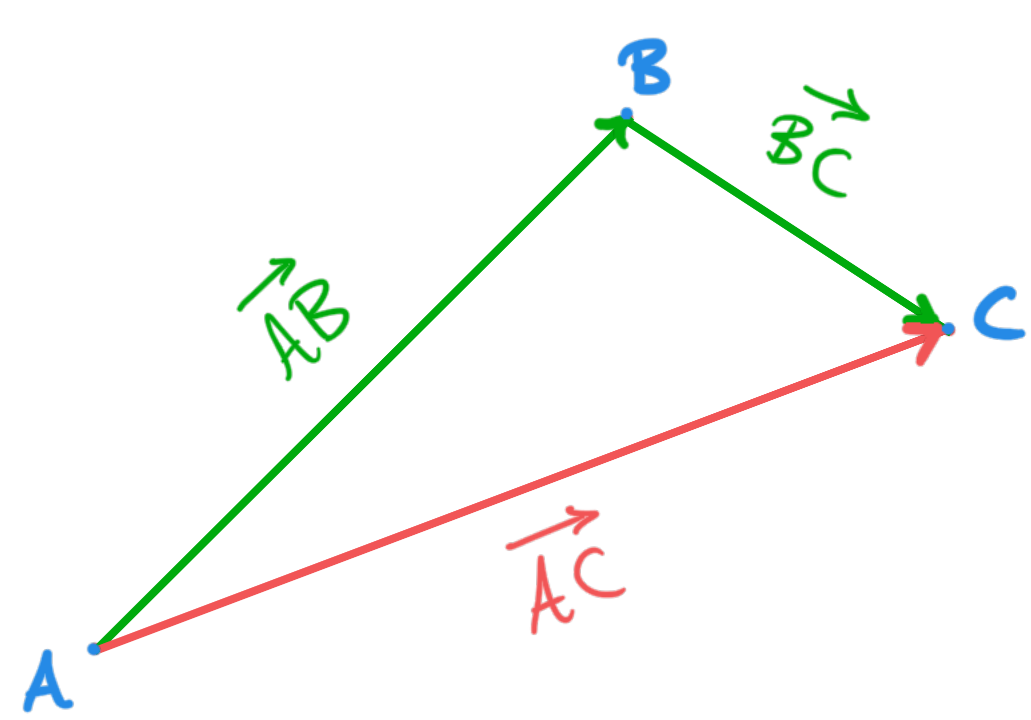
- Imagine a particle moves from a point A to a point B.

  So its displacement vector is given by $\vec{AB}$.

  Suppose the particle then moves from B to C.

  This displacement vector is given by $\vec{BC}$.

Fig. 2:



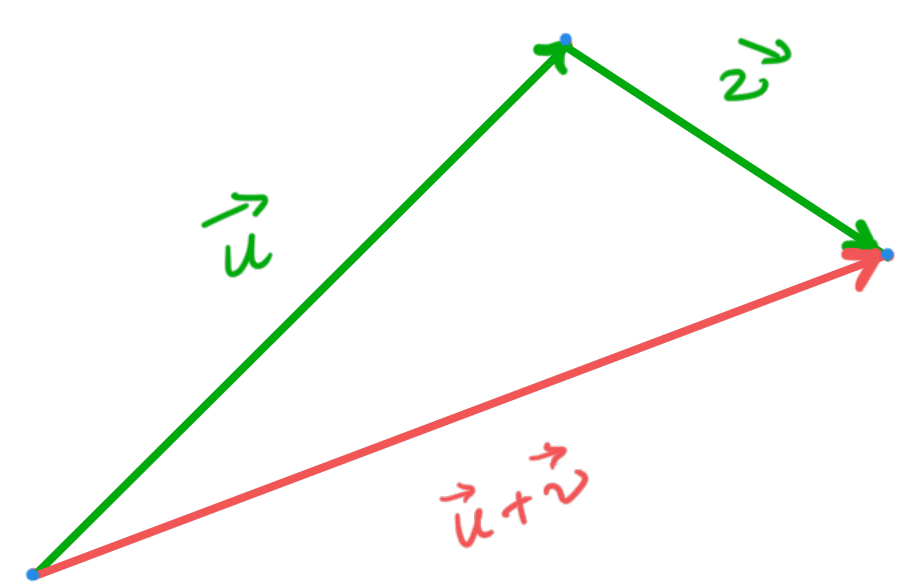
The resulting red vector $\vec{AC}$ is called the sum of $\vec{AB}$ and $\vec{BC}$ and we write:

$$\boxed{\vec{AC} = \vec{AB} + \vec{BC}}$$

Definition: (Vector Addition)

