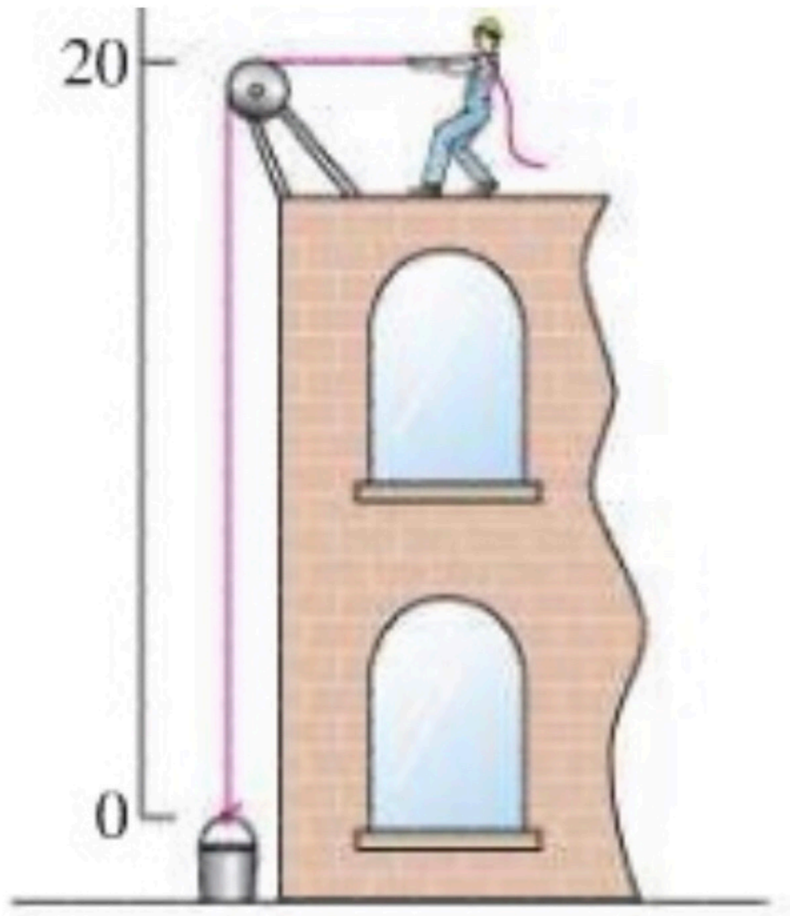
Example A 20 ft. rope weighing 0.08 lbs per foot is used to haul a bucket weighing 5 lb. to the top of a building. How much work is done in lifting the rope and bucket to the top of the building ?



We use Riemann sums to solve this problem, by breaking the interval $[0, 20]$ into thin slices and adding the work done on each slice to approximate the total work done in lifting the rope and bucket. We the take the limit of the sum as the width of the slices tends to 0 to find the work done in lifting the rope and bucket.

Work done $= \int_0^{20} \text{Force}(x)\, dx$

"The force required to hold the rope and bucket up at the start"

