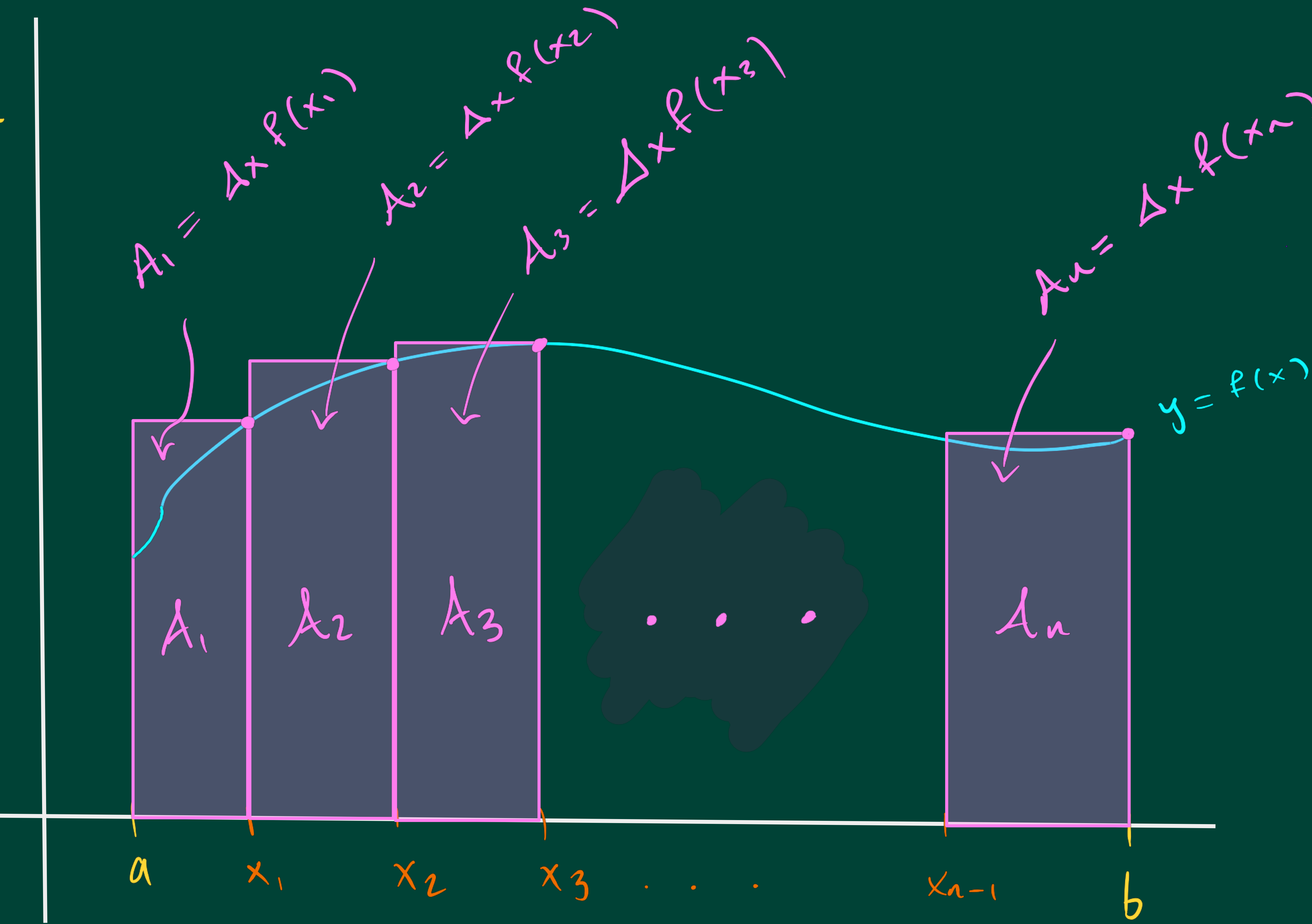
Right Endpoint Approximation:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n$$



$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

$$\boxed{\begin{aligned} R_n &= A_1 + \dots + A_n \\ &= \Delta x f(x_1) + \dots + \Delta x f(x_n) \\ &= \Delta x \left(f(x_1) + \dots + f(x_n) \right) \end{aligned}}$$

Midpoint Approximation:

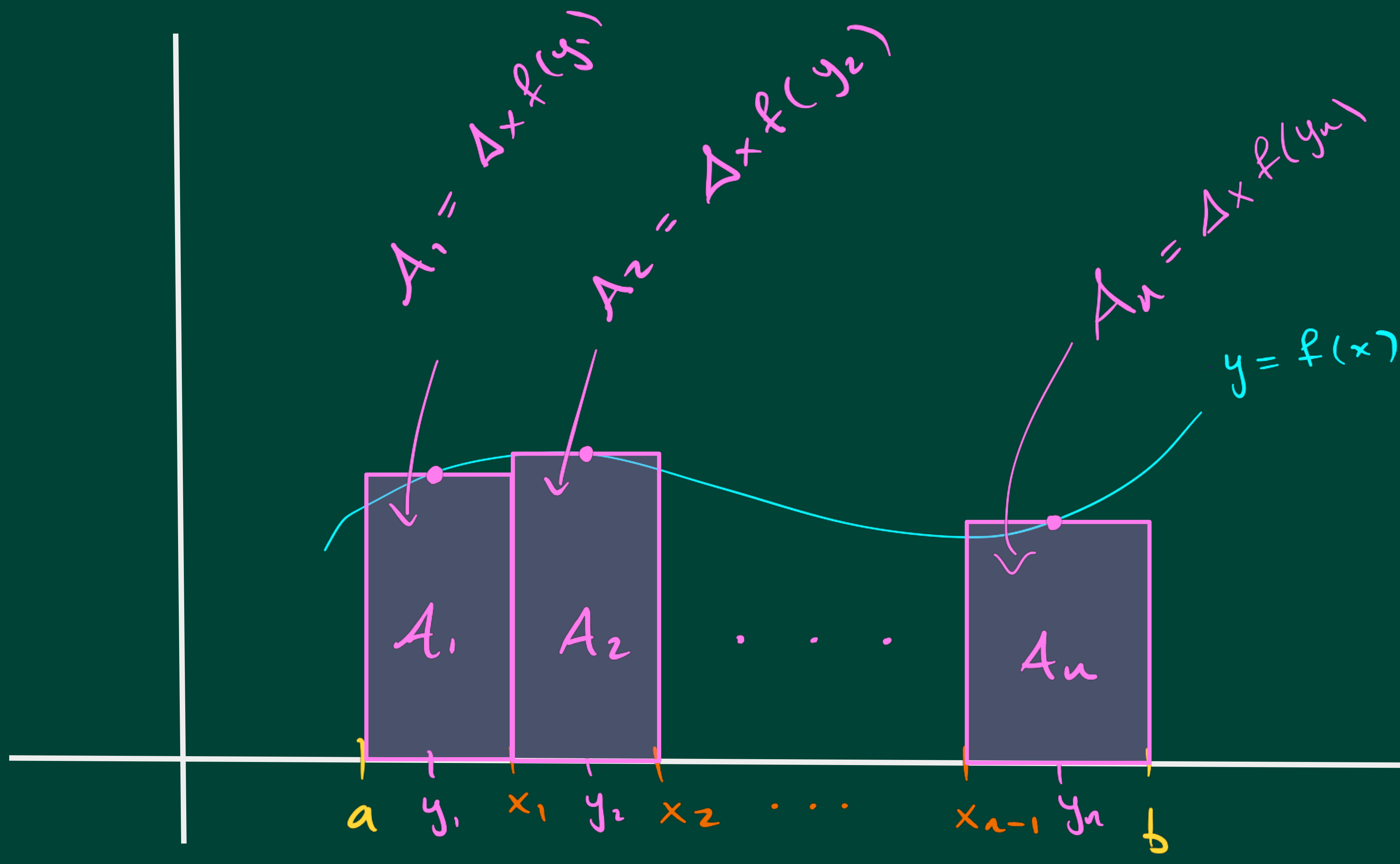
$$\Delta x = \frac{b-a}{n}$$

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$\vdots$$

$$x_i = a + i\Delta x$$



$$y_1 = \frac{a + x_1}{2} = \frac{a + a + \Delta x}{2} = \frac{2a + \Delta x}{2} = a + \frac{\Delta x}{2}$$

