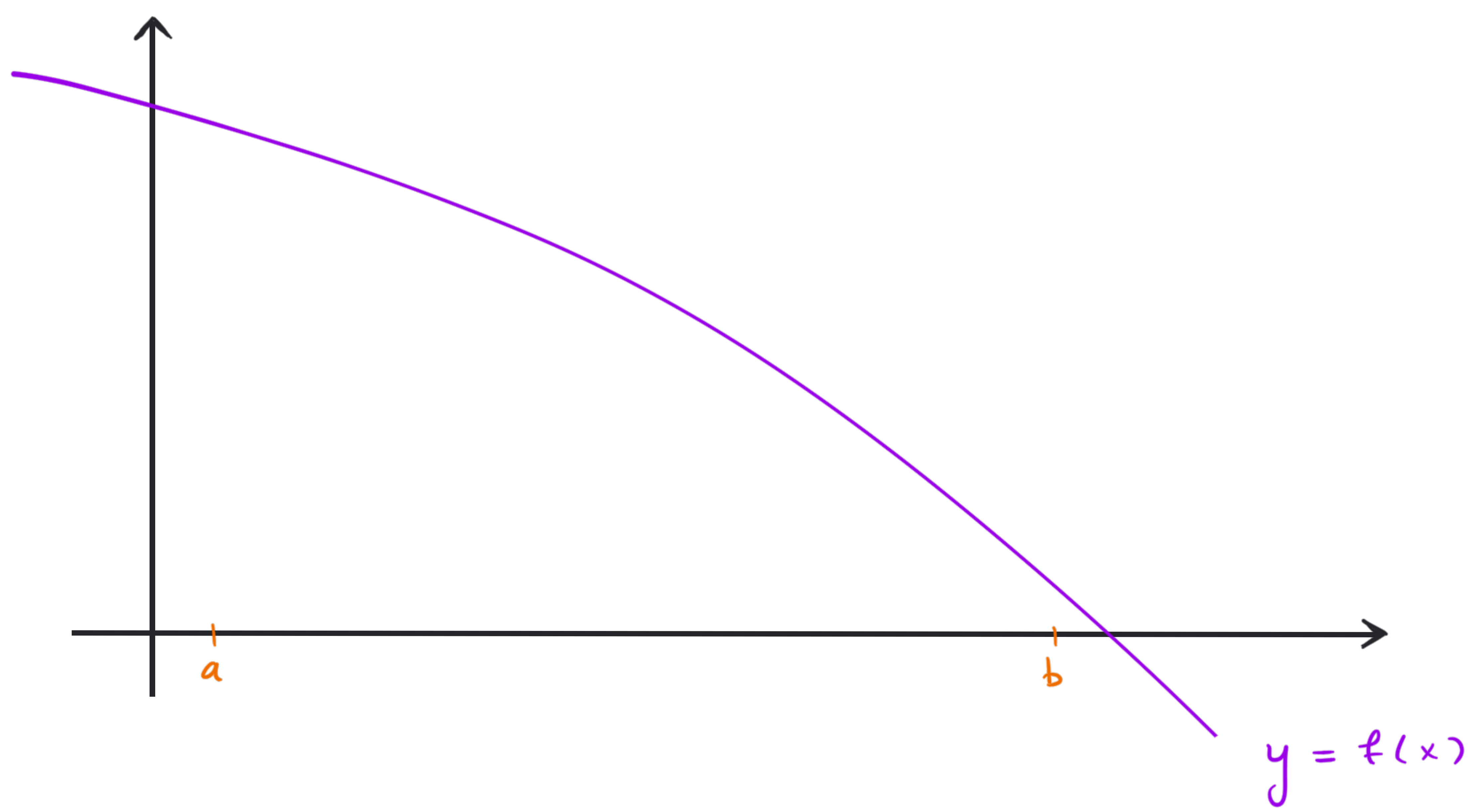
§24. Areas and distances:

Say you want to find the area under a curve $y = f(x)$ between two x-values a and b.

(with f continuous)