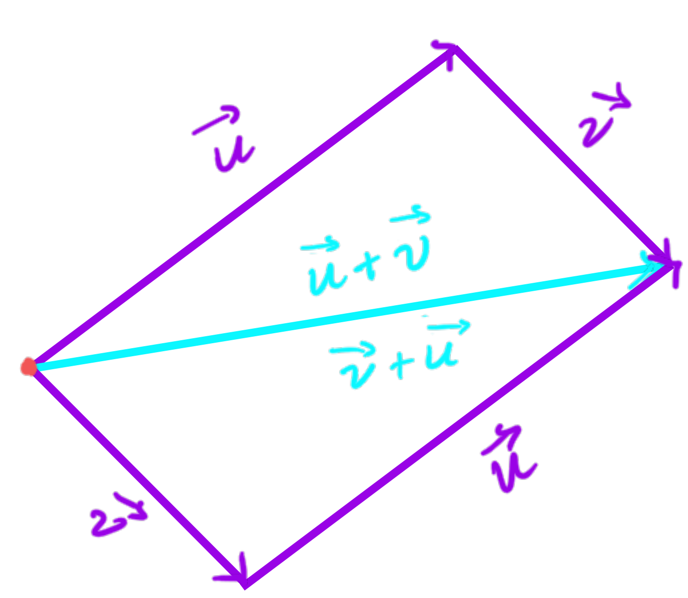
If $\vec{u}$ and $\vec{v}$ are vectors, positioned such that the initial point of $\vec{v}$ coincides with the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the inital point of $\vec{u}$ to the terminal point of $\vec{v}$.

Fig. 3: (Triangle Law)



We can see by symmetry that $\vec{u}+\vec{v} = \vec{v}+\vec{u}$:

Fig. 4: (Parallelogram Law)



Question: What should we get if we add $\vec{v}$ and $\vec{v}$?

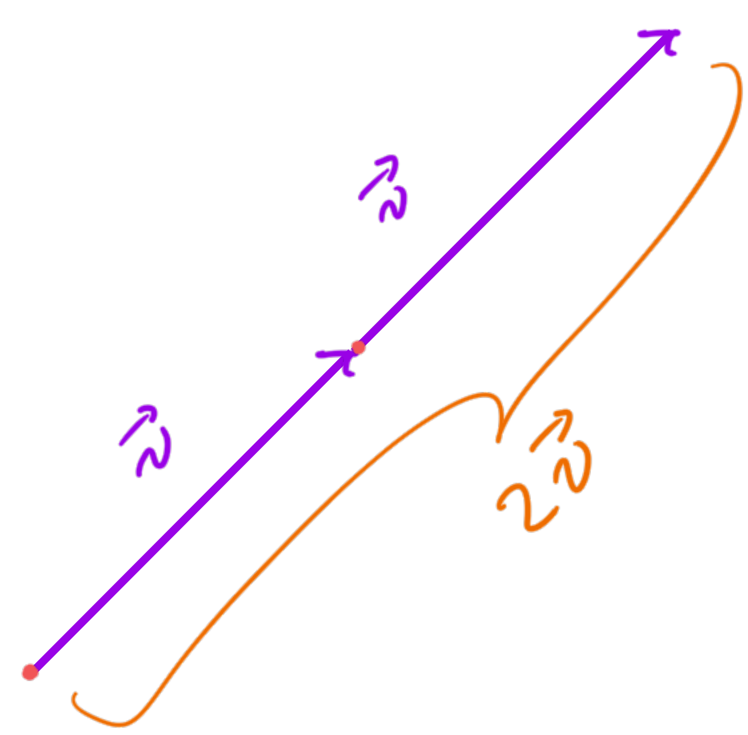
Algebraically, we should want: $\vec{v} + \vec{v} = 2\vec{v}$.

But what does "$2\vec{v}$" mean?

What does it mean to multiply a vector by a real number?

The answer is motivated by the geometry:

Fig. 5:



Direction? No change.

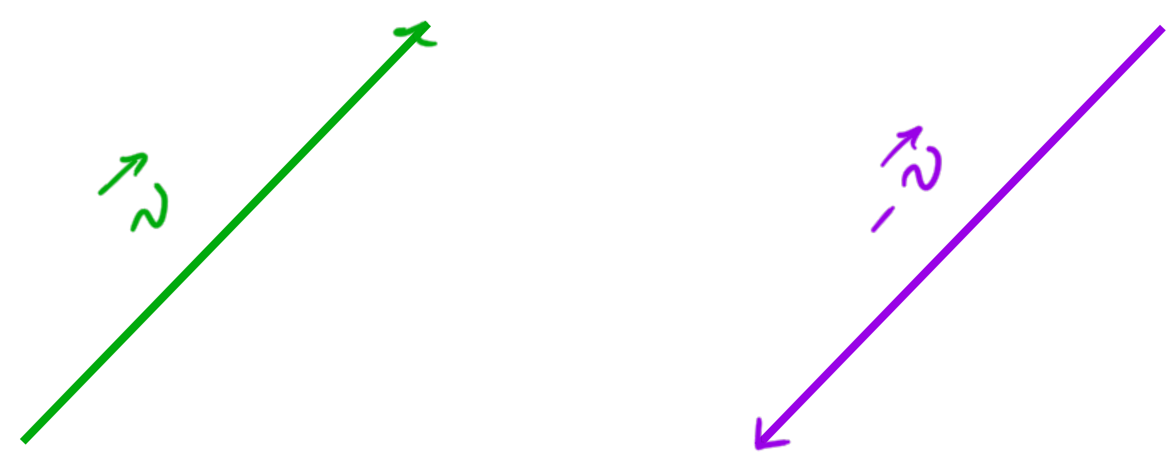
Length? Twice as long.