$$y_2 = \frac{x_1 + x_2}{2} = \frac{(a + \Delta x) + (a + 2\Delta x)}{2} = a + \frac{3\Delta x}{2}$$

$$\vdots$$

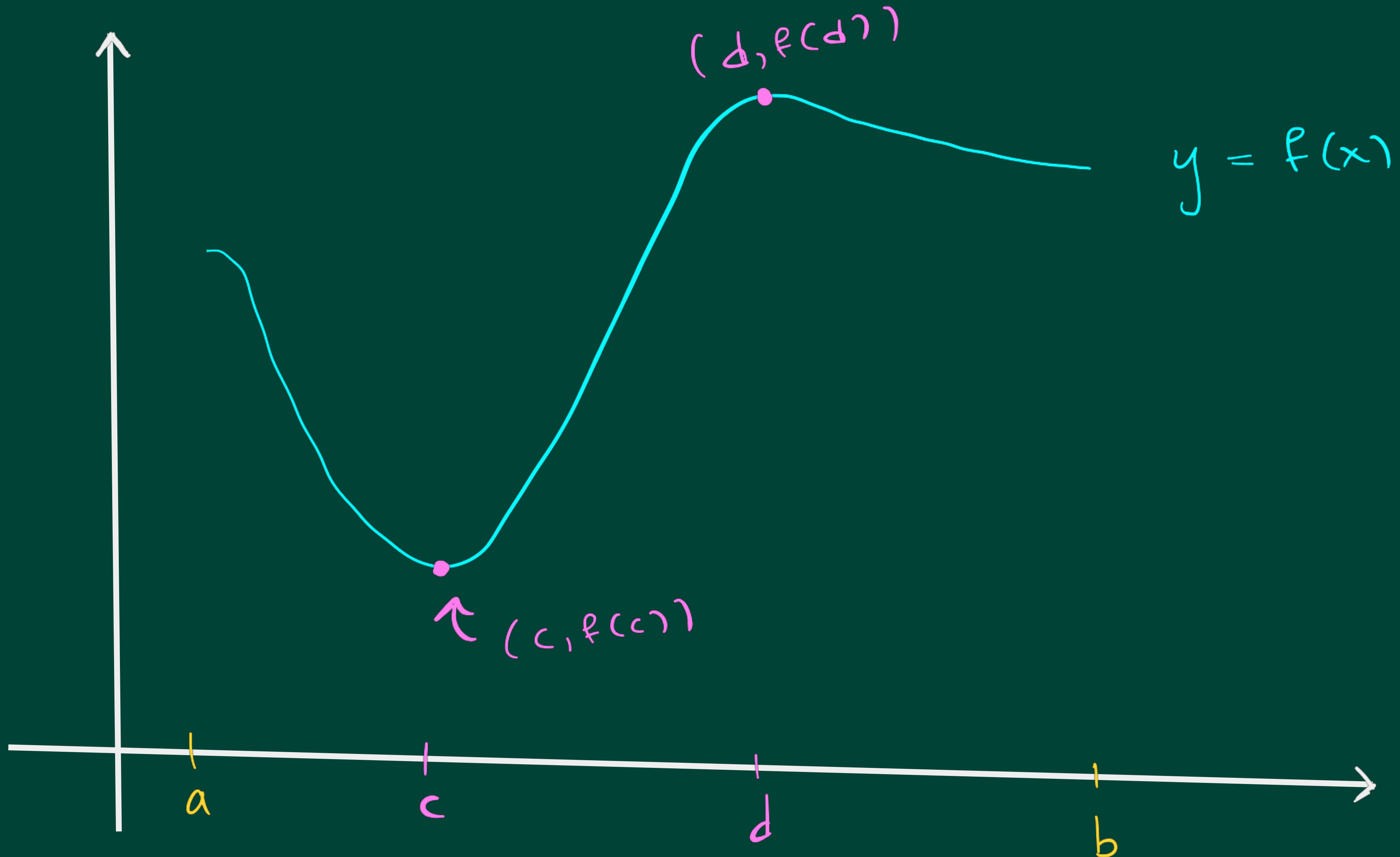
$$y_i = \frac{x_i + x_{i+1}}{2} = \frac{(a + i\Delta x) + (a + (i+1)\Delta x)}{2} = a + \frac{(2i+1)\Delta x}{2}$$

$$M_n = \Delta x\, f(y_1) + \cdots + \Delta x\, f(y_n) = \Delta x \left(f(y_1) + \cdots + f(y_n) \right)$$

$$= \Delta x \left(\sum_{i=1}^{n} f(y_i) \right)$$

$$= \Delta x \left(\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2} \right) \right)$$

$$= \Delta x \left(\sum_{i=1}^{n} f\left(a + (2i+1) \frac{\Delta x}{2} \right) \right)$$



