§ 25. The Definite Integral:

Definition: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function.

If $\lim_{n\to\infty} S_n = A \in \mathbb{R}$ for any sequence of Riemann sums with $\lim_{n\to\infty} \Delta x_k = 0$, we say f is integrable.

We define the definite integral of f over $[a,b]$ to be this limit:

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} S_n$$

Remarks:

1) Any continuous function is integrable.
2) Hence, for any continuous function, we may use any sequence of Riemann sums we want (left/right endpoint, midpoint, etc.) to compute the definite integral $\int_a^b f(x)\,dx$.

Examples:

1) Evaluate R_8 (the right endpoint approximation with 8 equal subintervals) of the function $f(x) = x^3 - 1$ on $[0,2]$.

Soln: $a = 0$, $b = 2$, $n = 8$.

So $\Delta x = \frac{2-0}{8} = \frac{1}{4}$, $x_k = k\Delta x = k/4$

$x_1 = 1/4$, $f(x_1) = (1/4)^3 - 1 = 1/64 - 1 = -63/64$

$x_2 = 1/2$, $f(x_2) = (1/2)^3 - 1 = 1/8 - 1 = -7/8$

$x_3 = 3/4$, $f(x_3) = (3/4)^3 - 1 = 27/64 - 1 = -37/64$

$x_4 = 1$, $f(x_4) = (1)^3 - 1 = 0$

$x_5 = 5/4$, $f(x_5) = (5/4)^3 - 1 = \frac{125}{64} - 1 = \frac{61}{64}$

$x_6 = 3/2$, $f(x_6) = (3/2)^3 - 1 = 27/8 - 1 = 19/8$

$x_7 = 7/4$, $f(x_7) = (7/4)^3 - 1 = 343/64 - 1 = 279/64$

$x_8 = 2$, $f(x_8) = (2)^3 - 1 = 8 - 1 = 7$

$$R_8 = \sum_{k=1}^{8} f(x_k)\Delta x = \left(f(x_1) + \dots + f(x_8) \right) \frac{1}{4} = 3.0625$$

2) Find a formula for R_n, for arbitrary n.

3) Find $\int_0^2 (x^3 - 1)\, dx$.

Net Signed Area

We saw in the previous section that if $f(x) > 0$ is a continuous function on the interval $[a, b]$, then the definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = A \quad (\text{where } \Delta x \to 0 \text{ as } x \to 0)$$

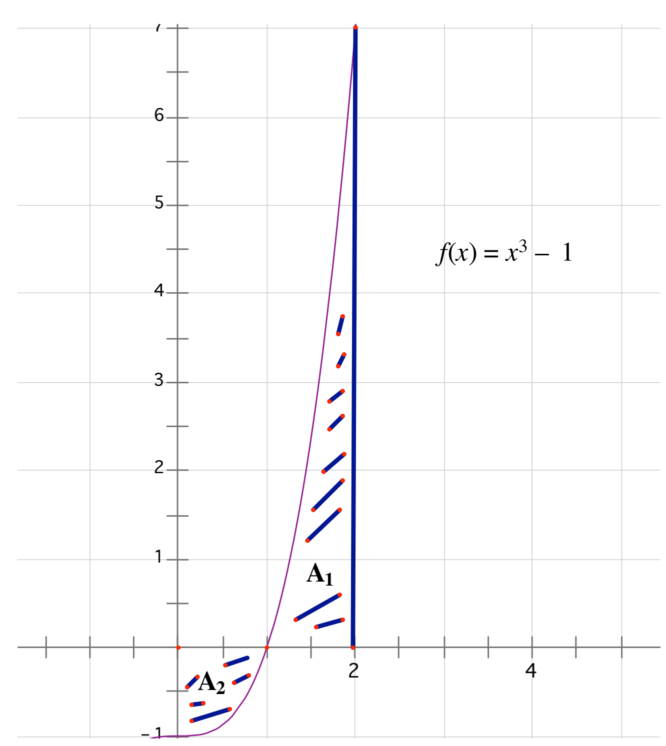
gives the **area under the curve** $y = f(x)$ **over the interval** $[a, b]$**.**

When $f(x)$ has both positive and negative values on the interval $[a, b]$, the definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = A_1 - A_2 \quad (\text{where } \Delta x \to 0 \text{ as } x \to 0)$$

gives the **net area** or net signed area, $A_1 - A_2$, where A_1 is sum of the areas of the regions between the graph of $f(x)$ and the $x-$ axis which are above the x-axis and A_2 is the sum of the areas of the regions between the graph of $f(x)$ and the $x-$ axis which are below the x-axis.

