§27. The Indefinite Intergal:

Last time we saw how knowing the antiderivative of a continuous function $f(x)$ allows us to compute the definite integral: $\int_a^b f(x)\,dx$.

Recall: If F is an antiderivative of f:

1) $\int f(x)\,dx = F(x) + C$

2) $F'(x) = f(x)$

Remarks:

1) $\int f(x)\,dx$ is called the indefinite integral or general antiderivative of f, and it is a family of functions.

2) $\int_a^b f(x)\,dx$ is called the definite integral of f over $[a,b]$, and it is a number.

Below is a list of useful antiderivatives:

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int kdx = kx + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \csc x \cot x dx = -\csc x + C$

Remark: $1/x^n$, $n > 0$ and $\tan(x)$ are not cts on all of $\mathbb{R}$. For functions which are not continuous everywhere, we can find antiderivatives over intervals where the function is continuous, and represent the antiderivative as a piecewise defined function.

Example:

$$\int \frac{1}{x^2}\, dx = \begin{cases} \frac{-1}{x} + C_1, & \text{for } x < 0 \\ \frac{-1}{x} + C_2, & \text{for } x > 0 \end{cases}$$

Example Find the indefinite integral:

$$\int x^3 + 2x + 5dx$$

Example Find the indefinite integral:

$$\int \frac{x^{5/2} + 5x^3 + 3}{x^2} dx$$

Example Find the indefinite integral

$$\int \frac{\sin x}{\cos^2 x} dx$$

Example Find the values of the definite integrals listed below:

$$\int_1^2 x^3 + 2x + 5dx, \qquad \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x} dx, \qquad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x + 2\cos x dx$$

Interpretation and application of the Definite Integral:

Recall: (F.T.C. – Part 2)

1) $$\int_a^b f(x)\,dx = F(b) - F(a)$$

for f continuous on $[a,b]$, F an antiderivative of f.

2) Since $F'(x) = f(x)$, we can rewrite this as:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad (*)$$

Remark: If we consider $F(x)$ as a quantity:

1) $F(b) - F(a)$ is the net change over $[a,b]$.

2) $F'(x)$ is the rate of change.

So, now $(*)$ says:

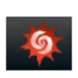
$$\boxed{\int_a^b (\text{rate of change})\,dx = \text{net change between } a \text{ and } b}$$

Some common applications of the definite integral are as follows:

- If $V(t)$ is the volume of water that has passed through a pipe at time t, then $V'(t)$ is the rate of flow of water at time t and
$$\int_{t_1}^{t_2} V'(t)dt$$
is the volume of water that has passed through the pipe between time t_1 and time t_2.

- If the rate of growth of a population is given by $P'(t)$, then the net change in the population during the time period from t_1 to t_2 is given by
$$\int_{t_1}^{t_2} P'(t)dt = P(t_2) - P(t_1).$$

- If the cost of producing x units of a commodity is given by $C(x)$ and the marginal cost of producing x units of a commodity is $C'(x)$, then the increase in cost from raising production levels from $x = x_1$ to $x = x_2$ is
$$\int_{x_1}^{x_2} C'(x)dx = C(x_2) - C(x_1).$$

-  If an object is moving along a straight line with position function $s(t)$ and velocity $s'(t) = v(t)$ then
$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$
is the net change in position or displacement of the object during the time period from t_1 to t_2.

- To calculate the distance that the object above travels during the time period from t_1 to t_2, we must integrate the speed function. The distance travelled by the object during the time period from t_1 to t_2 is
$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance travelled.}$$

Example A particle moves along a straight line. The velocity at time t is given by the $v(t) = t^2 - 4$ m/s.

(a) Find the displacement of the particle during the time period $0 < t < 3$.

(b) Find the distance travelled during this time period.

Example Water flows from a tank at the rate of $r(t) = 100 - 2t$ gallons per minute. How much water flows from the tank in the first 5 minutes?

Example The acceleration of a particle moving in a straight line is given by $a(t) = 2t + 1$ m/s^2. It is known that the initial velocity of the particle is $v(0) = 3$, find the velocity on the interval $0 \leq t \leq 10$ and find the distance travelled in the first 10 minutes.