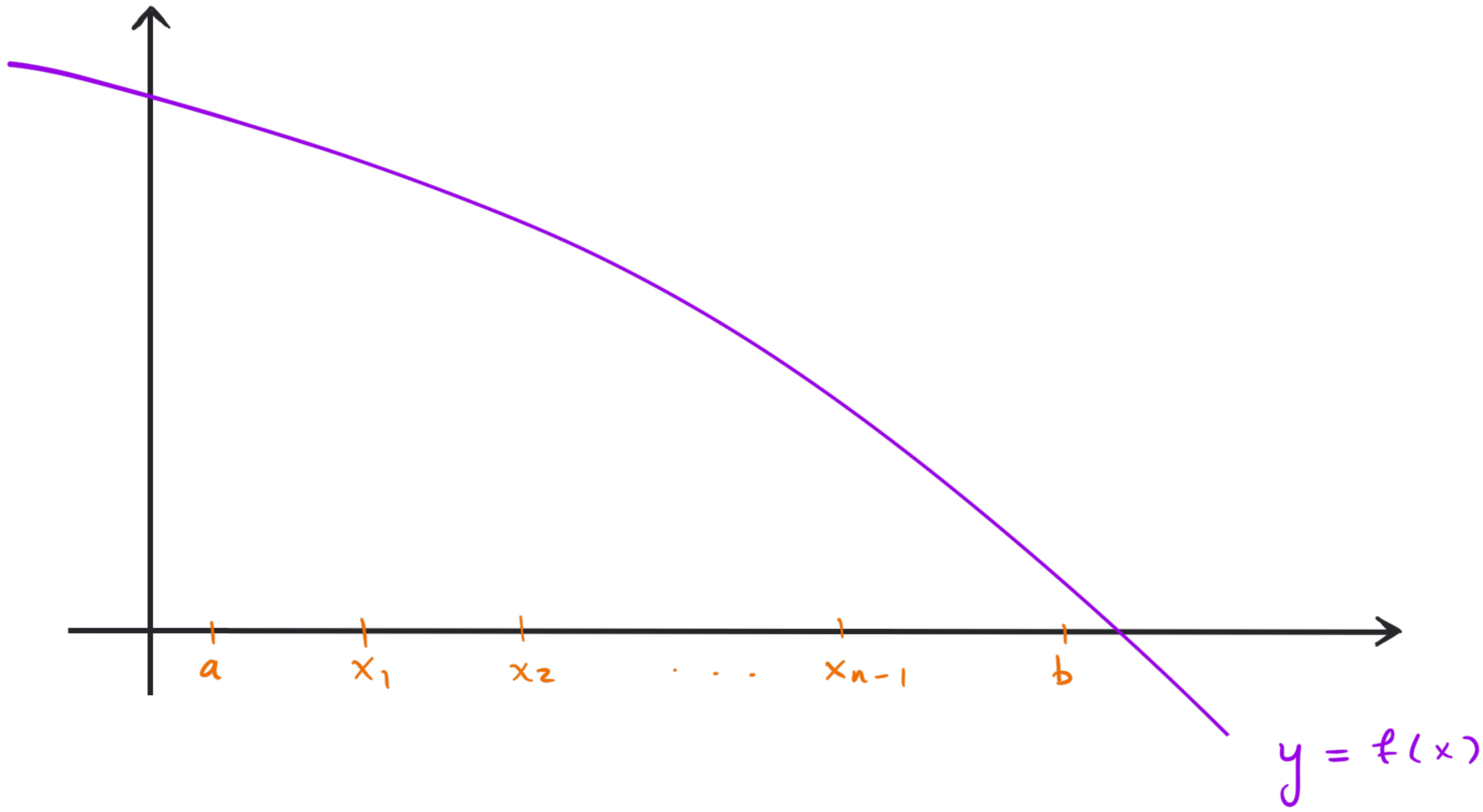


Method:

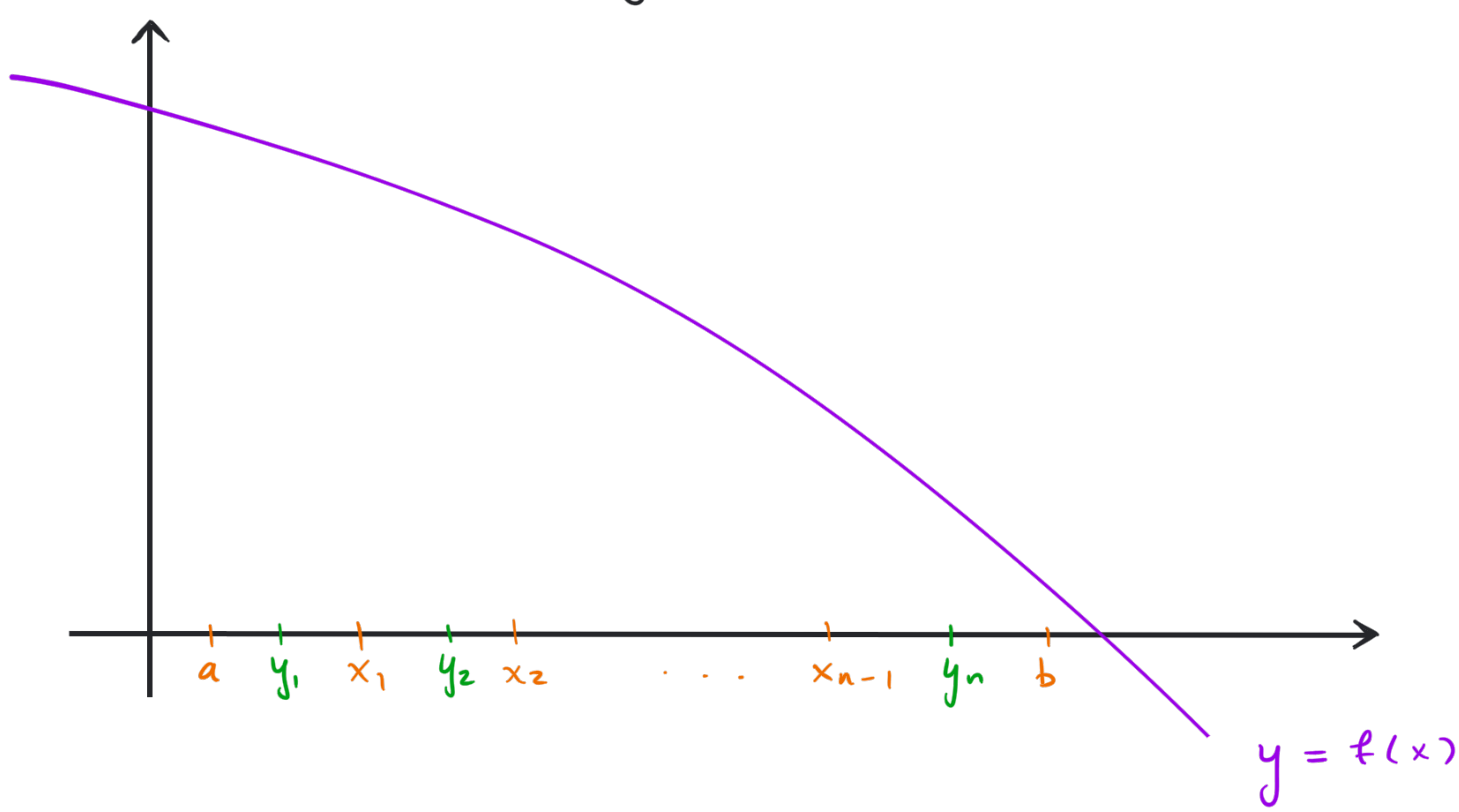
1) Divide the interval $[a, b]$ into smaller sub-intervals:

$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$

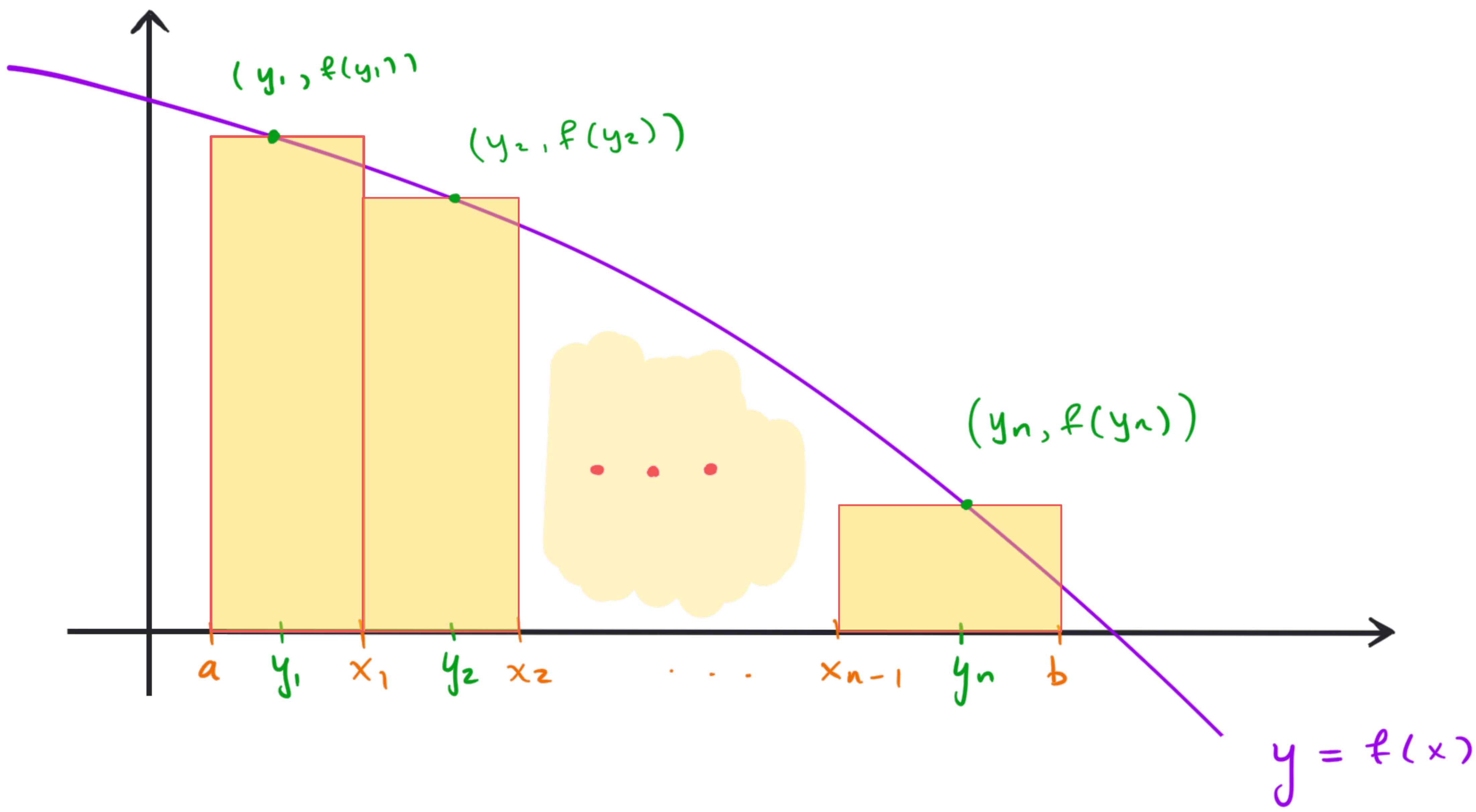
for some n.



2) Pick values $y_k \in [x_{k-1}, x_k]$



3) Draw rectangles of width $(x_k - x_{k-1})$ and height $f(y_k)$:



Idea: The sum of all the areas of the rectangles $\approx$ area under the graph between a and b $=: \mathcal{A}$.