Example In the case of $f(x) = x^3 - 1$ on the interval $[0, 2]$, the graph is shown below:



Example Using the net signed area interpretation of the definite integral and geometry to evaluate the following definite integrals:

$$\int_{-3}^{3} \sqrt{9 - x^2}dx, \qquad \int_0^1 xdx, \qquad \int_{-1}^{1} xdx$$

It is important to be able to recognize the definite integral when we encounter it, because we will develop useful methods by which we can calculate the definite integral without taking limits of Riemann sums later.

Example Express the following limit of Riemann sums as a definite integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{x_i} \Delta x, \quad [\pi, 2\pi].$$

where $x_i = \pi + i\Delta x$ and $\Delta x = \frac{\pi}{n}$.

Integrability

As it turns out all continuous functions on an interval $[a, b]$ are integrable, in fact if a function has just a finite number of jump discontinuities on an interval $[a, b]$, it is integrable on $[a, b]$.

Theorem **If f is continuous on $[a, b]$ or if f has only a finite number of jump discontinuities on $[a, b]$**, then f is integrable on $[a, b]$, that is the definite integral $\int_a^b f(x)dx$ exists and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x.$$

Note that the sum for which the limit above is calculated is R_n, the right endpoint approximation to $\int_a^b f(x)dx$. We could equally well use the limit of the left endpoint approximation or the midpoint approximation. In fact **if the value of a definite integral is unknown, the midpoint approximation is frequently used to approximate it.** We will study other methods of approximation in Calculus 2

Midpoint Rule If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\bar{x_i})\Delta x = \Delta x(f(\bar{x_1}) + f(\bar{x_2}) + \cdots + f(\bar{x_n})),$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{and} \quad \bar{x_i} = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

Example Use the midpoint rule with $n = 4$ to approximate $\int_0^{2\pi} \sin(\frac{x}{2})dx$.
Fill in the tables below:
$\Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}$

x_i	$x_0 = 0$	$x_1 = \frac{\pi}{2}$	$x_2 = \pi$	$x_3 = \frac{3\pi}{2}$	$x_4 = 2\pi$

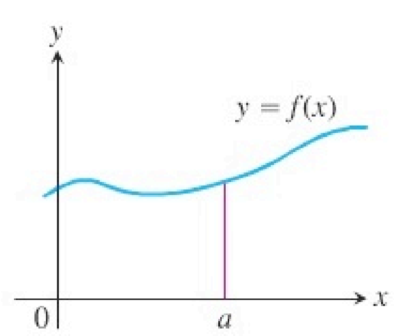
$\bar{x_i} = \frac{1}{2}(x_{i-1} + x_i)$	$\bar{x_1} = \frac{\pi}{4}$	$\bar{x_2} = \frac{3\pi}{4}$	$\bar{x_3} = \frac{5\pi}{4}$	$\bar{x_4} = \frac{7\pi}{4}$
$f(\bar{x_i}) = \sin\frac{\bar{x_i}}{2}$	$\sin\frac{\pi}{8}$	$\sin\frac{3\pi}{8}$	$\sin\frac{5\pi}{8}$	$\sin\frac{7\pi}{8}$

$M_4 = \sum_1^4 f(\bar{x_i})\Delta x = \left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8}\right)\Delta x = \left(0.3827 + 0.9239 + 0.9239 + 0.3827\right)\frac{\pi}{2} \approx 4.1048$

Properties of the Definite Integral

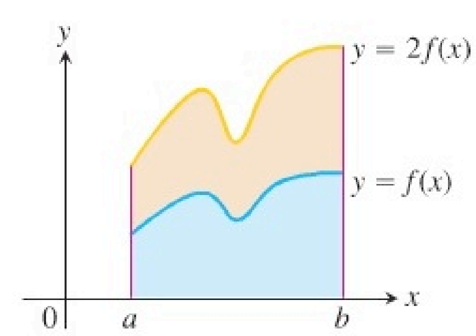
If f and g are integrable functions on $[a, b]$ (in particular if they are continuous) and if c is a constant, we have the following properties of the definite integrals:

1. Order of integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
2. Zero Width Interval: $\int_a^a f(x)dx = 0$.
3. Integral of a constant: $\int_a^b cdx = c(b-a)$
4. Constant multiple: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
5. Sum and Difference: $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
6. Additivity: $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.
7. Min-Max inequality: If f has maximum value M on $[a, b]$ and minimum value m on $[a, b]$, then
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$
8. Domination: if $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
   if $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f(x)dx \geq 0$.



