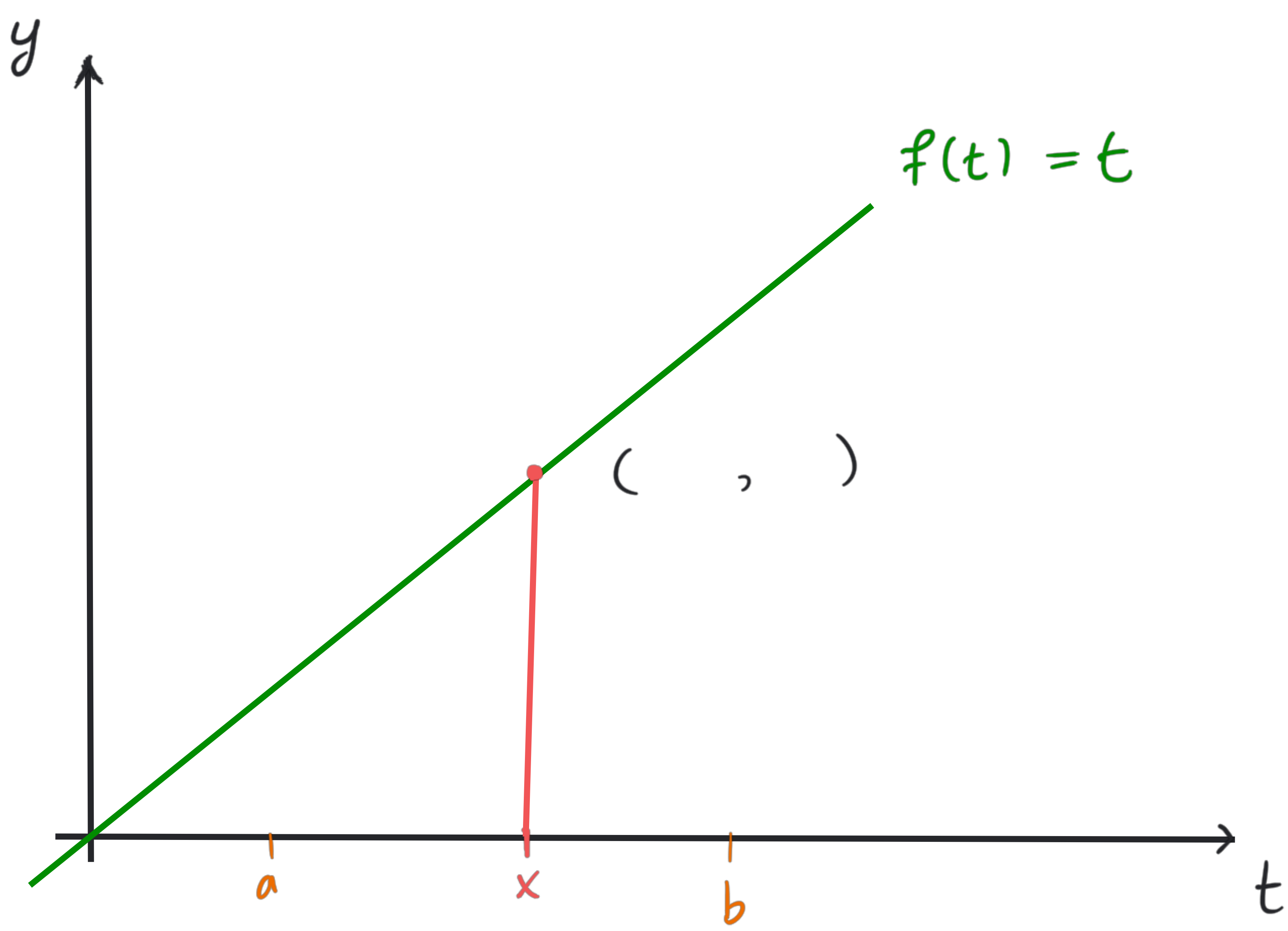
§ 26. The Fundamental Theorem of Calculus:

Roughly speaking, the Fundamental Theorem of Calculus says that differentiation and integration are inverse processes.

Example: Let $f(t) = t$. Take $x \in [a,b]$.



$$\frac{d}{dx}\left\{ \int_a^x f(t)\,dt \right\} = \frac{d}{dx}\left\{ \int_a^x t\,dt \right\} = \underline{\qquad\qquad}$$

Theorem: (F.C.T. - Part 1)

If f is continuous on $[a,b]$, then for $x \in [a,b]$:

$$\boxed{\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

Proof: Omitted.

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Remark: This shows $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

Example: Find the derivatives of the functions below:

(a) $g(x) = \int_0^x \sqrt{9+t^2}\,dt$

(b) $h(x) = \int_5^x \frac{1}{\sqrt{1+\cos^2(t)}}\,dt$

Remarks:

1) It can also be shown that

$$\frac{d}{dx}\left[\int_b^x f(t)\,dt\right] = f(x)$$

2) However,

$$\frac{d}{dx}\left[\int_x^b f(t)\,dt\right] = \frac{d}{dx}\left[-\int_b^x f(t)\,dt\right] = -f(x)$$

Example: Find the derivative of $F(x) = \int_x^1 \frac{1}{3+\cos(u)}\,du$.

Remark: We can also use chain rule with the F.T.C.

Example: Find $g'(x)$ if $g(x) = \int_1^{x^2} \frac{1}{3+\cos(t)}\,dt$.

Theorem: (F.T.C. – Part 2)

Say f is continuous on $[a,b]$ with antiderivative F. Then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Hence, if f is continously differentiable on $[a,b]$, then:

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

Proof: Omitted.

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Examples: Evaluate the following integrals:

$$\int_{-1}^{1} x^2 dx, \qquad \int_{1}^{3} \frac{1}{x^2} dx \qquad \int_{0}^{\frac{\pi}{2}} \cos x dx \qquad \int_{0}^{\frac{\pi}{4}} \sqrt{x} + 2\sec^2 x dx$$

<u>Question</u>: Why doesn't the above method work for $\int_{-1}^{1} \frac{1}{t^2}\, dt$?