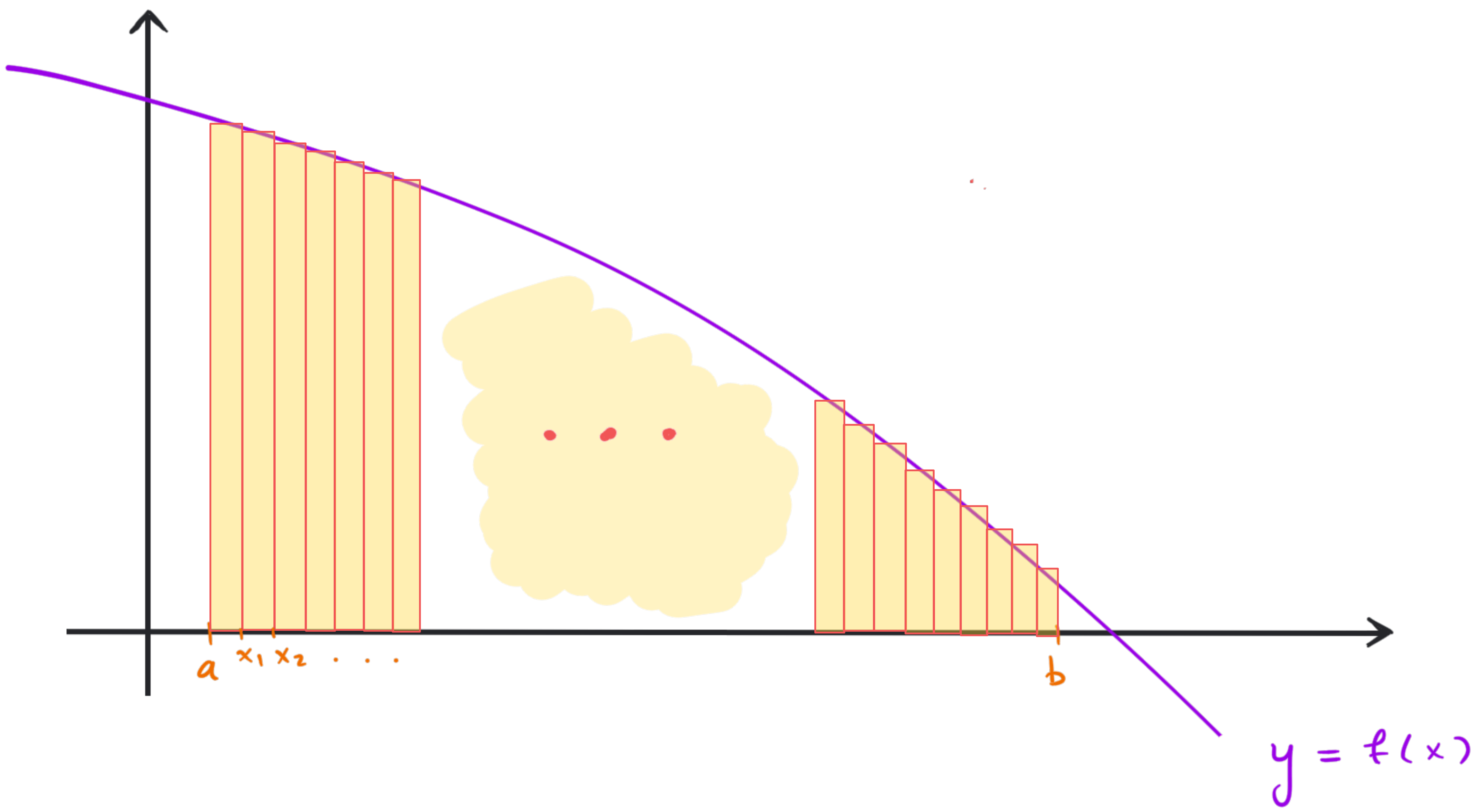
NB

i.e. $\boxed{\mathcal{A} \approx (x_1 - a) f(y_1) + (x_2 - x_1) f(y_2) + \cdots + (b - x_{n-1}) f(y_n)}$

Question: Can we make this approximation better?

Step 4: Do the process for a larger n.

i.e. make the subdivisions finer:



$$A \approx (x_1 - a) f(y_1) + (x_2 - x_1) f(y_2) + \dots + (b - x_{n-1}) f(y_n)$$

Step 5: Limit $n \to \infty$ in this process, REQUIRING

that $\Delta x_k := (x_k - x_{k-1})$, $1 \leq k \leq n$

(i.e. the "gap size") has

$$\lim_{n\to\infty} \Delta x_k = 0$$

i.e. "gap sizes" go to zero.

Notation:

1) If all the subintervals are of the same length: Δx (i.e. $x_k - x_{k-1} = \Delta x$ for all k), then

$$A \approx \Delta x\, f(y_1) + \Delta x\, f(y_2) + \dots + \Delta x\, f(y_n)$$

$$= \left(f(y_1) + f(y_2) + \dots + f(y_n) \right) \Delta x$$

2) These formulas are shorter using Σ notation:

$$\boxed{A \approx \sum_{k=1}^{n} (x_k - x_{k-1})\, f(y_k)}$$

(with the convention that $x_0 = a$, $x_n = b$).

If "gap-size" is fixed:

$$\boxed{A \approx \sum_{k=1}^{n} f(y_k)\, \Delta x}$$

Remark:

$$\boxed{\Delta x = \frac{b-a}{n}}$$

Definition: Any of these sums

$$\sum_{k=1}^{n} (x_k - x_{k-1}) f(y_k)$$

is called a Riemann Sum.

If you have divided $[a,b]$ into n intervals, it is called an nth Riemann Sum, denoted S_n.

Whole point: We can see the "error" decreases as we take finer and finer subdivisions.