Question: What should $-\vec{v}$ be?

Algebraically, we should want $\vec{v} + (-\vec{v}) = \vec{0}$.

Geometrically, this means we want a vector, $-\vec{v}$, such that if we follow $\vec{v}$ and then $-\vec{v}$, our displacement is $\vec{0}$:

Fig. 6:



Direction? Opposite.

Length? Same.

This motivates our definition:

Definition: (Scalar Multiplication)

- If $c > 0$, then $c\vec{v}$ is a vector that is $|c|$ times as long as $\vec{v}$, in the same direction as $\vec{v}$.
- If $c < 0$, then $c\vec{v}$ is a vector that is $|c|$ times as long as $\vec{v}$, in the opposite direction to $\vec{v}$.

Remark: For any vector $\vec{v}$: $0 \cdot \vec{v} = \vec{0}$.

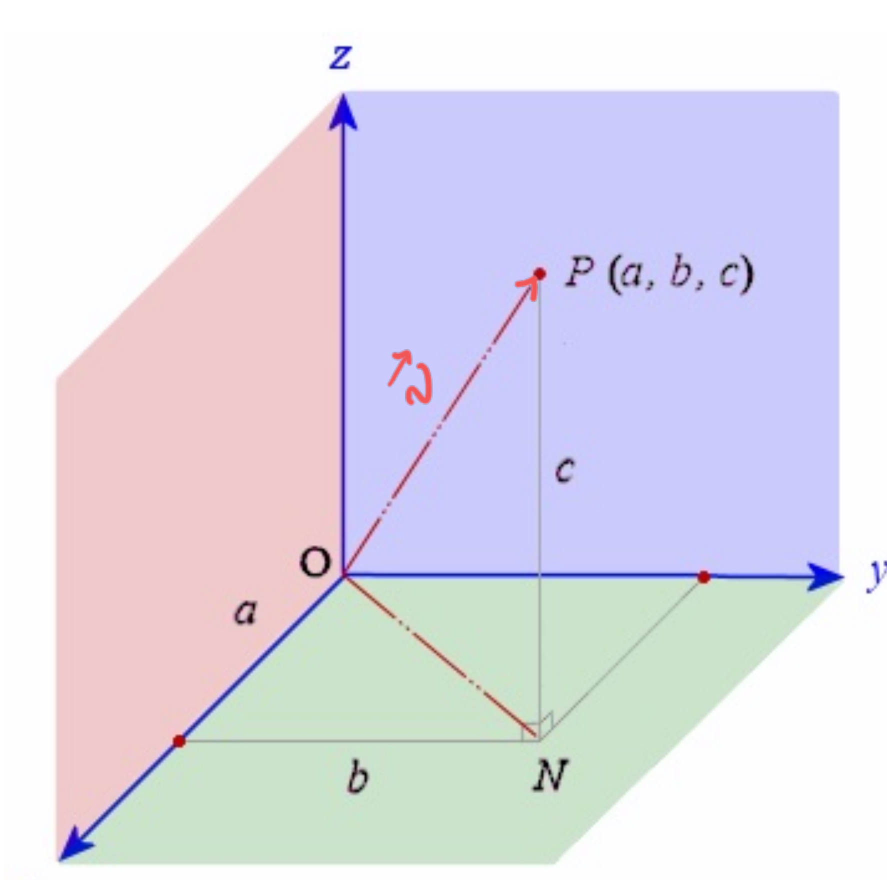
Exercise: With $\vec{a}$ and $\vec{b}$ as below, draw $\vec{a} - 2\vec{b}$.

Components:

This is a way to treat vectors algebraically.