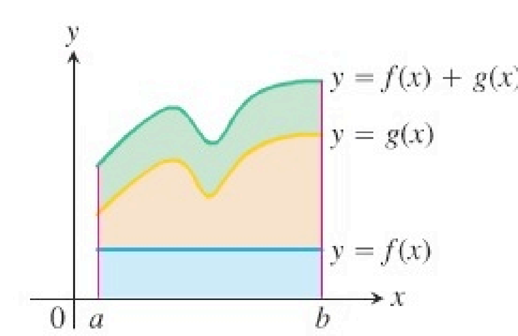
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



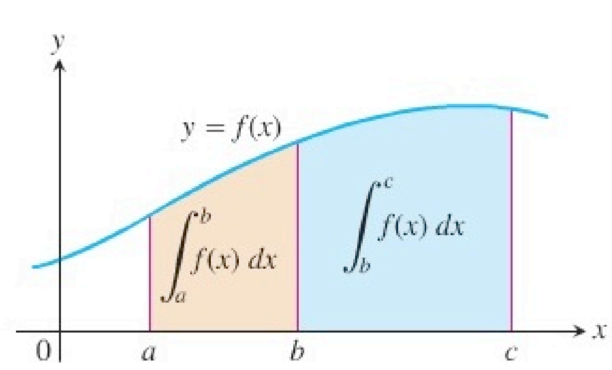
(b) *Constant Multiple:* ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



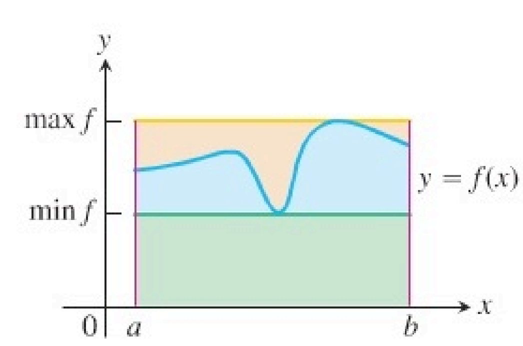
(c) *Sum:* (*areas add*)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



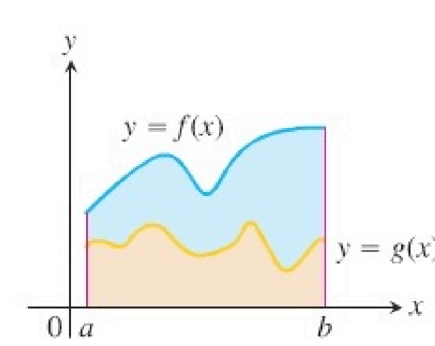
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \geq g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Example Recall that we have calculated the following integrals using limits of Riemann sums or geometry:

$$\boxed{\int_0^2 (x^3-1)dx = 2, \quad \int_{-3}^3 \sqrt{9-x^2}dx = \frac{9\pi}{2}, \quad \int_0^1 (1-x^2)dx = \frac{2}{3}, \quad \int_0^1 xdx = \frac{1}{2}.}$$

Using these results to evaluate the following integrals:

(a) $\int_0^1 x^2 dx$ (note $1-(1-x^2)=x^2$.)

(b) $\int_0^1 3x^2+2x+5dx$.

(c) $\int_3^{-3}\sqrt{9-x^2}dx$

(d) $\int_0^1(x^3-1)dx+\int_1^2(x^3-1)dx$

(e) Use property 7 to find upper and lower bounds for the definite integral

$$\int_0^2 1-x^3+\cos(10x)\ dx$$

(f) Find $\int_1^1 x^{100}+x^2+35\ dx$

(g) Use property 8 to find a lower bound for $\int_{-3}^3\sqrt{9-x^2}+x^4+x^6\ dx$.

Old Exam Questions Exam 3 Fall 2007 : # 6, 10, Exam 3 Fall 2008 : # 7, 10, 11(a),