Hence:

NB

$$\boxed{A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(y_k)\Delta x_k}$$

provided the "gap sizes" $\longrightarrow 0$.

Crucial question: Does the way we split up the interval / the y_k's we pick affect our answer in the limit?

Answer: No (as long as f is continuous and gap sizes go to zero).

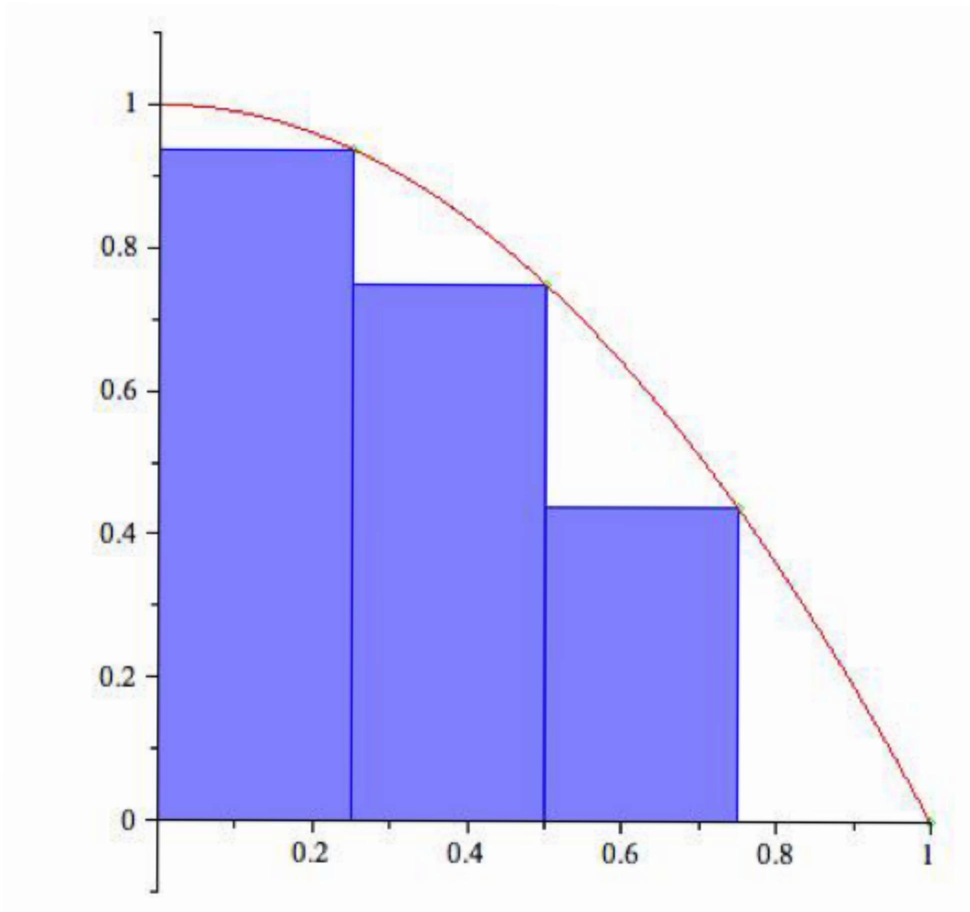
Remark: Sometimes we don't pick our y_k's randomly: If we make the gap size constant: $\Delta x = \frac{b-a}{n}$, and:

1) If we make $y_k = x_k$, we call this the Right endpoint approximation:

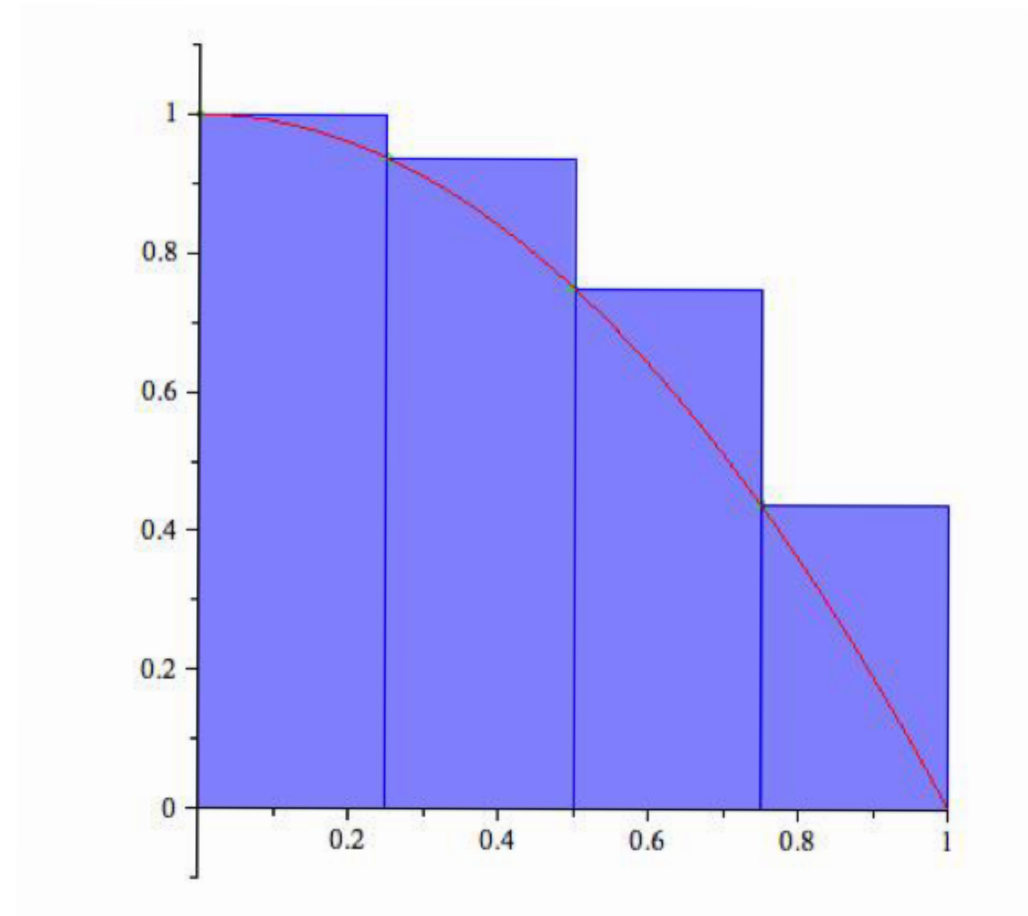
$$\boxed{R_n := \sum_{k=1}^{n} f(x_k)\,\Delta x}$$