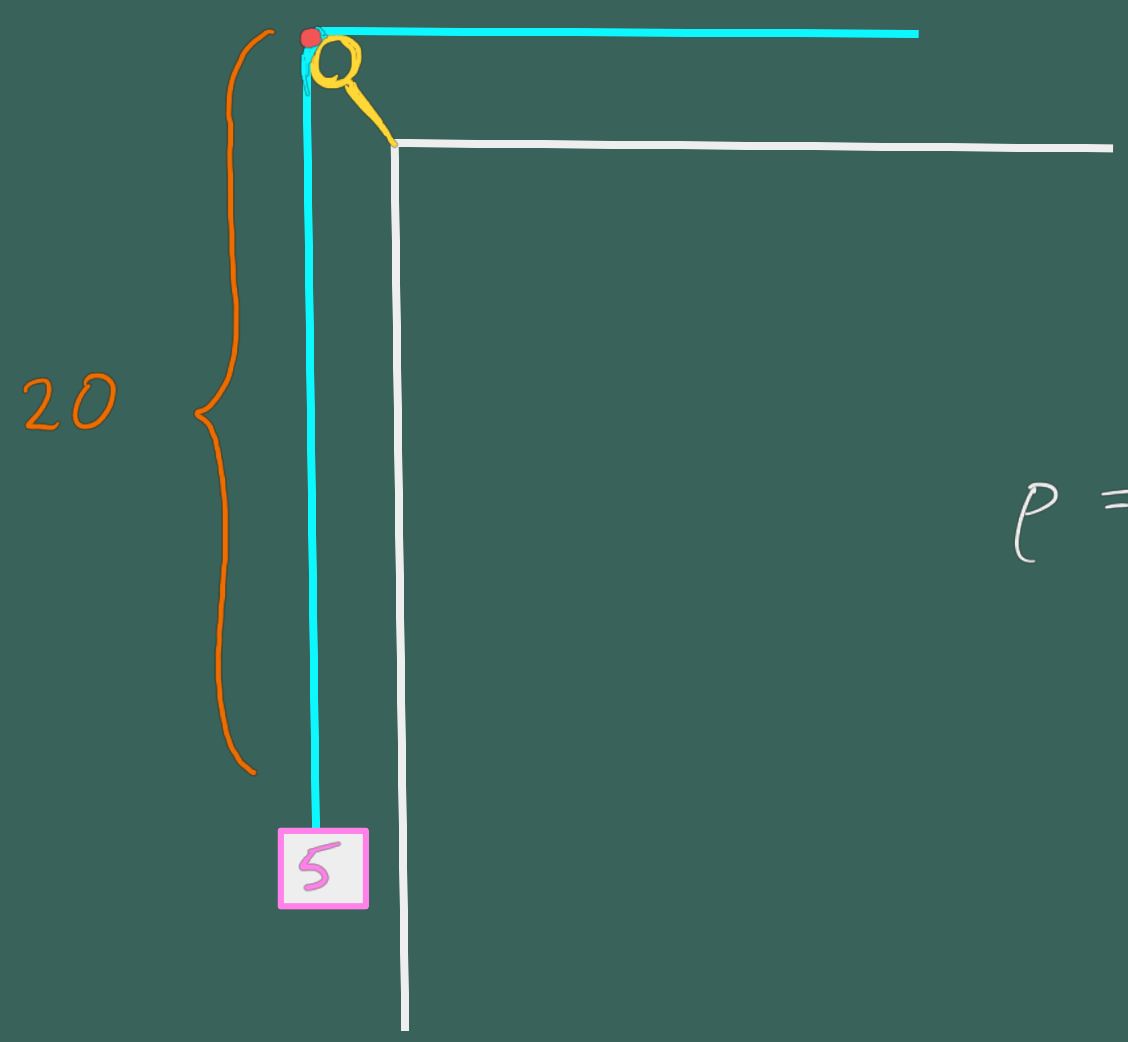
$= (\text{Weight of bucket}) + (\text{Weight of rope})$

$= 5 + 20\left(\frac{8}{100}\right)$

"The force required to hold the rope and bucket up after you've pulled up x ft of rope"