Note:

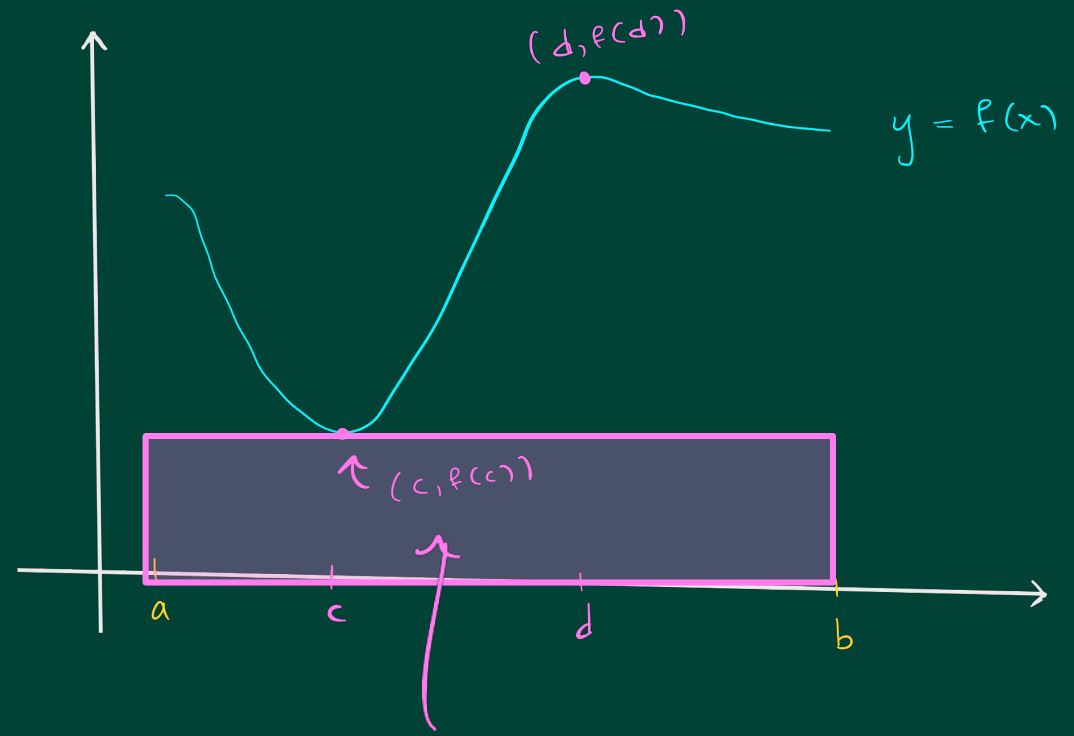
1) $f(c) \leq f(x)$, for all $x \in [a,b]$

2) $f(x) \leq f(d)$ for all $x \in [a,b]$

Then: Let $m := f(c)$, $M := f(d)$. So

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

As a picture:



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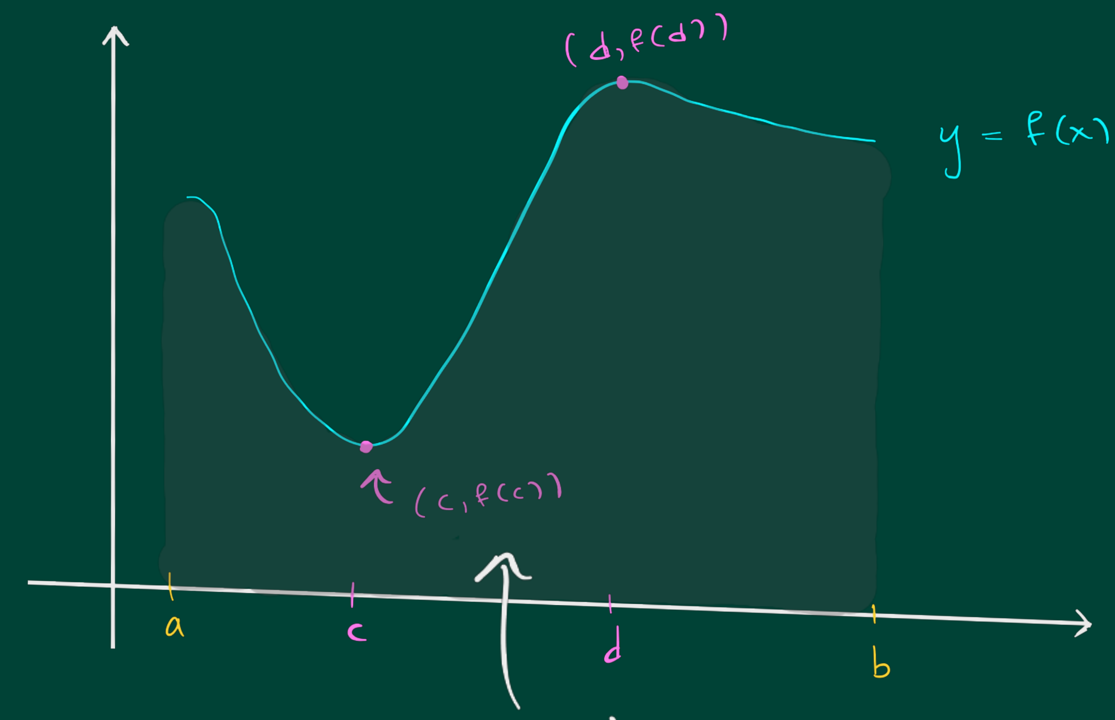

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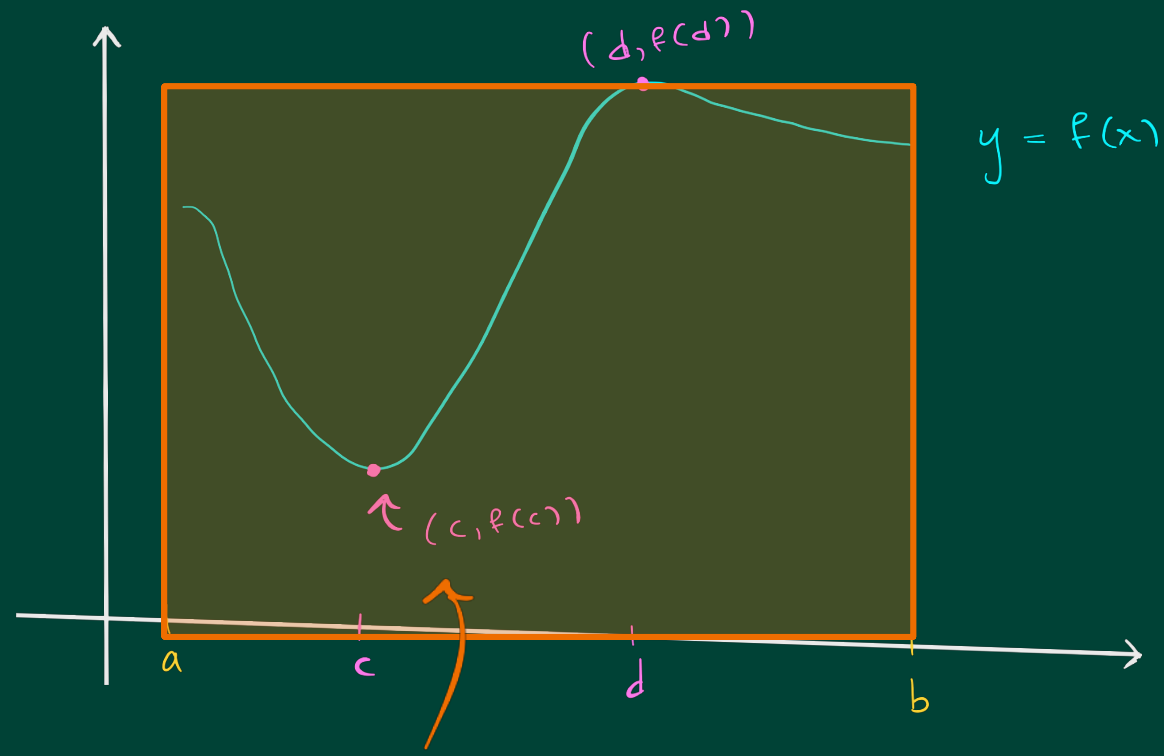