If we take a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, based at the origin, we can write down the coordinates of the terminal point:

Fig. 7: ($\mathbb{R}^3$)



We define $\vec{v} = \overrightarrow{OP}$ to have components:

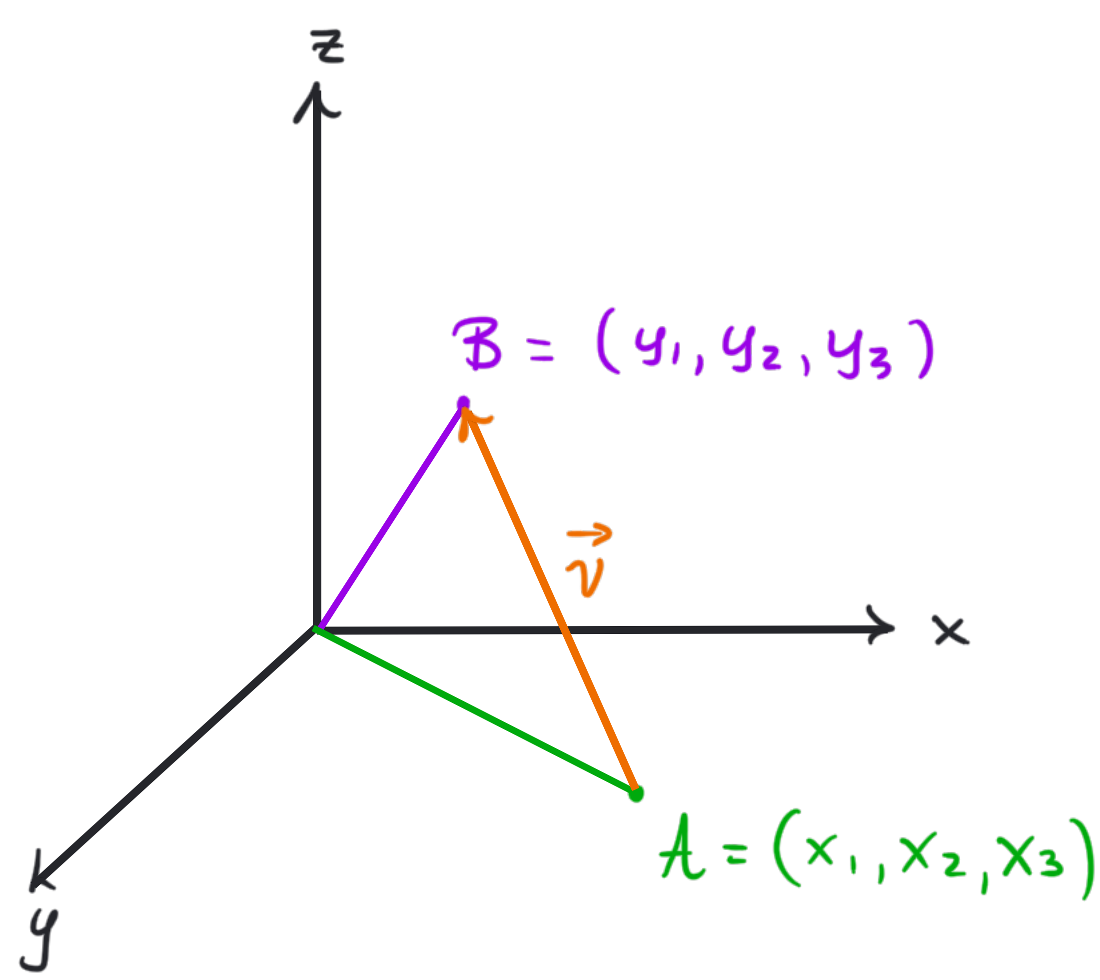
$$\vec{v} = (a, b, c).$$

- We can clearly see that if we instead base $\vec{v}$ at a point $A = (x_1, x_2, x_3)$, that it should terminate at the point $B = (x_1 + a, x_2 + b, x_3 + c)$.

- Let's say we have two points $A = (x_1, x_2, x_3)$, and $B = (y_1, y_2, y_3)$.

What should the components of $\vec{v} = \overrightarrow{AB}$ be?

Fig. 8:



Geometrically we see that: $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

So we should have $\vec{v} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

which, in components would be:

$$\boxed{\vec{v} = (y_1, y_2, y_3) - (x_1, x_2, x_3) = (y_1 - x_1, y_2 - x_2, y_3 - x_3)}$$

