§ 23. Antiderivatives:

Idea: We have established how to take certain functions and find their derivatives:

e.g.

$$x^3 \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow 3x^2$$

$$\sin(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \cos(x)$$

$$f \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow f'$$

Question: Can we reverse the process?

$$\begin{tikzcd}
f \arrow[r] & \boxed{\frac{d}{dx}} \arrow[r] & f' \arrow[dl, bend left] \\
& \boxed{?} \arrow[ul, bend left] &
\end{tikzcd}$$

Answer: Kind of.

Definitions: A function F is called an antiderivative for g on an interval I if $F'(x) = g(x)$ for all $x \in I$.

Remark: Sometimes we can "spot" antiderivatives.

Example: Find an antiderivative of $f(x) = x^3$.

$F(x) =$

Question: Can you think of another?

Recall: We proved two theorems as results of the Mean Value Theorem:

Theorem 1: If F is an antiderivative of $f(x) = 0$, then $F(x) = c$ for some constant c.

Theorem 2: If F and G are antiderivatives of a function f, then there is a constant c such that $G(x) = F(x) + C$ for all x.

Definition: The general antiderivative of a function f is $F(x) + C$ where $F(x)$ is an antiderivative of f.

Remark: The traditional notation for the general antiderivative of a function f is:

$$\boxed{\int f(x)\,dx}$$

NB

If you specify a value of an antiderivative it makes it unique.

i.e. requiring $F(a) = b$ for some a and b.

Example:

(a) Find the general antiderivative of x^3.

(b) Find the unique antiderivative $F(x)$, such that $F(0) = 1$.

(c) Can you draw the curves $y = F(x)$ for different antiderivatives?

Remark: By "reversing" some of the rules for differentiation, we get rules for anti-derivatives:

1) $\int c f(x)\,dx = c\int f(x)\,dx$

2) $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

Below is a table of important anti-derivatives:

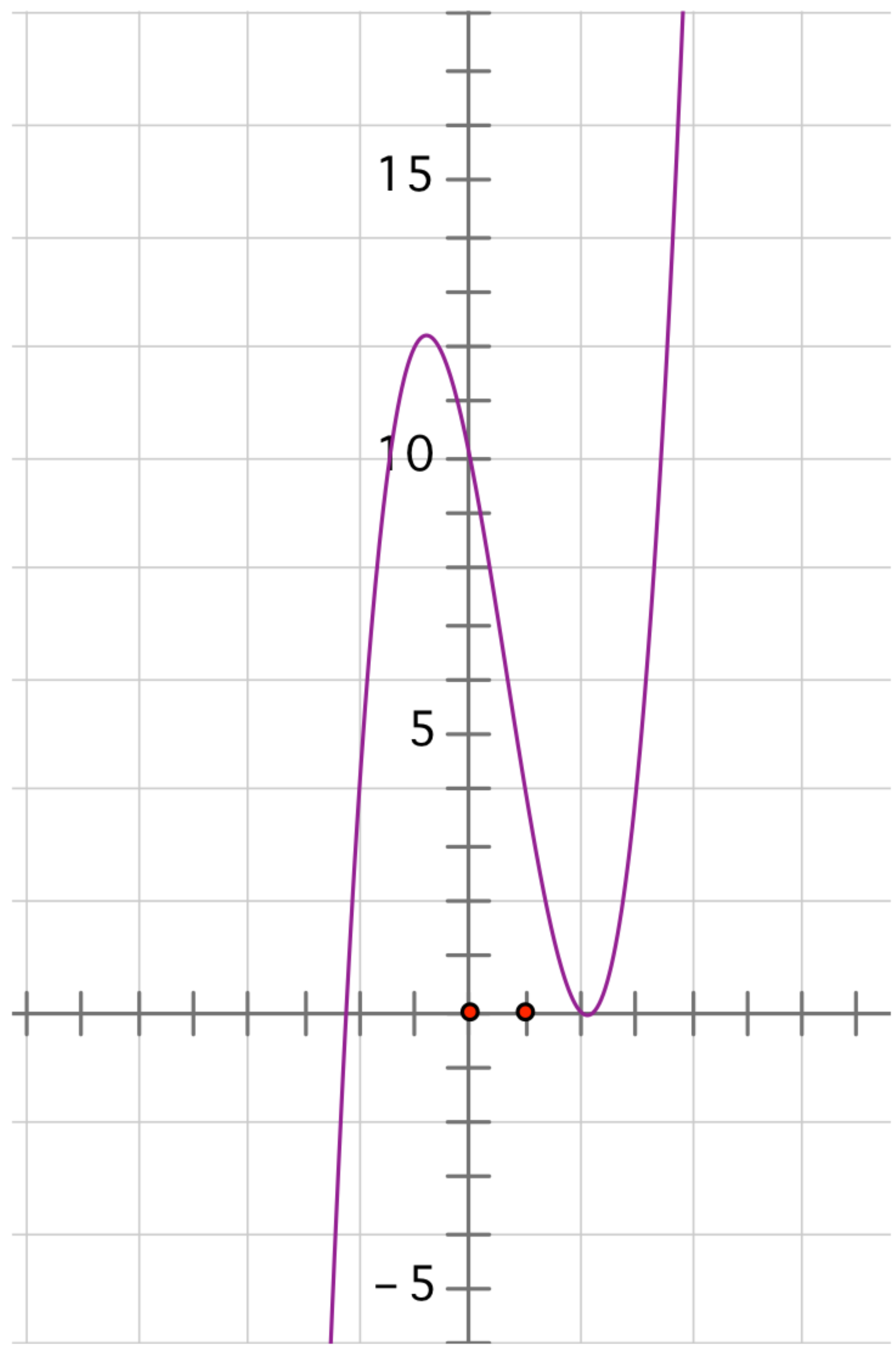
$f(x)$	$\int f(x)\,dx$
$x^n, \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$

