§ 28. Integration by Substitution:

Recall: (Chain Rule)

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\,g'(x)$$

Remark: 'Reversing' this rule gives us:

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

Example: $f(g(x)) = (x^2+1)^2$

(i) Find $(f\circ g)'(x)$.

(ii) $\int 4x(x^2+1)\,dx =$

The Substitution Rule:

If g is a differentiable and f is continuous on the range of $g = I$, some interval, then:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$, $du = g'(x)\,dx$.

Method:

1) Identify g (check if one piece of the integrand differentiates to another piece).
2) Let $u = g(x)$, $du = g'(x)\,dx$
3) Integrate w.r.t. u.
4) Fill back in $g(x)$ for u.

Example: Find $\int 2x\sqrt{x^2+1}\,dx$.

Sometimes your substitution may result in an integral of the form $\int f(u)c\, du$ for some constant c, which is not a problem.

Example Find the following:

$$\int x^3\sqrt{x^4+1}\, dx, \qquad \int \sin^3 x \cos x\, dx, \qquad \int x\sin(x^2+3)\, dx$$

Sometimes the appropriate substitution is non-obvious and you may have to work a little harder to put the resulting integral in the form $\int f(u)du$:

Example Find the following :

$$\int \frac{x^3}{\sqrt{x^2+1}}\, dx$$

The Definite Integral

The Substitution Rule For Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof If F is an antiderivative for f, we have

$$\int_a^b f(g(x))g'(x)\,dx = \int_a^b F'(g(x))g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)).$$

On the other hand, letting $u = g(x)$, we have

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

This gives us two options for calculating a definite integral using substitution:

1. We can calculate the antiderivative in terms of x and use the original limits of integration to evaluate the definite integral or

2. we can change the limits of integration when we make the substitution, calculate the antiderivative in terms of u and evaluate using the new limits of integration.

Example Evaluate the following definite integral using both methods

$$\int_0^1 2x\sqrt{x^2+1}\,dx$$

Method 1 In our example above, we calculated $\int 2x\sqrt{x^2+1}\,dx = \frac{2(x^2+1)^{3/2}}{3} + C$. Using the fundamental theorem of calculus, we get

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \frac{2(x^2+1)^{3/2}}{3}\Big|_0^1 = \frac{2(2)^{3/2} - 2(1)^{3/2}}{3} = \frac{4\sqrt{2}-2}{3}.$$

Method 2 As in the example above, we substitute $u = x^2 + 1$. When we change the variable, we also change the limits of integration. When $x = 0$, $u = u(x) = u(0) = 1$, when $x = 1$, $u = u(x) = u(1) = 2$. Our transformed integral is now given by

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \int_1^2 \sqrt{u}\,du = \frac{2u^{3/2}}{3}\Big|_1^2 = \frac{2(2)^{3/2} - 2(1)^{3/2}}{3} = \frac{4\sqrt{2}-2}{3}.$$

Example Evaluate the following definite integrals:

$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx, \qquad \int_2^3 x\sqrt{x^2+1}\,dx.$$

Even and Odd Functions

Sometimes we can use symmetry to make evaluation of integrals easier:
If f is an even function ($f(x) = f(-x)$), then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
If f is an odd function ($f(x) = -f(-x)$), then $\int_{-a}^{a} f(x)dx = 0$

Example Evaluate the following definite integrals:

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \tan^5 x dx, \qquad \int_{-1}^{1} x^4 + x^2 + 1 dx.$$

Extra Examples (Please attempt these before you check the solutions)

Example Find the following indefinite integrals:

$$\int \frac{x}{\sqrt{x^2+1}}\, dx, \qquad \int \sin(2x+1)\, dx$$

Example (tricky - ish) Find the following :

$$\int \sin^2 x \cos^3 x \, dx$$

Example Evaluate the following definite integrals:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta \cos\theta \, d\theta, \qquad \int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx \quad \text{(Use results from previous example)}$$

Extra Examples Solutions

Example Find the following indefinite integrals:

$$\int \frac{x}{\sqrt{x^2+1}}\,dx, \qquad \int \sin(2x+1)\,dx$$

Ex 1.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx$$

Let $u = x^2 + 1$,
$du = 2xdx \rightarrow xdx = \frac{du}{2}$.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}\frac{\sqrt{u}}{(1/2)} + C = \sqrt{x^2+1} + C$$

Ex 2.

$$\int \sin(2x+1)\,dx$$

Let $u = 2x + 1$,
$du = 2dx \rightarrow dx = \frac{du}{2}$.

$$\int \sin(2x+1)\,dx = \int \sin(u)\,\frac{du}{2} = \frac{1}{2}\int \sin u\,du = \frac{-\cos u}{2} + C$$

Example (tricky - ish) Find the following :

$$\int \sin^2 x \cos^3 x\,dx$$

We let $u = \sin x$ and replace the extra $\cos^2 x$ by $1 - u^2$. We get $du = \cos x\,dx$ and

$$\int \sin^2 x \cos^3 x\,dx = \int u^2(1-u^2)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Example Evaluate the following definite integrals:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta\cos\theta\ d\theta, \qquad \int_1^2 \frac{x}{\sqrt{x^2+1}}\ dx \quad \text{(Use results from previous example)}$$

Ex 1:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3\theta\cos\theta\ d\theta$$

Let $u = \sin\theta$,
then $du = \cos\theta\ d\theta$.
Changing the limits, we get
$u(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$u(\frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin\theta)^3\cos\theta\ d\theta = \int_{u(\frac{\pi}{4})}^{u(\frac{\pi}{3})} u^3 du = \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} u^3 du = \frac{u^4}{4}\bigg|_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \frac{1}{4}\left[\frac{(\sqrt{3})^4}{16} - \frac{1}{(\sqrt{2})^4}\right]$$
$$= \frac{1}{4}\left[\frac{9}{16} - \frac{1}{4}\right] = \frac{1}{4}\left[\frac{5}{16}\right] = \frac{5}{64}.$$

Ex. 2 (using method 1): Above, we saw that

$$\int \frac{x}{\sqrt{x^2+1}}\ dx = \sqrt{x^2+1} + C$$

So

$$\int_1^2 \frac{x}{\sqrt{x^2+1}}\ dx = \sqrt{x^2+1}\bigg|_1^2 = \sqrt{4+1} - \sqrt{1+1} = \sqrt{5} - \sqrt{2}.$$