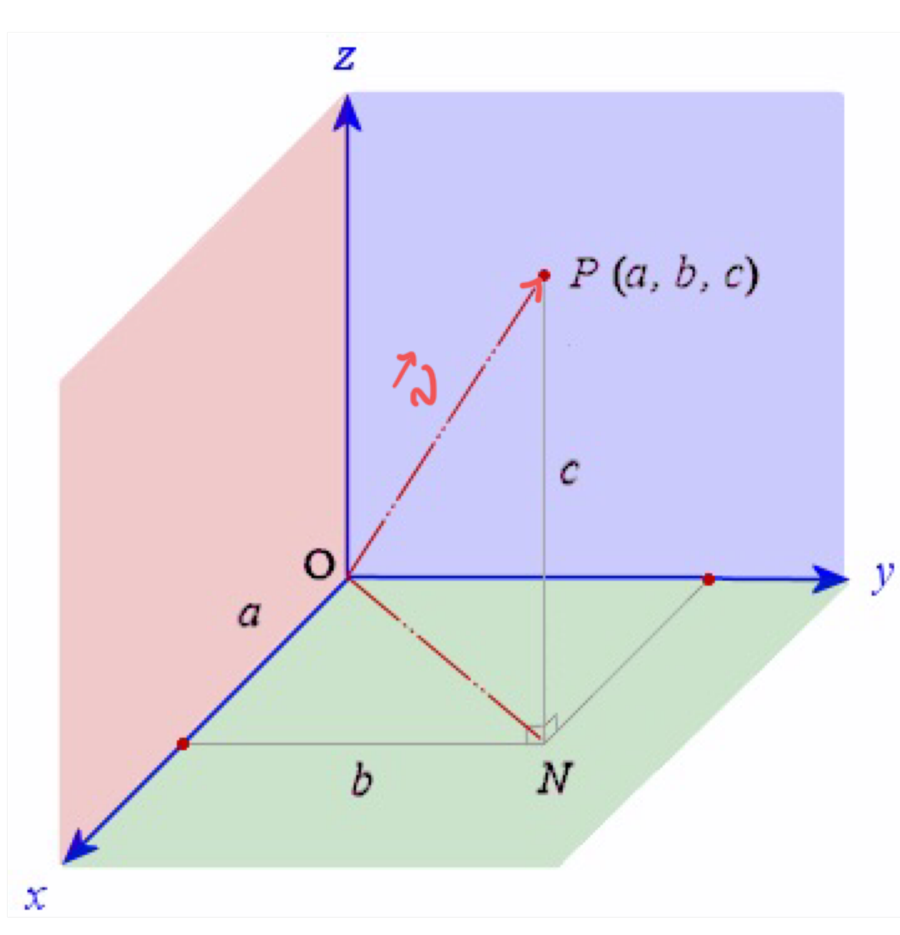
Main Point: $\vec{v}$ represented in components is:

$$\vec{v} = \begin{pmatrix} \text{displacement } \vec{v} \\ \text{causes in} \\ x\text{-direction} \end{pmatrix}, \begin{pmatrix} \text{displacement } \vec{v} \\ \text{causes in} \\ y\text{-direction} \end{pmatrix}, \begin{pmatrix} \text{displacement } \vec{v} \\ \text{causes in} \\ z\text{-direction} \end{pmatrix}$$

In particular, this representation doesn't care about where $\vec{v}$ is based.

Remark: Clearly the length of $\vec{v}$ is doesn't care about where $\vec{v}$ is based either.

Hence, we compute the length of $\vec{v}$, which we denote by $\|\vec{v}\|$, by using its representation in components:



Using pythagoras, we can see:

$$\boxed{\|\vec{v}\| = \sqrt{a^2+b^2+c^2}}$$

Questions: How do we add / subtract / scale vectors algebraically?

Answers: If $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$,

then:

(i) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

(iii) For $c \in \mathbb{R}$: $c\vec{a} = (ca_1, ca_2, ca_3)$

Remark: We could just have easily done all this in the 2-dimensional case, $\mathbb{R}^2$: $\vec{v} = (x_1, x_2)$, or the 4-dimensional case, $\mathbb{R}^4$: $\vec{v} = (x_1, x_2, x_3, x_4)$,

$\vdots$

or the n-dimensional case, $\mathbb{R}^n$: $\vec{v} = (x_1, \dots, x_n)$.

General Properties of vectors:

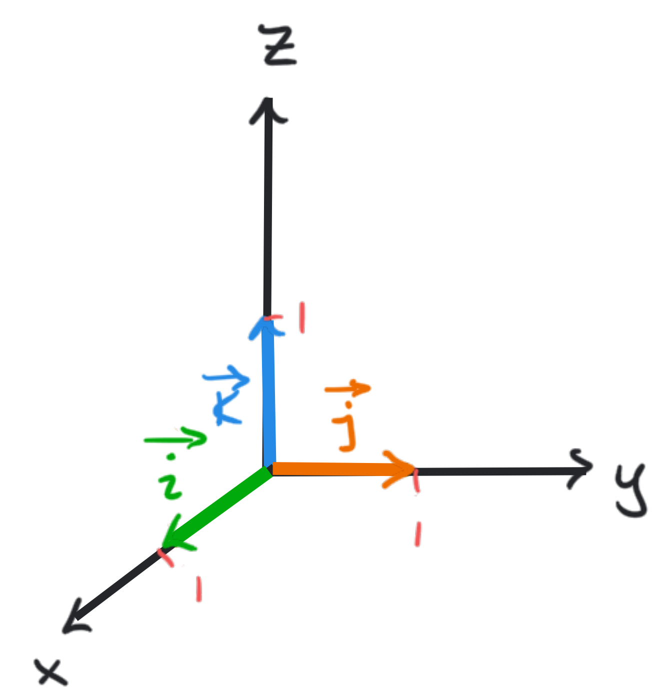
1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Remark: Every one of these properties can be justified geometrically.