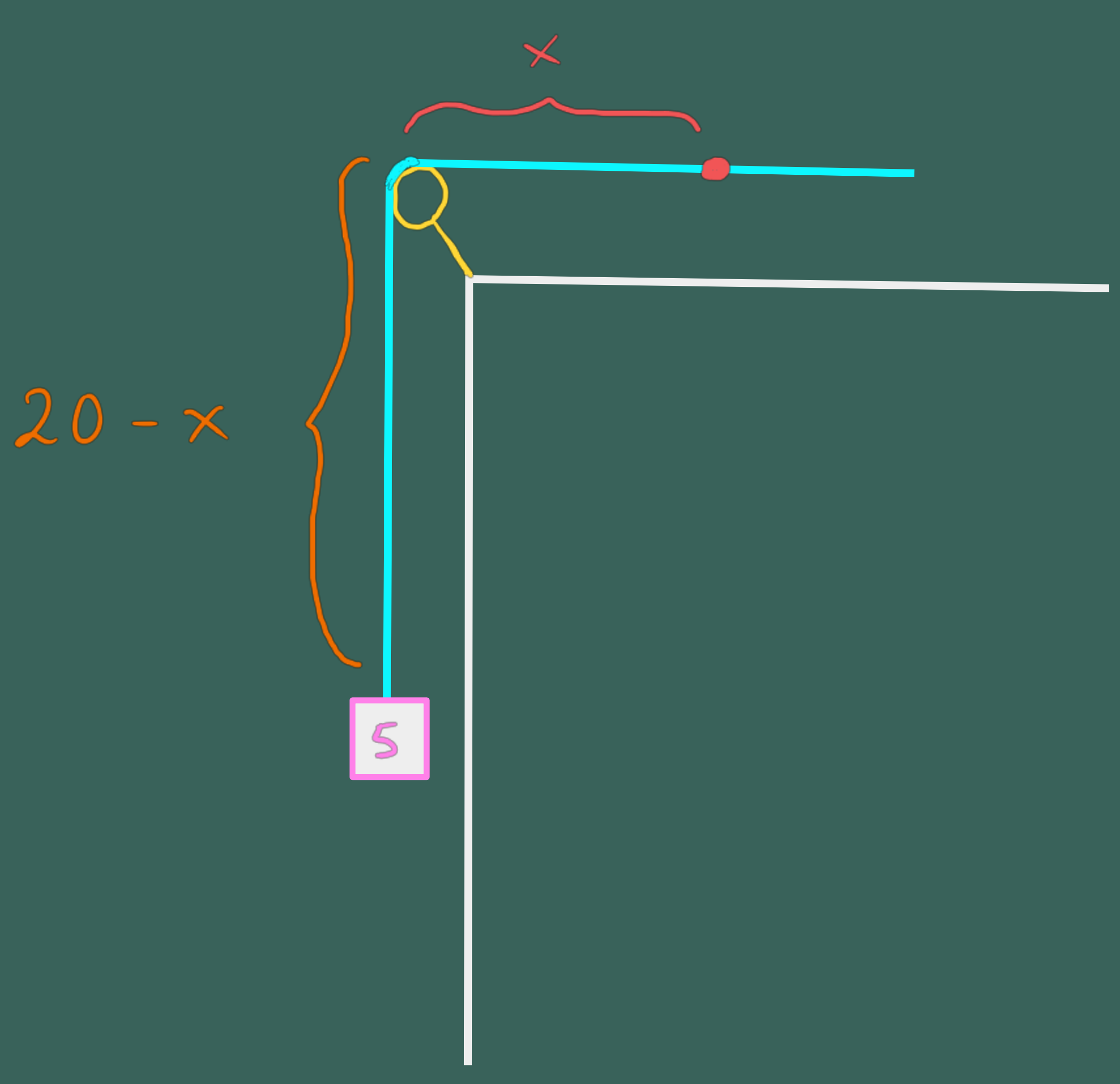
$= (\text{Weight of bucket}) + (\text{weight of rope left})$

$= 5 + (20 - x)\left(\frac{8}{100}\right)$



$$\rho = \frac{8}{100} \text{ ft-lb}/\text{ft}$$

$$F_{start} = 5 + 20\left(\frac{8}{100}\right)$$



$$F_{\text{After } x \text{ ft of rope lifted}} = 5 + (20 - x)\left(\frac{8}{100}\right)$$

"To get the total work done we "add up" all the forces"

$$W = \int_0^{20} 5 + (20 - x)\left(\frac{8}{100}\right) \, dx$$

$$= \int_0^{20} \left(5 + \frac{8}{5} - \frac{8x}{100}\right) dx$$

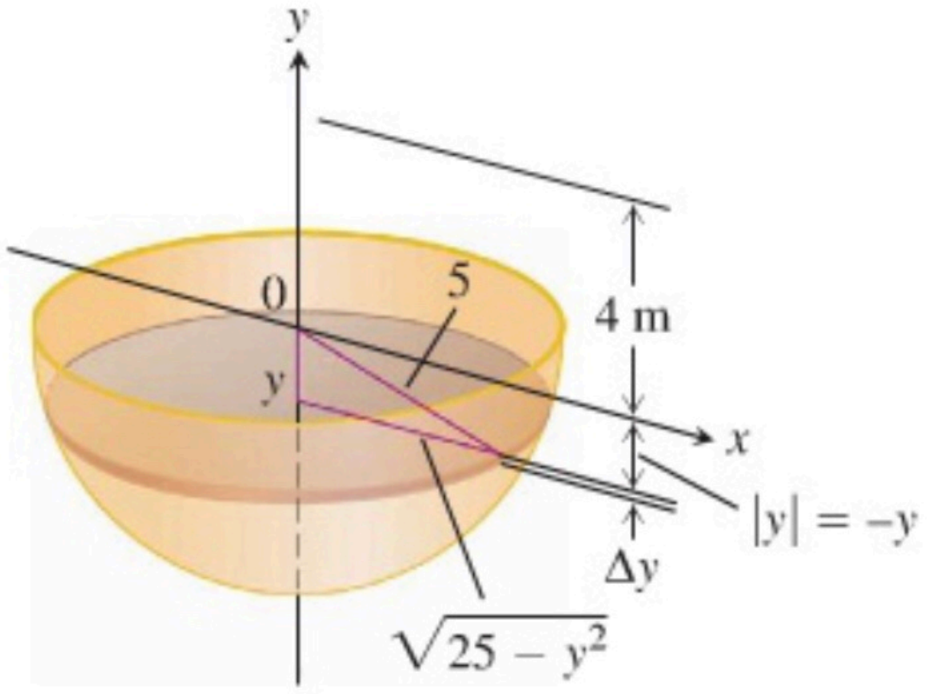
$$= \int_0^{20} \frac{33}{5} - \frac{8x}{100} \, dx$$