Examples: (1) Find all functions F such that

$$F'(x) = 3\cos(x) + \sqrt[3]{x^2} + 4x^2$$

(2) Find $G(x)$ if $G'(x) = x^2 + 4x + 1$ and $G(0) = 1$.

Example The graph of a function $f(x)$ is shown in the diagram below, sketch the graph of an antiderivative of f.



Example A particle moving in a straight line has acceleration $a(t) = \cos t + \sin t$ where t is measured in seconds. We have the initial position is $s(0) = 0$ and initial velocity $v(0) = 5$ ft/s. Find the position function of the particle.

Example A ball is thrown vertically upwards with a speed of 10 ft/s from the edge of a cliff which is 400 ft. above the beach at its base. Let $h(t)$ denote its height in feet above the beach below t seconds later. When does the rock reach its maximum height and when does it hit the beach below? (Note acceleration due to gravity is $-32 ft/s^2$).

Solution Let $h(t)$ denote the height of the ball above the beach below at t seconds after it is thrown. After the ball is thrown, the only force acting on it is the force of gravity, therefore we know that

$$\boxed{h''(t) = -32 \text{ ft}/s^2.}$$

From the information given, we also have that $\boxed{h(0) = 400}$ and $\boxed{h'(0) = 10}$.
Now we know that $h'(t)$ is an antiderivative for $a(t) = -32$ ft/s^2. Since

$$\int -32 dt = -32t + C,$$

we must have that $h'(t) = -32t + C$ and since $h'(0) = C = 10$, we must have

$$h'(t) = -32t + 10.$$

The rock reached MAXIMUM HEIGHT when $h'(t) = 0$, that is when $10 = 32t$ or $t \approx 0.3125$ seconds. To find when the rock hits the beach below, we must find $h(t)$. This is an antiderivative for $h'(t)$.

$$\int h'(t)dt = \int (-32t + 10)dt = -16t^2 + 10t + D,$$

where D is a constant. Therefore $h(t) = -16t^2 + 10t + D$, for some constant D and since $h(0) = 400$, we have $D = 400$ and

$$h(t) = -16t^2 + 10t + 400.$$

The Rock HITS THE BEACH when $h(t) = 0$, that is when $-16t^2 + 10t + 400 = 0$ or

$$t = \frac{-10 \pm \sqrt{25700}}{-32} = 5.323 \text{ or } -4.64.$$

Since we are unable to throw rocks back in time, we have $\boxed{t = 5.323}$.

Extra Example A car driver fully applies the brakes producing a constant deceleration of $22 ft/s^2$ and producing skid marks (in a straight line) measuring 200 ft. as it comes to a halt. How fast was the car traveling when the brakes were first applied?

Please attempt this problem before you look at the solution on the next page.

Solution This car travels in a straight line, from the time the brakes are hit.
Let $S(t)$ denote the distance the car has travelled in feet t seconds after the brakes have been hit.
From the information given, we know that

$$\boxed{s(0) = 0} \text{ and } \boxed{s(t_1) = 200},$$

where t_1 is the number of seconds it takes for the car to come to a halt.
We let $v(t)$ denote the velocity of the car at time t and $a(t)$, the acceleration at time t. From the information given, we have

$$v(t_1) = 0 \quad \text{and} \quad a(t) = -22ft/s^2$$

for $0 \leq t \leq t_1$.

$$\boxed{\text{We want to find } v(0).}$$

Since $a(t) = v'(t)$, we have

$$v(t) = \int a(t)dt = \int (-22)dt = -22t + C.$$

Since $v(t_1) = 0$, we have $\boxed{C = 22t_1}$.
Since $s'(t) = v(t)$, we get

$$s(t) = \int v(t)dt = \int (-22t + C)dt = -11t^2 + Ct + D,$$

where D is a constant. Now since $s(0) = 0$, we get $\boxed{D = 0}$ and

$$s(t) = -11t^2 + Ct.$$

Using the fact that $C = 22t_1$ and $s(t_1) = 200$, we get

$$s(t_1) = -11t_1^2 + 22t_1^2 = 200 \quad \text{and} \quad 11t_1^2 = 200.$$

This gives that $\boxed{t_1 = \sqrt{\frac{200}{11}}}$ and

$$v(0) = C = 22t_1 = 22\sqrt{\frac{200}{11}} \approx 93.81 \ ft/s \approx 63.28 \ m.p.h.$$