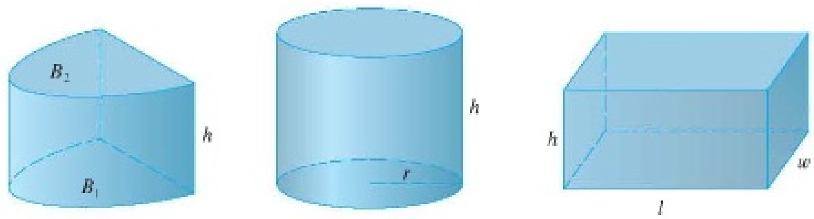
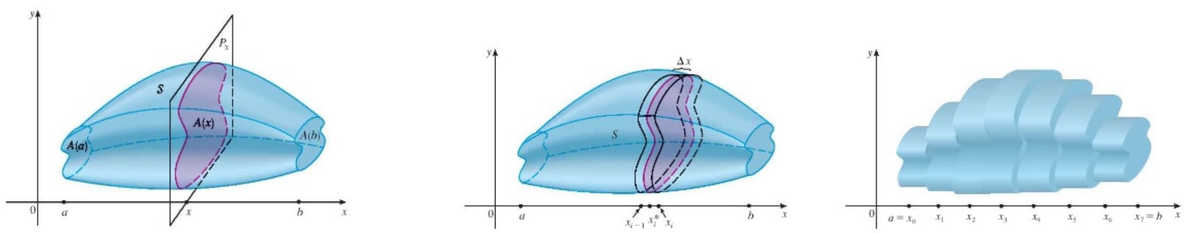
Volumes by Disks and Washers

Volume of a cylinder A cylinder is a solid where all cross sections are the same. The volume of a cylinder is $A \cdot h$ where A is the area of a cross section and h is the height of the cylinder.



(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \pi r^2 h$ (c) Rectangular box $V = lwh$

For a solid S for which the cross sections vary, we can approximate the volume using a Riemann sum.



The areas of the cross sections (taken perpendicular to the x-axis) of the solid shown on the left above vary as x varies. The areas of these cross sections are thus a function of x, $A(x)$, defined on the interval $[a, b]$. The volume of a slice of the solid above shown in the middle picture, is approximately the volume of a cylinder with height Δx and cross sectional area $A(x_i^*)$. In the picture on the right, we use 7 such slices to approximate the volume of the solid. The resulting Riemann sum is

$$V \approx \sum_{i=1}^{7} A(x_i^*)\Delta x.$$

The volume is the limit of such Riemann sums:

$$\boxed{V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.}$$

Thus if we have values for the cross sectional area at discrete points $x_0, x_1, \ldots, x_n$, we can estimate the volume from the data using a Riemann sum. On the other hand if we have a formula for the function $A(x)$ for $a \leq x \leq b$, we can find the volume using the Fundamental theorem of calculus, or in the event that we cannot find an antiderivative for $A(x)$, we can estimate the volume using a Riemann sum.

$$\boxed{V = \int_a^b A(x)dx.}$$

Example The base of a solid is the region enclosed by the curve $y = \frac{1}{x}$ and the lines $y = 0$, $x = 1$ and $x = 3$. Each cross section perpendicular to the x-axis is an isosceles right angled triangle with the hypotenuse across the base. Find the volume of the solid.

Solids of revolution, Method of disks

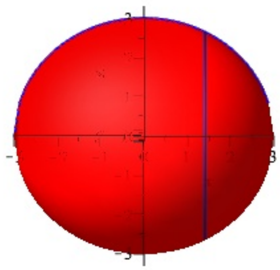
Let f be a continuous function on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Let R denote the region between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. When this region is revolved around the x-axis, it generates a solid, S, with circular cross sections of radius $f(x)$. The area of the cross section of S at x is the area of a circle with radius $f(x)$;

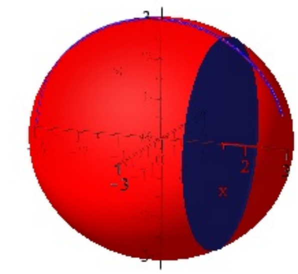
$$\boxed{A(x) = \pi[f(x)]^2}$$

and the volume of the solid (of revolution) generated by R is

$$\boxed{V = \int_a^b \pi[f(x)]^2 dx.}$$

Example Find the volume of a sphere of radius 3.



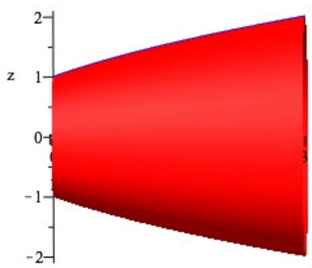


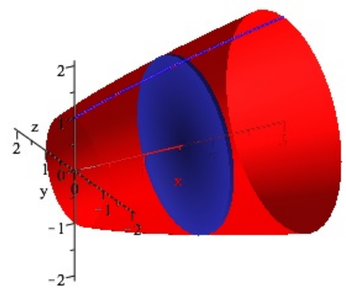
What is the equation of the curve, $y = f(x)$ which generates the sphere as a solid of revolution as described above?