$$= \frac{33x}{5} - \frac{8}{100}\frac{x^2}{2} \Big|_0^{20}$$

$$= \frac{33(20)}{5} - \frac{8}{100}\frac{(20)^2}{2}$$

$$= 132 - 16 = \underline{116 \text{ ft-lbs}}$$

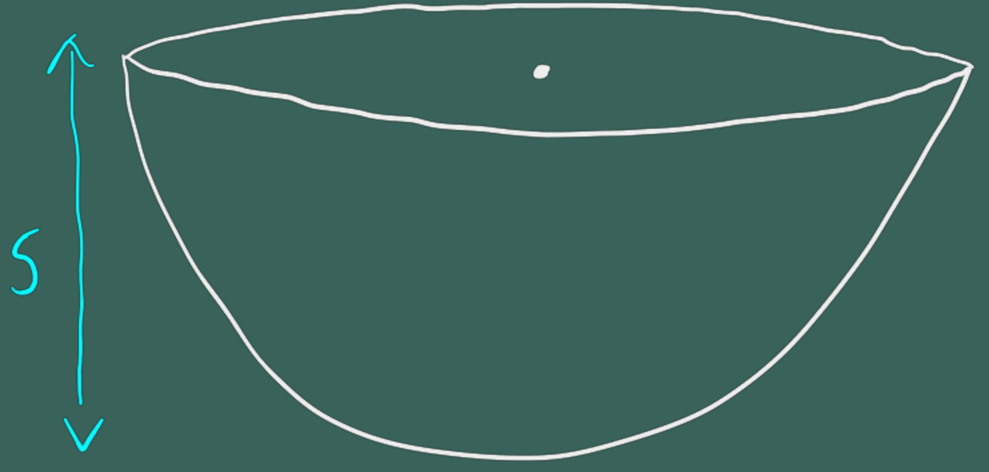
Example How much work is necessary to empty a full hemispherical water reservoir of radius 5m by pumping the water to a height of $4m$ above the top of the reservoir. (The density of water is $1000kg/m^3$ and the force of gravity is $9.8\ m/s^2$.)



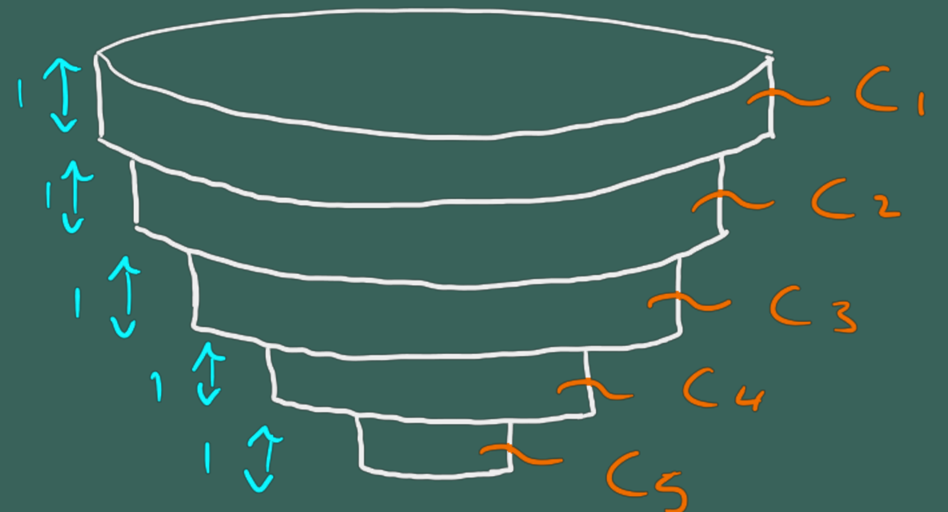
$\int_{-5}^{0} (4-y)\,\pi$

Step 1:

e.g. $n=5$



$\approx$



Approximate semi-circle with stacked cylinders of equal height. (In general, take n cylinders).