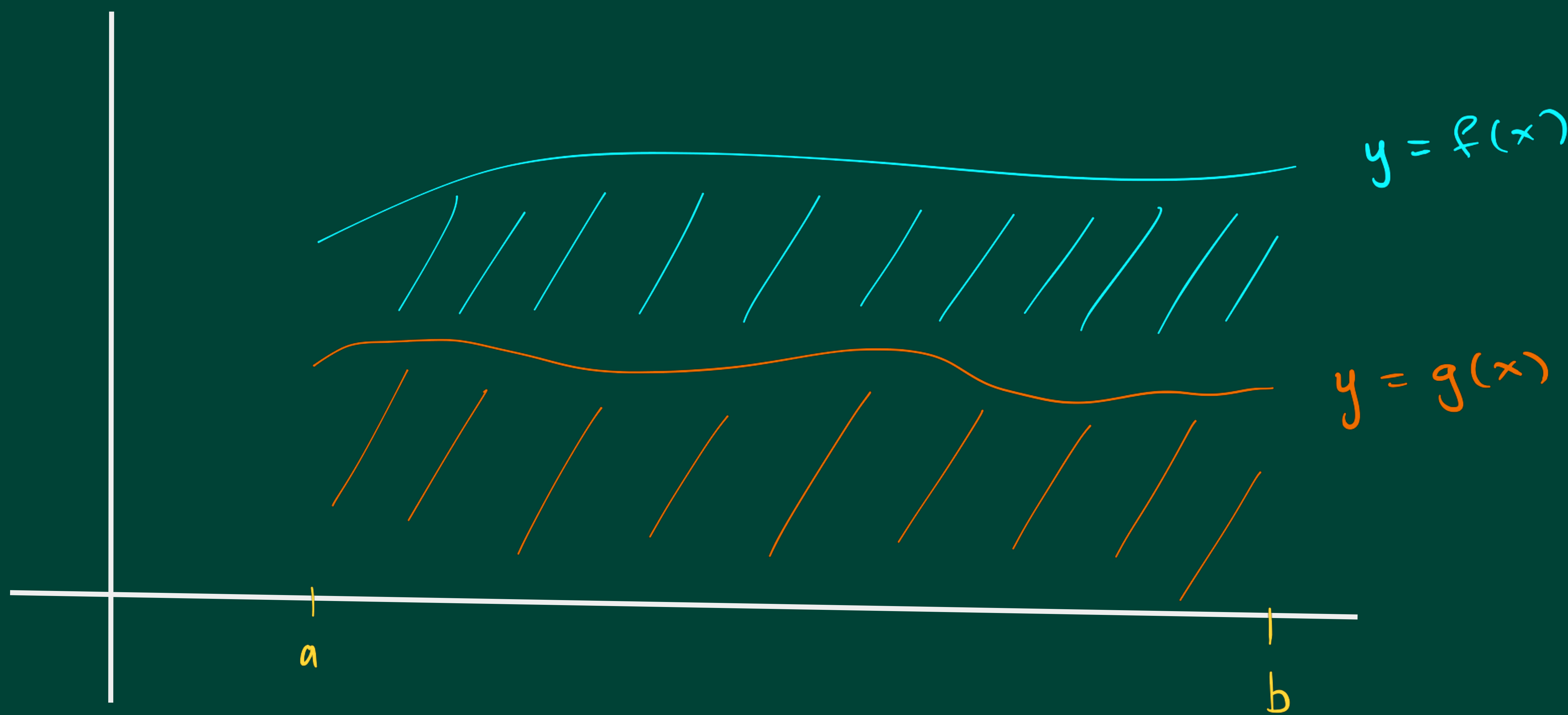
$$\text{Area} = f(c)(b-a) = m(b-a)$$

$$\text{Area} = \int_a^b f(x)\,dx$$

$$\text{Area} = f(d)(b-a) = M(b-a)$$

$$\Rightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

•) If $g(x) \le f(x)$ for all $x \in [a,b]$



$$\boxed{\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx}$$