Standard Basis Vectors:

Three vectors in $\mathbb{R}^3$ play a special role:

$$\left.\begin{aligned} \vec{i} &= (1,0,0) \\ \vec{j} &= (0,1,0) \\ \vec{k} &= (0,0,1) \end{aligned}\right\} \text{Standard Basis Vectors for } \mathbb{R}^3$$



Why?

Because any vector $\vec{v}$ can be represented, algebraically as a linear combination of $\vec{i}$, $\vec{j}$, $\vec{k}$:

$$\vec{v} = (a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$$

$$= a\vec{i} + b\vec{j} + c\vec{k}$$

Remark: It is useful to think of $\vec{i}, \vec{j}, \vec{k}$ as the "building blocks" for $\mathbb{R}^3$.

Exercise: If $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{j} + 2\vec{k}$, represent $\vec{u}+\vec{v}$ in as a linear combination of $\vec{i}, \vec{j}, \vec{k}$ and compute $\|\vec{u} + \vec{v}\|$.

Definition: A unit vector is a vector with length 1.

Example: $\vec{i}$, $\vec{j}$ and $\vec{k}$ are all unit vectors.

Remark: For any non-zero vector $\vec{u}$, there is a unit vector pointing in the same direction as $\vec{u}$. This vector is usually denoted as $\hat{u}$, and is given algebraically by:

$$\boxed{\hat{u} = \frac{1}{\|\vec{u}\|} \cdot \vec{u}}$$

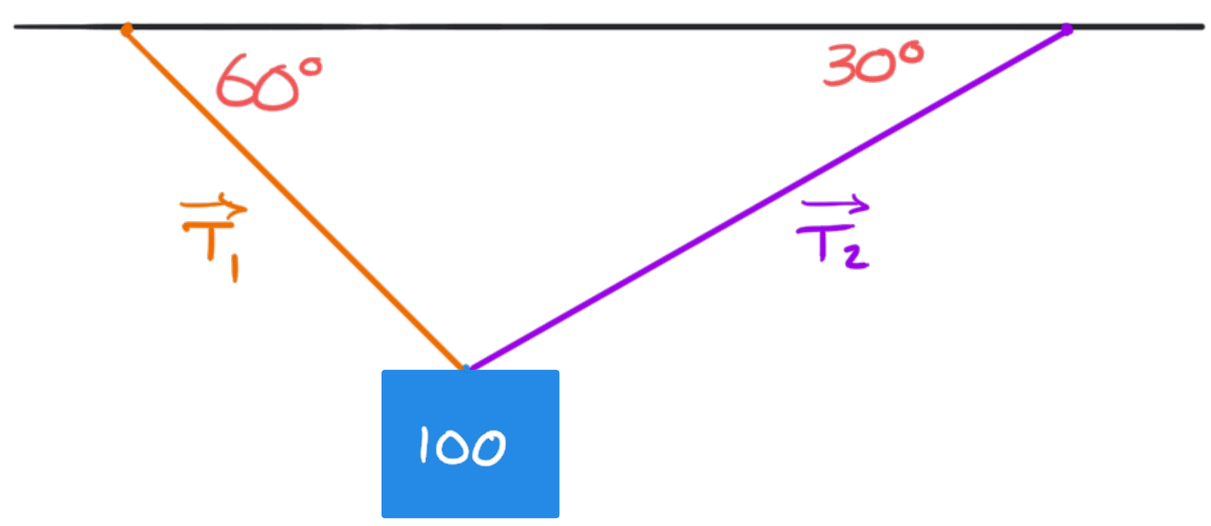
This process $(\vec{u} \rightsquigarrow \hat{u})$ is called normalizing $\vec{u}$.

Remark: Unit vectors are sometimes referred to as directions.

Exercise: Normalize $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$.

Applications:

1. A 100 lb weight hangs from two wires as shown:



Find the tension forces T_1 and T_2 in both wires and the magnitudes of the tensions, assuming the system is in equilibrium.