Step 2: Find work done in raising each cylinder 4m above the top of the tank.

To do this we need to find the weight of each cylinder:

$W = mg = (\text{Volume})(\text{mass density})\ g$

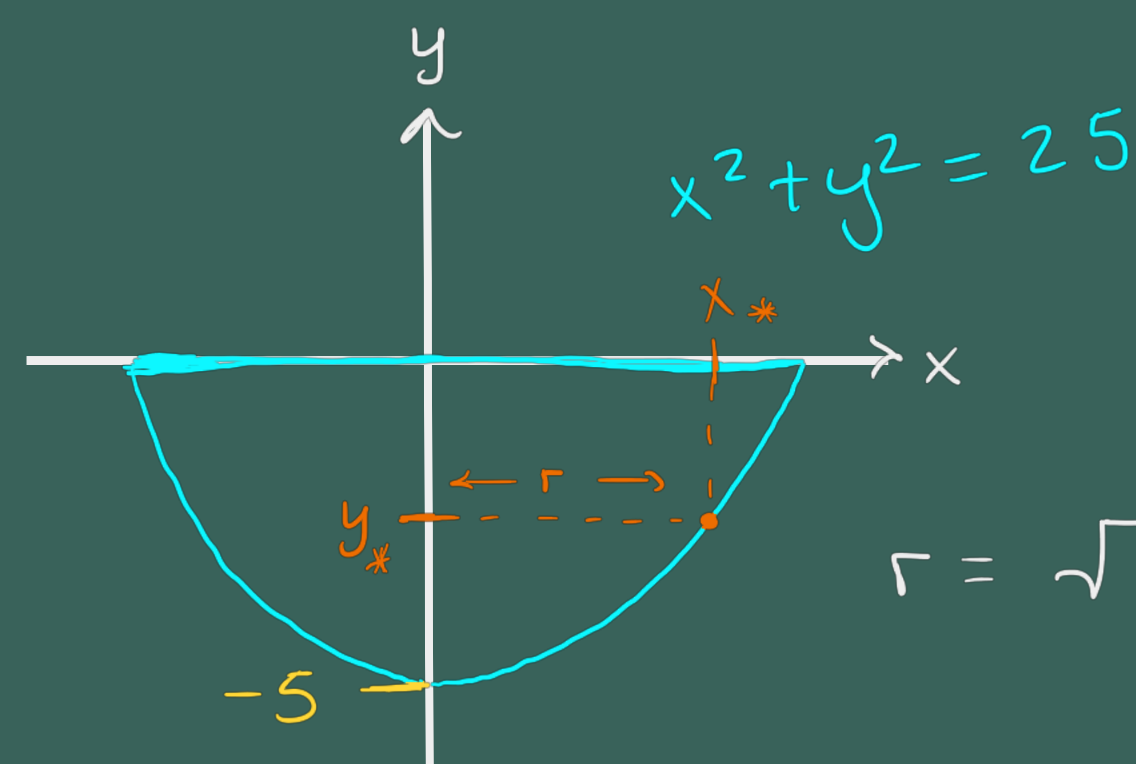
Volume of cylinder $= \pi r^2 h$

All cylinders are of height $\frac{5}{n}$.