What is the area of a cross section of the sphere at x, where $-3 \leq x \leq 3$?

What is the volume of the sphere?

Example Find the volume of the solid obtained from revolving the region bounded by the curve $y = \sqrt{x+1}$, $x = 0$, $x = 3$ and $y = 0$ (the x axis) about the x axis.





Method of Washers

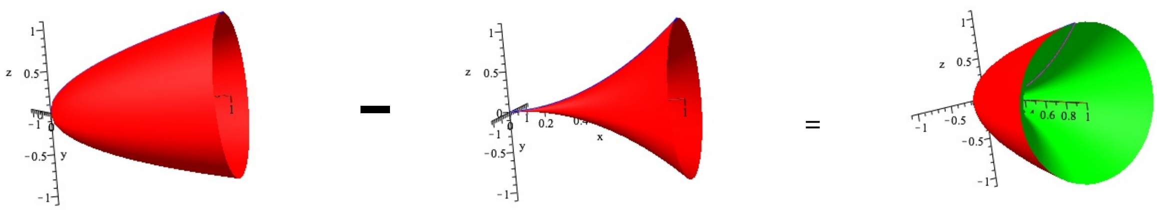
Let $f(x)$ and $g(x)$ be continuous functions on the interval $[a, b]$ with $f(x) \geq g(x) \geq 0$. Let R denote the region bounded above by $y = f(x)$, below by $y = g(x)$ and the lines $x = a$ and $x = b$. Let S be the solid obtained by revolving the region R around the x axis. The cross sections of S are washers with area is given by

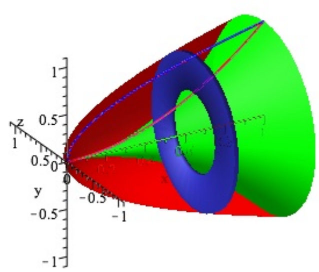
$$\boxed{A(x) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi[f(x)^2] - \pi[g(x)]^2.}$$

The volume of S is given by

$$\boxed{V = \int_a^b \pi[f(x)^2] - \pi[g(x)]^2 dx = \int_a^b \pi[f(x)^2 - g(x)^2] dx}$$

Example Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = 1$ about the x axis. We see from the pictures below how the formula is derived:





Rotating about a line $y = c$

We may also rotate a region between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ around a line of the form $y = c$ to generate a solid, S. Let us assume that $|f(x) - c| \geq |g(x) - c| \geq 0$ for $a \leq x \leq b$. The cross sections of S are washers with area

$$\boxed{A(x) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi(f(x) - c)^2 - \pi(g(x) - c)^2.}$$

Hence the volume of such a solid is given by

$$\boxed{V = \int_a^b \pi(f(x) - c)^2 - \pi(g(x) - c)^2 dx.}$$

Example What is the volume of the solid generated by rotating the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = 1$ about the line $y = -1$.

$$V = \int_0^1 \pi(\sqrt{x} - (-1))^2 - \pi(x^2 - (-1))^2 dx = \pi \int_0^1 (\sqrt{x} + 1)^2 - (x^2 + 1)^2 dx$$

$$= \pi \int_0^1 (x + 2\sqrt{x} + 1) - (x^4 + 2x^2 + 1)dx = \pi \int_0^1 x + 2\sqrt{x} + 1 - x^4 - 2x^2 - x dx$$

$$= \pi \left[\frac{x^2}{2} + 2 \cdot \frac{2}{3} \cdot x^{3/2} - \frac{x^5}{5} - 2\frac{x^3}{3} \right]_0^1 = \pi \left[\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right] = \pi \frac{29}{30}$$

Working with respect to the y axis

Example Let S be a solid bounded by the parallel planes perpendicular to the y axis, $y = c$ and $y = d$. If for each y in the interval $[c, d]$ the cross sectional area of S perpendicular to the y axis is $A(y)$, the volume of the solid S is

$$\boxed{V = \int_c^d A(y)dy}$$

(Provided that $A(y)$ is an integrable function of y)

Example Find the volume of a pyramid with height 10 in. and square base whose sides have length 4 in.

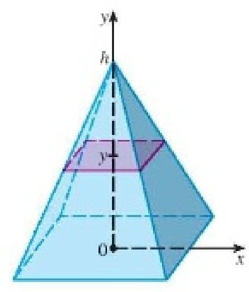
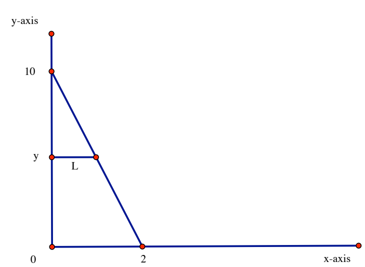


FIGURE 16

Each cross section of the pyramid perpendicular to the y axis is a square. To determine the length of the side of the square at y, we consider the triangle below, bounded by the y axis, the x axis and the line along the side of the pyramid directly above the x axis. The length of the side of the cross sectional square at y is $2L$ and the cross sectional area at y is $A(y) = 4L^2$. We would like to express this in terms of y.