2) If we make $y_k = x_{k-1}$, we call this the Left endpoint approximation:

$$\boxed{L_n := \sum_{k=1}^{n} f(x_{k-1})\,\Delta x}$$



Right endpoint approx.



Left endpoint approx.

Remark: $\quad A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$

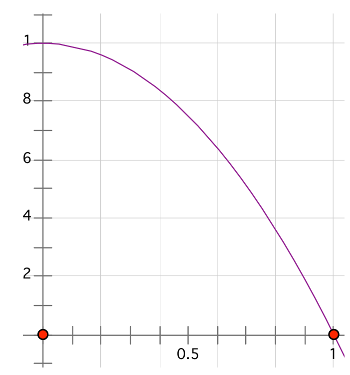
(provided that gap size $\to 0$).

Calculating Limits of Riemann sums

The following formulas are sometimes useful in calculating Riemann sums. I have attached some visual proofs at the end of the lecture.

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{(2n+1)n(n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2}$$

Let us now consider **Example 1**. We want to find $A =$ the area under the curve $y = 1 - x^2$ on the interval $[a, b] = [0, 1]$.



We know that $A = \lim_{n\to\infty} R_n$, where R_n is the right endpoint approximation using n approximating rectangles.

We must calculate R_n and than find $\lim_{n\to\infty} R_n$.

1. We divide the interval $[0, 1]$ into n strips of equal length $\Delta x = \frac{1-0}{n} = 1/n$. This gives us a partition of the interval $[0, 1]$,

$$x_0 = 0, \quad x_1 = 0 + \Delta x = 1/n, \quad x_2 = 0 + 2\Delta x = 2/n, \quad \ldots, \quad x_{n-1} = (n-1)/n, \quad x_n = 1.$$

2. We will use the right endpoint approximation R_n.

3. The heights of the rectangles can be found from the table below:

x_i	$x_0 = 0$	$x_1 = 1/n$	$x_2 = 2/n$	$x_3 = 3/n$	$\ldots$	$x_n = n/n$
$f(x_i) = 1 - (x_i)^2$	1	$1 - 1/n^2$	$1 - 2^2/n^2$	$1 - 3^2/n^2$	$\ldots$	$1 - n^2/n^2$

4.

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x =$$

$$\left(1 - \frac{1}{n^2}\right)\frac{1}{n} + \left(1 - \frac{2^2}{n^2}\right)\frac{1}{n} + \left(1 - \frac{3^2}{n^2}\right)\frac{1}{n} + \cdots + \left(1 - \frac{n^2}{n^2}\right)\frac{1}{n} =$$

$$\frac{1}{n} - \frac{1}{n^2}\left(\frac{1}{n}\right) + \frac{1}{n} - \frac{2^2}{n^2}\left(\frac{1}{n}\right) + \frac{1}{n} - \frac{3^2}{n^2}\left(\frac{1}{n}\right) + \cdots + \frac{1}{n} - \frac{n^2}{n^2}\left(\frac{1}{n}\right) =$$

5. Finish the calculation above and find $A = \lim_{n\to\infty} R_n$ using the formula for the sum of squares and calculating the limit as if R_n were a rational function with variable n.

Also $\quad A = \lim_{n\to\infty} L_n$

From Part 3, we have $\Delta x = 1/n$ and

$$L_n = \frac{1}{n} + \Big(1 - \frac{1}{n^2}\Big)\frac{1}{n} + \Big(1 - \frac{2^2}{n^2}\Big)\frac{1}{n} + \Big(1 - \frac{3^2}{n^2}\Big)\frac{1}{n} + \cdots + \Big(1 - \frac{(n-1)^2}{n^2}\Big)\frac{1}{n}$$

$$\frac{1}{n} + \frac{1}{n} - \frac{1}{n^2}\Big(\frac{1}{n}\Big) + \frac{1}{n} - \frac{2^2}{n^2}\Big(\frac{1}{n}\Big) + \frac{1}{n} - \frac{3^2}{n^2}\Big(\frac{1}{n}\Big) + \cdots + \frac{1}{n} - \frac{(n-1)^2}{n^2}\Big(\frac{1}{n}\Big) =$$

grouping the $\frac{1}{n}$'s together, we get

$$= \frac{n}{n} - \frac{1}{n}\Big[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\Big]$$

$$= 1 - \frac{1}{n^3}\big[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2\big]$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = 1 - \frac{1}{n^3}\Big[\frac{(2(n-1)+1)(n-1)((n-1)+1)}{6}\Big]$$

$$= 1 - \frac{1}{n^3}\Big[\frac{(2n-1)(n-1)(n)}{6}\Big]$$

$$= 1 - \frac{n}{6n^3}(2n-1)(n-1)$$

$$= 1 - \frac{2n^2 + \text{ smaller powers of } n}{6n^2}$$

So

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\Big[1 - \frac{2n^2 + \text{ smaller powers of } n}{6n^2}\Big] = 1 - \frac{2}{6} = 2/3.$$