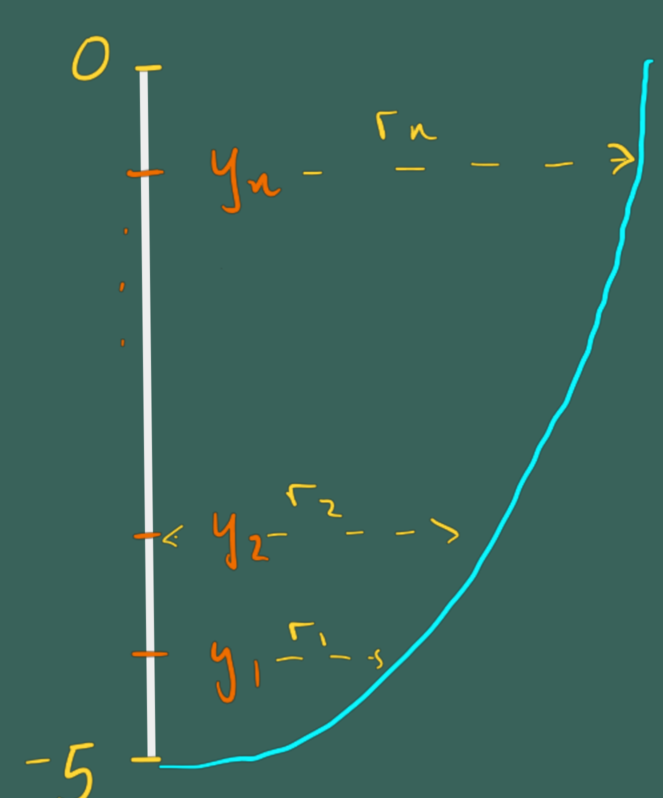
Volume of cylinder $= \pi r^2 \left(\frac{5}{n}\right)$

Radius?

Side Profile is a circle :



$r = \sqrt{25-y^2}$



$\Rightarrow$ Radius of kth cylinder

$= \sqrt{25 - y_k^2}$

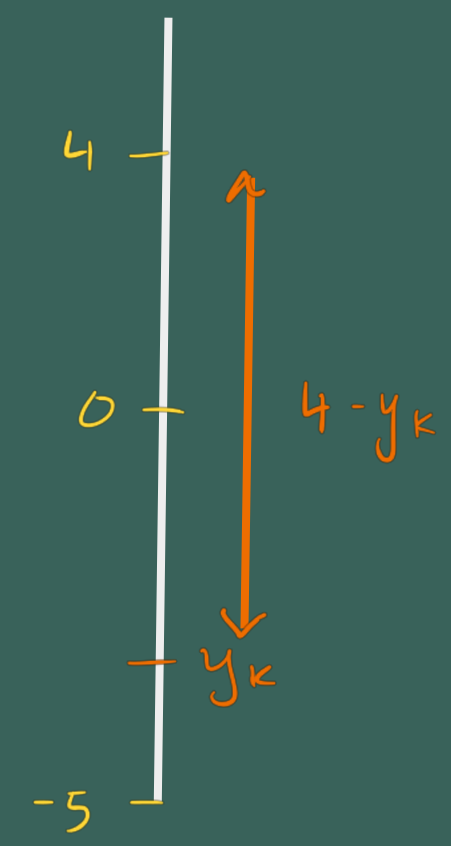
Weight of kth cylinder

$$= \left(\pi \underbrace{\left(\sqrt{25 - y_k^2}\right)^2}_{r^2} \underbrace{\left(\frac{5}{n}\right)}_{h} \right) 1000g$$

Work done on kth cylinder

$$= (\text{Weight})(\text{Distance})$$

$$= (1000\pi g)\left(\sqrt{25 - y_k^2}\right)^2 \left(\frac{5}{n}\right)(4 - y_k)$$



Work done in lifting all cylinders

$$= \sum_{k=1}^{n} (1000\pi g)(25 - y_k^2)(4 - y_k)\,\Delta y$$

Work done on actual hemisphere

$$= \lim_{\Delta y \to 0} \sum_{k=1}^{n} (1000\pi g)(25 - y_k^2)(4 - y_k)\,\Delta y$$

$$= \int_{-5}^{0} 1000\pi g\,(25 - y^2)(4 - y)\,dy$$

$$= 1000\pi g \int_{-5}^{0} (100 - 25y - 4y^2 + y^3)\,dy$$

$$= 1000\pi g \left(100y - \frac{25y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right)\Bigg|_{-5}^{0}$$

$$= 1000\pi g \left(0 - \left[100(-5) - \frac{25(-5)^2}{2} - \frac{4(-5)^3}{3} + \frac{(-5)^4}{4}\right]\right)$$

$$= 1000\pi g \left(500 + \frac{625}{2} + \frac{500}{3} + \frac{625}{4} \right)$$

$$= 1000\pi g \left(\frac{13625}{12} \right)$$

$$= \frac{3406250\pi g}{3} \text{ Joules}$$