Area Between Curves

In this section we calculate the area between two curves.

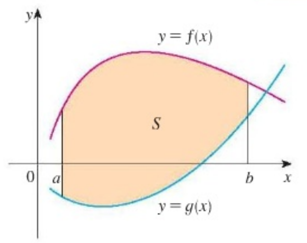
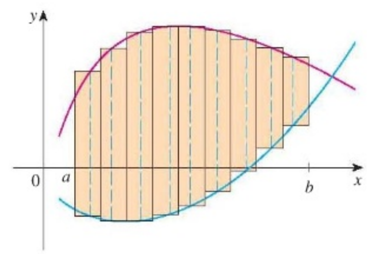


FIGURE 1
$S = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$



(b) Approximating rectangles

If we wish to estimate the area or the region shown above, between the curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, we can use n approximating rectangles of width $\Delta x = \frac{b-a}{n}$ as shown in the picture on the right. We can choose the height of each approximating rectangle to be $f(x_i^*) - g(x_i^*)$ where x_i^* is some point in the interval $[x_{i-1}, x_i]$. The sum of these rectangles

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

is an approximation of the area of the region S shown in the diagram. Using the limiting process as before we get the area of the region S is given by

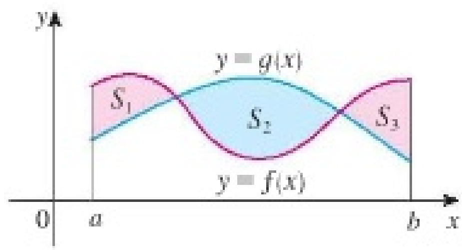
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \lim_{\Delta x\to 0} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.$$

From our definition of the definite integral we get that the above limit is a definite integral:

$$\boxed{A = \int_a^b \big(f(x) - g(x)\big)dx}.$$

Example Sketch the region bounded above by $y = x^3 + 2$, below by $y = 1 - x^2$ and on the sides by the lines $x = 0$ and $x = 1$ and calculate its area.

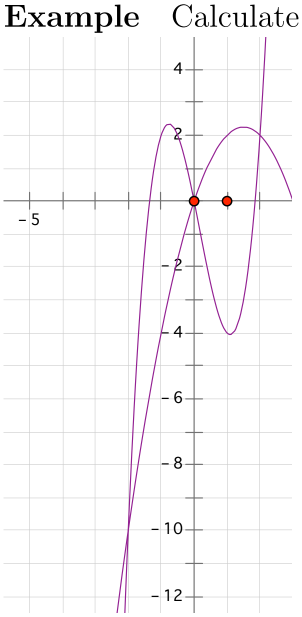
Example Sketch the region enclosed by the curves $y = 2x^2$ and $y = 1 - 2x^2$ and find its area.



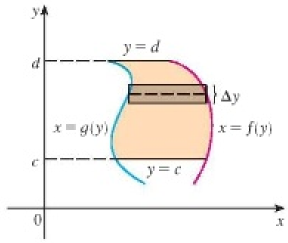
In the picture above the curves cross and it is not difficult to see that the area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\boxed{\int_a^b \big|f(x) - g(x)\big| dx.}$$

Example Calculate the area between the curves $y = -x^2 + 3x$ and $y = 2x^3 - x^2 - 5x$.



If we are dealing with functions of y, the area between the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ can found by using the same methods and an integral with respect to y.



In this case The area between the curves is given by

$$\boxed{A = \int_c^d \big(f(y) - g(y)\big)\,dy}$$

Example Find the area enclosed by the parabola $x = y^2$ and the line $x = y + 2$.

Example Find the area enclosed by the curves $x = \cos y$, $x = 2 - \cos y$ and the lines $y = 0$ and $y = \pi$.

A sketch of these two curves will show that between $y = 0$ and $y = \pi$, the curves meet only at $y = 0$. Also on that interval $2 - \cos x \geq \cos x$. Therefore the area between the curves for $0 \leq y \leq \pi$ is given by

$$\int_0^\pi 2 - \cos y - \cos y \; dy = \int_0^\pi 2 - 2\cos y \; dy = 2y - 2\sin y \Big|_0^\pi = 2\pi.$$