Riemann Sums in Action: Distance from Velocity/Speed Data

To estimate distance travelled or displacement of an object moving in a straight line over a period of time, from discrete data on the velocity of the object, we use a Riemann Sum. If we have a table of values:

time $= t_i$	$t_0 = 0$	t_1	t_2	...	t_n
velocity $= v(t_i)$	$v(t_0)$	$v(t_1)$	$v(t_2)$	...	$v(t_n)$

where $\Delta t = t_i - t_{i-1}$, then we can approximate the displacement on the interval $[t_{i-1}, t_i]$ by $v(t_{i-1}) \times \Delta t$ or $v(t_i) \times \Delta t$. Therefore the total displacement of the object over the time interval $[0, t_n]$ can be approximated by

$$\boxed{\text{Displacement} \approx v(t_0)\Delta t + v(t_1)\Delta t + \cdots + v(t_{n-1})\Delta t \quad \text{Left endpoint approximation}}$$

or

$$\boxed{\text{Displacement} \approx v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t \quad \text{Right endpoint approximation}}$$

These are obviously Riemann sums related to the function $v(t)$, hinting that there is a connection between the area under a curve (such as velocity) and its antiderivative (displacement). This is indeed the case as we will see later.

When we use speed = |velocity| instead of velocity. the above formulas translate to

$$\boxed{\text{Distance Travelled} \approx |v(t_0)|\Delta t + |v(t_1)|\Delta t + \cdots + |v(t_{n-1})|\Delta t}$$

and

$$\boxed{\text{Distance Travelled} \approx |v(t_1)|\Delta t + |v(t_2)|\Delta t + \cdots + |v(t_n)|\Delta t}$$

Example The following data shows the speed of a particle every 5 seconds over a period of 30 seconds. Give the left endpoint estimate for the distance travelled by the particle over the 30 second period.

time in s $= t_i$	0	5	10	15	20	25	30
velocity in m/s $= v(t_i)$	50	60	65	62	60	55	50

$$L = |v(t_0)|\Delta t + |v(t_1)\Delta t + \cdots + |v(t_6)|\Delta t$$

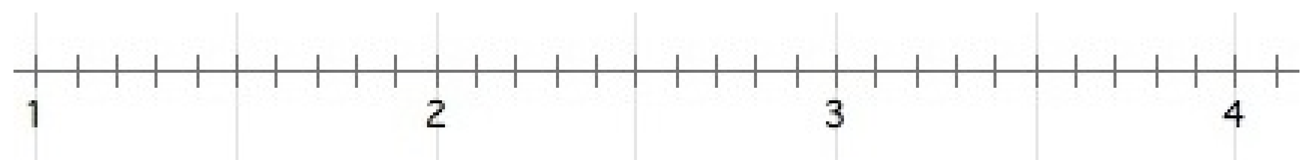
$$= 50(5) + 60(5) + 65(5) + 62(5) + 60(5) + 55(5)$$

$$= 5[50 + 60 + 65 + 62 + 60 + 55] = 1760m.$$

The above sum is a Riemann sum, telling us that the distance travelled is approximately the area under the (absolute vale of velocity) curve.... .... hmmmmm intetresting........ remember speed $= |v(t)| =$ derivative of distance travelled. ....

Extra Example Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 4$ using six approximating rectangles and

$\Delta x = \frac{b-a}{n} =$____________ , where $[a, b] = [1, 4]$ and $n = 6$.
Mark the points $x_0, x_1, x_2, \ldots, x_6$ which divide the interval $[1, 4]$ into six subintervals of equal length on the following axis:



Fill in the following tables:

x_i	$x_0 =$	$x_1 =$	$x_2 =$	$x_3 =$	$x_4 =$	$x_5 =$	$x_6 =$
$f(x_i) = 1/x_i$							

(a) Find the corresponding right endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$R_6 =$

(b) Find the corresponding left endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$L_6 =$

(c) Fill in the values of $f(x)$ at the midpoints of the subintervals below:

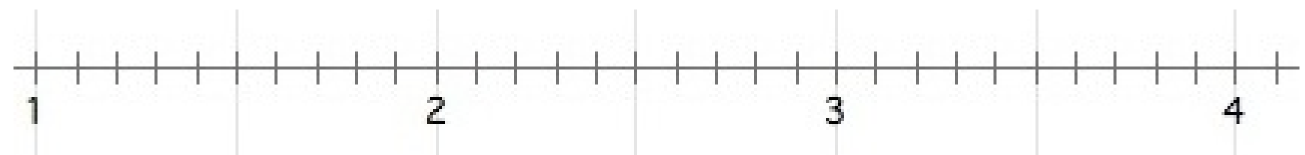
midpoint $= x_i^m$	$x_1^m =$	$x_2^m =$	$x_3^m =$	$x_4^m =$	$x_5^m =$	$x_6^m =$
$f(x_i^m) = 1/x_i^m$						