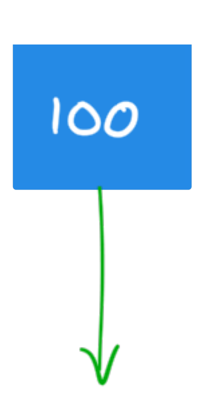
Sol^n: As our system is in equilibrium, we have:

$$\vec{T_1} + \vec{T_2} + \vec{w} = \vec{0} \quad , \text{ or :}$$

$$\vec{T_1} + \vec{T_2} = -\vec{w} = 100\,\vec{j} \qquad (+)$$

Force Diagrams:

$\vec{w}$:



$$\Rightarrow \quad \vec{w} = -100\,\vec{j}$$

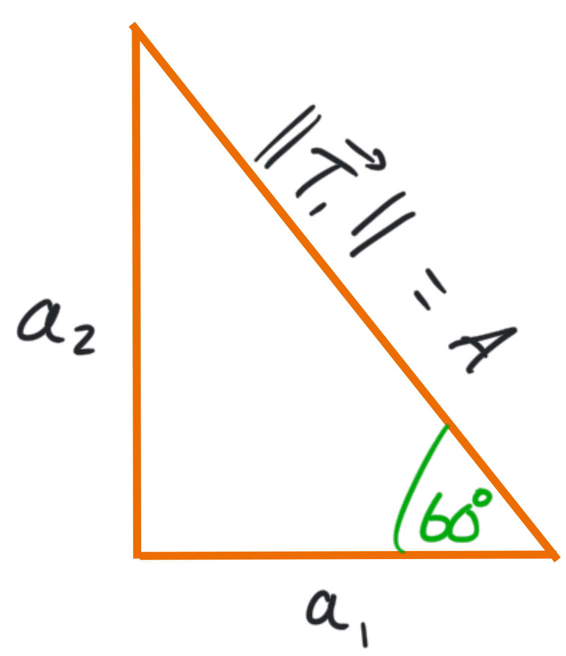
Let us denote the magnitude of the tension forces :

$$\|\vec{T}_1\| =: A \quad \& \quad \|\vec{T}_2\| =: B.$$

So, breaking our forces up into components :

$$\vec{T}_1 = a_1 \vec{i} + a_2 \vec{j} \quad \text{and} \quad \vec{T}_2 = b_1 \vec{i} + b_2 \vec{j}$$

$\vec{T}_1$:

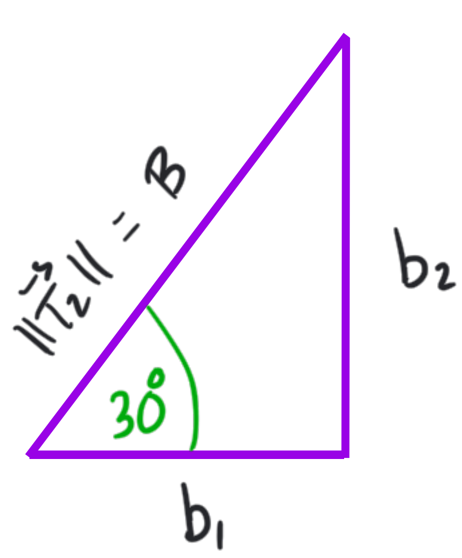


$$\Rightarrow \quad a_1 = A\cos 60° = \frac{A}{2}$$

$$a_2 = A\cos 30° = \frac{\sqrt{3}}{2}A$$

Hence : $\boxed{\vec{T}_1 = -\frac{A}{2}\vec{i} + \frac{\sqrt{3}}{2}A\vec{j}}$

$\vec{T}_2$:



$$\Rightarrow \quad b_1 = B\cos 30° = \frac{\sqrt{3}}{2}B$$

$$b_2 = B\sin 30° = \frac{B}{2}$$

Hence : $\boxed{\vec{T}_2 = \frac{\sqrt{3}}{2}B\vec{i} + \frac{B}{2}\vec{j}}$

So, rewriting (+):

$$\left(-\frac{A}{2}\vec{i} + \frac{\sqrt{3}}{2}A\vec{j}\right) + \left(\frac{\sqrt{3}}{2}B\vec{i} + \frac{B}{2}\vec{j}\right) = 100\vec{j} + 0\vec{i}$$

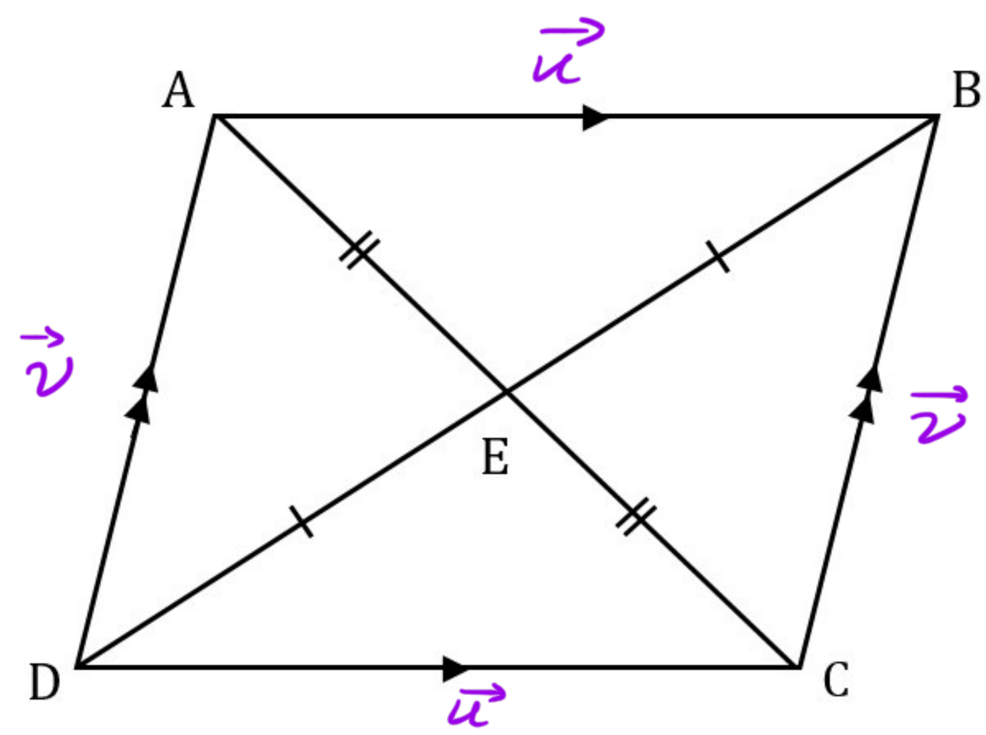
Equating components:

$$-\frac{A}{2} + \frac{\sqrt{3}}{2}B = 0 \quad \& \quad \frac{\sqrt{3}}{2}A + \frac{B}{2} = 100$$

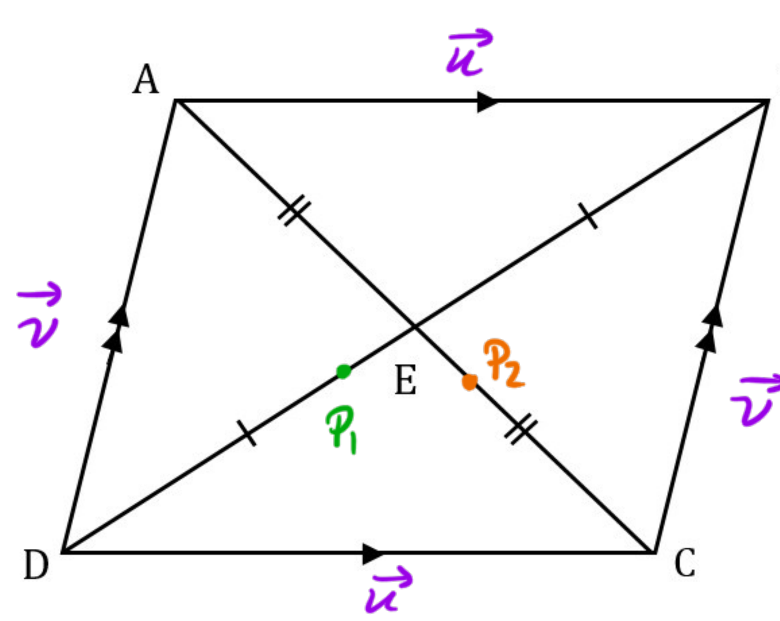
Solving this system yields: $A = 50\sqrt{3}$ & $B = 50$.

So
$$\boxed{\begin{aligned} \vec{T}_1 &= -25\sqrt{3}\,\vec{i} + 75\,\vec{j} \\ \vec{T}_2 &= 25\sqrt{3}\,\vec{i} + 25\,\vec{j} \end{aligned}}$$

2. Show that the diagonals of a parallelogram bisect eachother :



Sol$^{\underline{n}}$: Let's denote the halfway point of the cord DB by P_1 and the halfway point of the cord CA by P_2.



Our claim is that $P_1 = P_2$.

We can see $\overrightarrow{DB} = \vec{u} + \vec{v}$, so to get to P_1 we start at D and follow $\frac{1}{2}(\vec{u} + \vec{v})$.

Let's represent the vector $\overrightarrow{CA}$ by $\vec{w}$.

We can see that to get to P_2, starting from D, we follow $\vec{u}$ and then $\frac{1}{2}\vec{w}$.

So, in components:

$$P_1 = D + \frac{1}{2}(\vec{u} + \vec{v})$$

and

$$P_2 = D + \vec{u} + 1/2\,\vec{w}.$$

But we can see from the diagram that

$\vec{w} = -\vec{u} + \vec{v}$.

So $$\begin{aligned} P_2 &= D + \vec{u} + 1/2(-\vec{u} + \vec{v}) \\ &= D + 1/2(\vec{u} + \vec{v}) \\ &= P_1 \end{aligned}$$

— o —