By simiar triangles we have $\frac{10-y}{L} = \frac{10}{2}$. This gives $2(10 - y) = 10L$ and $L = \frac{10-y}{5}$. Therefore the cross sectional area at y is given by $A(y) = 4L^2 = \frac{4}{25}(10 - y)^2 = \frac{4}{25}(100 - 20y + y^2)$. By the formula, the volume of the pyramid is

$$\int_0^{10} \frac{4}{25}(100 - 20y + y^2)dy = \frac{4}{25}\int_0^{10}(100 - 20y + y^2)dy = \frac{4}{25}\left[100y - 10y^2 + y^3/3\right]_0^{10}$$

$$= 160/3$$

Solids of Revolution; Revolving around the y axis

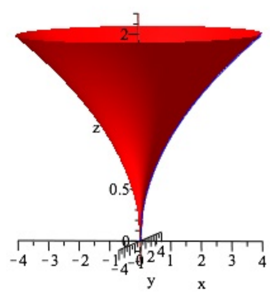
Let $f(y)$ be a continuous function on $[c, d]$ with $f(y) \geq 0$ for all $y \in [c, d]$. Let R denote the region between the curve $x = f(y)$ and the y-axis and the lines $y = c$ and $y = d$. When the region R is revolved around the y-axis, it generates a solid with circular cross sections of radius $f(y)$. The area of the cross section at y is the area of such a circle;

$$\boxed{A(y) = \pi[f(y)]^2}$$

and the volume of the solid (of revolution) generated by R is

$$\boxed{V = \int_c^d \pi[f(y)]^2 dy.}$$

Example Find the volume of the solid generated by revolving the region bounded by the curve $x = y^2$ and the lines $y = 0$, $y = 2$ and $x = 0$(the y axis) about the y axis.



$$V = \int_0^2 \pi y^4 dy = \pi \frac{y^5}{5}\bigg|_0^2 = \pi \frac{32}{5}.$$

Method of Washers with respect to y axis

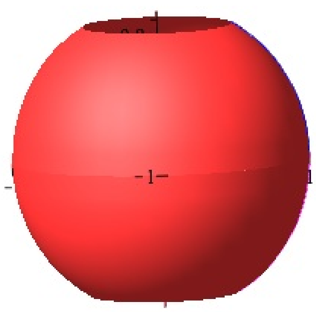
Let $f(y)$ and $g(y)$ be continuous functions on the interval $[c, d]$ with $f(y) \geq g(y) \geq 0$. Let R denote the region bounded by the curves $x = f(y)$, $x = g(y)$ and the lines $y = c$ and $y = d$. Let S be the solid obtained by revolving the region R around the y axis. The cross sections of S are washers with area is given by

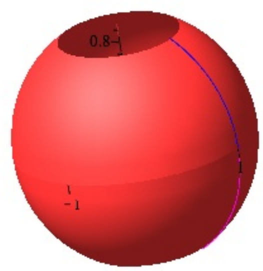
$$\boxed{A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi[f(y)^2] - \pi[g(y)]^2.}$$

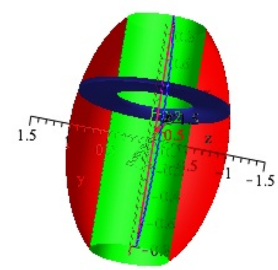
The volume of S is given by

$$\boxed{V = \int_c^d \pi[f(y)^2] - \pi[g(y)]^2 dx = \int_c^d \pi[f(y)^2 - g(y)^2]dy}$$

Example Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{1 - y^2}$ and the line $x = 1/2$ about the y axis.







The curve $x = \sqrt{1 - y^2}$ and the line $x = 1/2$ meet when $\sqrt{1 - y^2} = 1/2$ or $y^2 = 3/4$ giving us $y = \pm\frac{\sqrt{3}}{2}$. We see that a cross section of this solid is a washer with area

$$A(y) = \pi(\text{outer radius })^2 - \pi(\text{inner radius})^2 = \pi(\sqrt{1 - y^2})^2 - \pi(1/2)^2 = \pi(1 - y^2 - 1/4) = \pi(3/4 - y^2).$$

The volume is given by

$$V = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} A(y)dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} A(y)dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \pi(3/4 - y^2)dy$$

$$= \pi\left(3/4y - \frac{y^3}{3}\right)\Bigg|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} = \pi\left(\frac{3}{4}(\frac{\sqrt{3}}{2}) - \frac{(\frac{\sqrt{3}}{2})^3}{3}\right) - \left(\pi\left(\frac{3}{4}(\frac{-\sqrt{3}}{2}) - \frac{\left(\frac{-\sqrt{3}}{2}\right)^3}{3}\right)\right) = 2\pi\left(\frac{3}{4}(\frac{\sqrt{3}}{2}) - \frac{(\frac{\sqrt{3}}{2})^3}{3}\right) = \pi\frac{\sqrt{3}}{2}$$