Find the corresponding midpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$M_6 =$

Extra Example Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 4$ using six approximating rectangles and

$\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{1}{2}$, where $[a, b] = [1, 4]$ and $n = 6$.
Mark the points $x_0, x_1, x_2, \ldots, x_6$ which divide the interval $[1, 4]$ into six subintervals of equal length on the following axis:



Fill in the following tables:

x_i	$x_0 = 1$	$x_1 = 3/2$	$x_2 = 2$	$x_3 = 5/2$	$x_4 = 3$	$x_5 = 7/2$	$x_6 = 4$
$f(x_i) = 1/x_i$	1	2/3	1/2	2/5	1/5	2/7	1/4

(a) Find the corresponding right endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$R_6 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$
$$= \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{5}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} + \frac{2}{7}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}$$
$$= \frac{2}{6} + \frac{1}{4} + \frac{2}{10} + \frac{1}{6} + \frac{2}{14} + \frac{1}{8} = 1.217857$$

(b) Find the corresponding left endpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$L_6 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$
$$= 1\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{5}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} + \frac{2}{7}\cdot\frac{1}{2}$$
$$= \frac{1}{2} + \frac{2}{6} + \frac{1}{4} + \frac{2}{10} + \frac{1}{6} + \frac{2}{14} = 1.59285$$

(c) Fill in the values of $f(x)$ at the midpoints of the subintervals below:

midpoint $= x_i^m$	$x_1^m = 5/4$	$x_2^m = 7/4$	$x_3^m = 9/4$	$x_4^m = 11/4$	$x_5^m = 13/4$	$x_6^m = 15/4$
$f(x_i^m) = 1/x_i^m$	4/5	4/7	4/9	4/11	4/13	4/15

Find the corresponding midpoint approximation to the area under the curve $y = 1/x$ on the interval $[1, 4]$.

$$M_6 = \sum_{i=1}^{6} f(x_i^*)\Delta x$$
$$= \frac{4}{5}\cdot\frac{1}{2} + \frac{4}{7}\cdot\frac{1}{2} + \frac{4}{9}\cdot\frac{1}{2} + \frac{4}{11}\cdot\frac{1}{2} + \frac{4}{13}\cdot\frac{1}{2} + \frac{4}{15}\cdot\frac{1}{2} = 1.376934$$

Extra Example Find the area under the curve $y = x^3$ on the interval $[0, 1]$.
We know that $A = \lim_{n\to\infty} R_n$, where R_n is the right endpoint approximation using n approximating rectangles.

We must calculate R_n and than find $\lim_{n\to\infty} R_n$.

1. We divide the interval $[0, 1]$ into n strips of equal length $\Delta x = \frac{1-0}{n} = 1/n$. This gives us a partition of the interval $[0, 1]$,
$$x_0 = 0, \quad x_1 = 0 + \Delta x = 1/n, \quad x_2 = 0 + 2\Delta x = 2/n, \quad \ldots, \quad x_{n-1} = (n-1)/n, \quad x_n = 1.$$

2. We will use the right endpoint approximation R_n.

3. The heights of the rectangles can be found from the table below:

x_i	$x_0 = 0$	$x_1 = 1/n$	$x_2 = 2/n$	$x_3 = 3/n$	$\ldots$	$x_n = n/n$
$f(x_i) = (x_i)^3$	0	$1/n^3$	$2^3/n^3$	$3^3/n^3$	$\ldots$	n^3/n^3

4.
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x =$$
$$\left(\frac{1}{n^3}\right)\frac{1}{n} + \left(\frac{2^3}{n^3}\right)\frac{1}{n} + \left(\frac{3^3}{n^3}\right)\frac{1}{n} + \cdots + \left(\frac{n^3}{n^3}\right)\frac{1}{n} =$$
$$= \sum_{i=1}^{n} \frac{i^3}{n^4} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

5.
$$A = \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2}$$
$$= \lim_{n\to\infty} \frac{1}{4}\cdot\frac{(n+1)}{n}\cdot\frac{(n+1)}{n} = \frac{1}{4}.$$

Extra Example, estimates from data on rate of change The same principle applies to estimating Volume from discrete data on its rate of change:
Oil is leaking from a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

time in h	0	1	2	3	4	5	6	7	8
leakage in gal/h	50	70	97	136	190	265	369	516	720

The following gives the right endpoint estimate of the amount of oil that has escaped from the tanker after 8 hours:

$$R_8 = 70 \cdot 1 + 97 \cdot 1 + 136 \cdot 1 + 190 \cdot 1 + 265 \cdot 1 + 369 \cdot 1 + 516 \cdot 1 + 720 \cdot 1 = 2363 \text{ gallons.}$$

The following gives the right endpoint estimate of the amount of oil that has escaped from the tanker after 8 hours:

$$L_8 = 50 \cdot 1 + 70 \cdot 1 + 97 \cdot 1 + 136 \cdot 1 + 190 \cdot 1 + 265 \cdot 1 + 369 \cdot 1 + 516 \cdot 1 = 1693 \text{ gallons.}$$

Since the flow of oil seems to be increasing over time, we would expect that
$L_8 <$ true volume leaked $< R_8$ or the true volume leaked in the first 8 hours is somewhere between 1693 and 2363 gallons.

Visual proof of formula for the sum of integers: