§ 22. Newton's Method:

Motivation: Newton's method is a way of attempting to solve $f(x) = 0$ for a given function f. i.e. it is a method for finding roots of functions.

Remark: It doesn't always work.

Method:

1) Make a guess: $x_1 \in \mathbb{R}$.
2) The next "guess" is $x_2 := x_1 - \frac{f(x_1)}{f'(x_1)}$
3) The next "guess" is $x_3 := x_2 - \frac{f(x_2)}{f'(x_2)}$

$\vdots$

The $(n+1)$th "guess" is

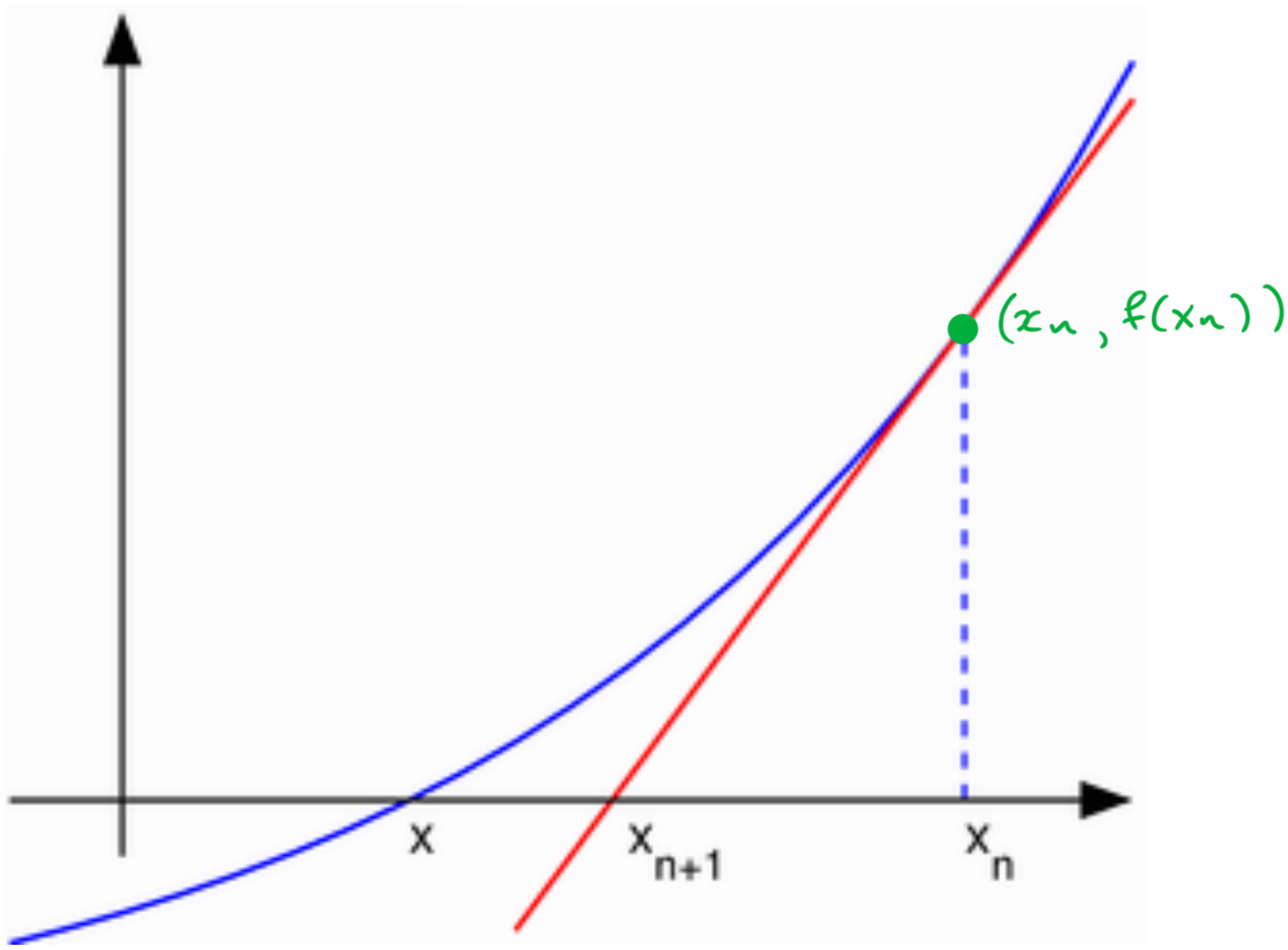
$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

NB

Remark: For this to make any sense at all, f needs to be differentiable at each x_n with $f'(x_n) \neq 0$.

Question: Why this formula?

Picture:



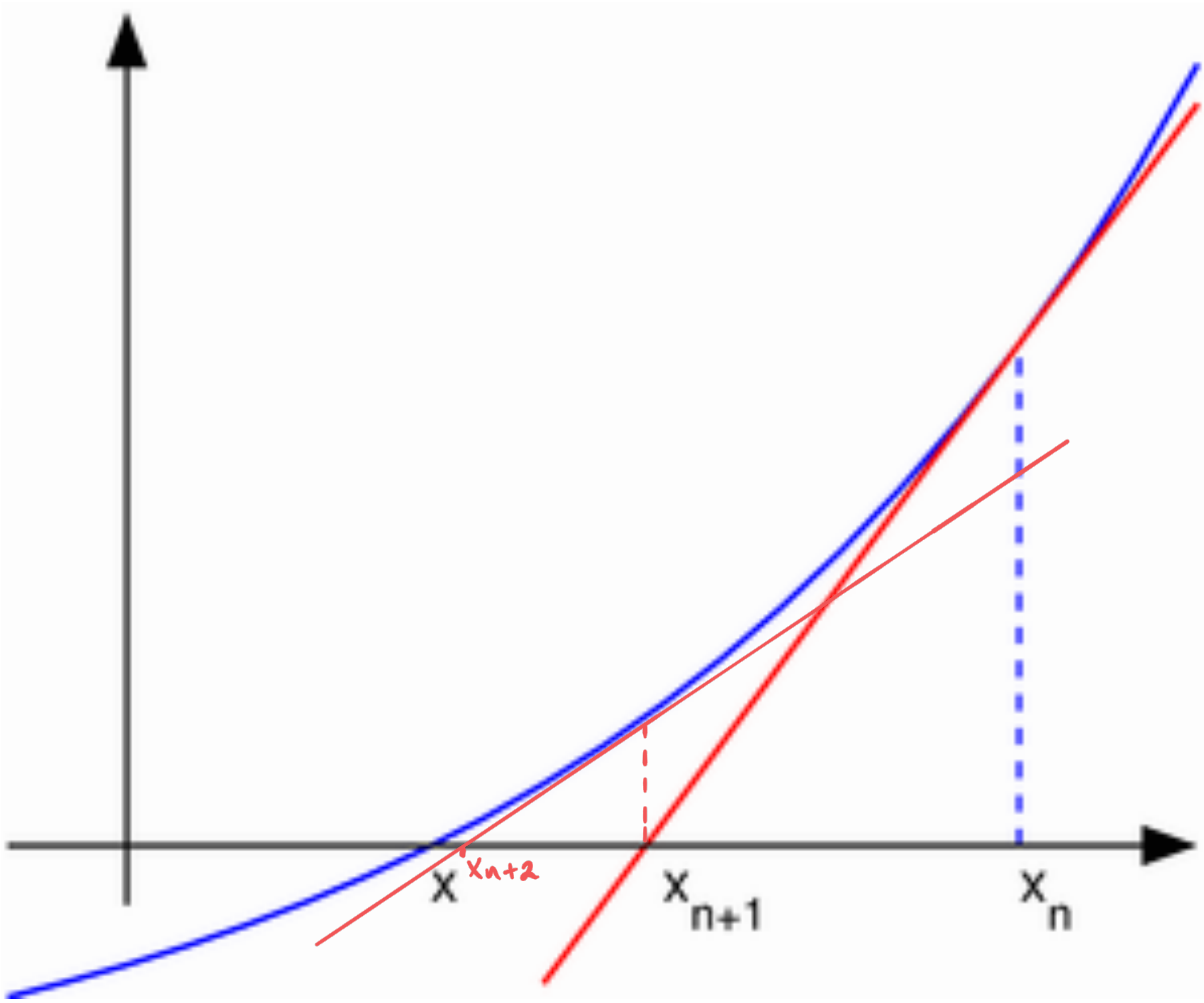
x_{n+1} is the x-intercept of the tangent line to f at x_n.

The equation of the tangent line (shown in red) is $L(x) = f'(x_n)(x - x_n) + f(x_n)$.

x_{n+1} being the x-intercept $\Rightarrow L(x_{n+1}) = 0$

$$\Rightarrow f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0$$

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Guesses can get better and better.

Remark: Sometimes your guesses can get worse and worse (or further and further apart).

In this scenario you should start over with a new guess.

Example 1 Use Newton's method to find the fourth approximation, x_4, to the root of the following equation

$$x^3 - x - 1 = 0$$

starting with $x_1 = 1$.

Note that if $f(x) = x^3 - x - 1$, then $f(1) = -1 < 0$ and $f(2) = 5 > 0$. Therefore by the Intermediate Value Theorem, there is a root between $x = 1$ and $x = 2$.

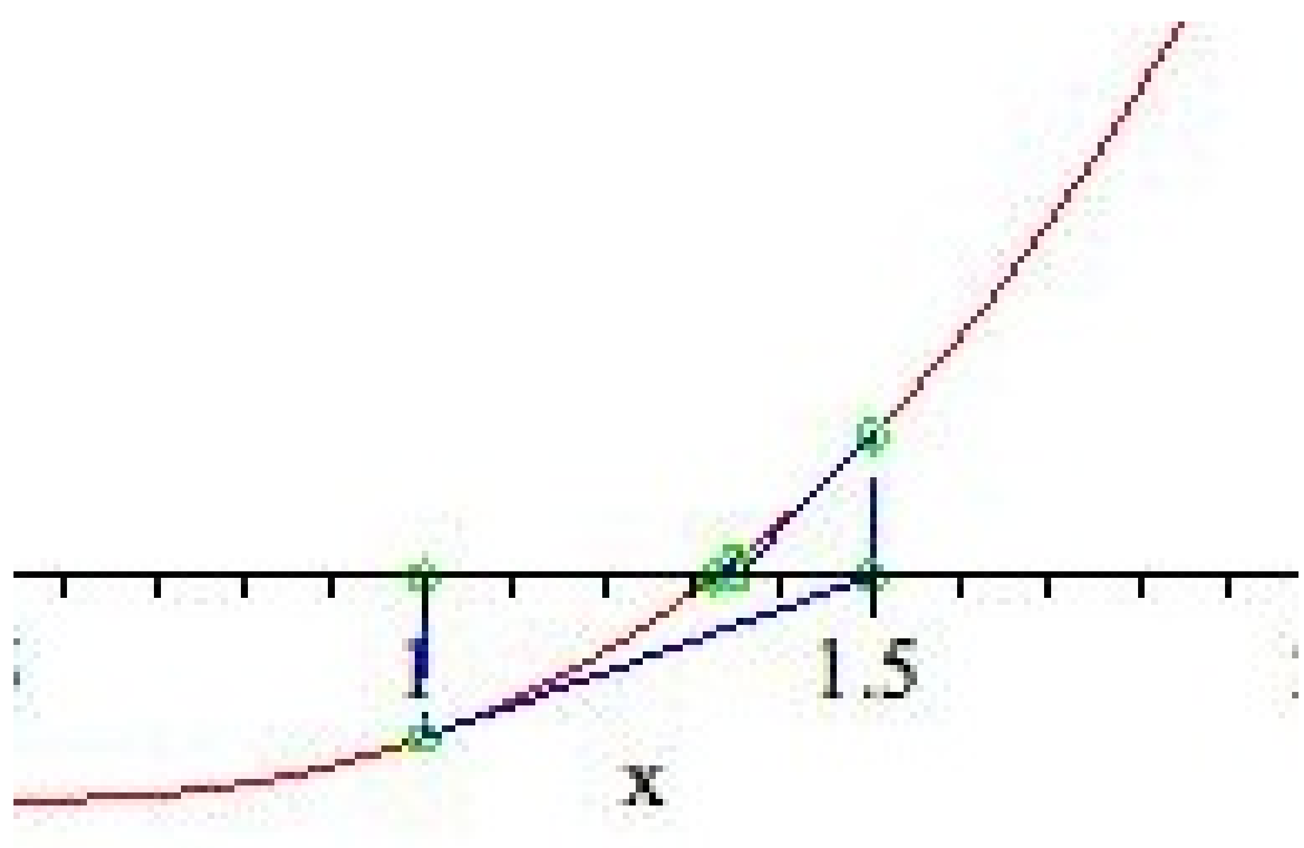
It is helpful to make a table for computation:

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1			
2				
3				

As a **rule of thumb**, when successive approximations x_n and x_{n+1} agree up to K decimal places, then our approximation is accurate up to K decimal places. For example the table below shows the approximations $x_1, x_2, \ldots, x_6$ for the above problem. We see that x_5 and x_6 agree up to 9 decimal places. By the rule of thumb above, the solution to the equation is equal to 1.324717957 when rounded off to 9 decimal places.

		$f(x) = x^3 - x - 1$	$f'(x) = 3x^2 - 1$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	2	1.5
2	1.5	0.875	5.75	1.347826087
3	1.347826087	0.100682173	4.449905482	1.325200399
4	1.325200399	0.002058362	4.268468292	1.324718174
5	1.324718174	0.000000924	4.264634722	1.324717957
6	1.324717957	-1.8672×10^{-13}	4.264632999	1.324717957

5.



Example 2 Use Newton's method to estimate $\sqrt{2}$ correct up to 5 decimal places.

Find an equation $f(x) = 0$ for which $\sqrt{2}$ is a solution.

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1				
2				
3				
4				
5				

Example 3 Do the curves $y = \cos x$ and $y = 3x$ meet?

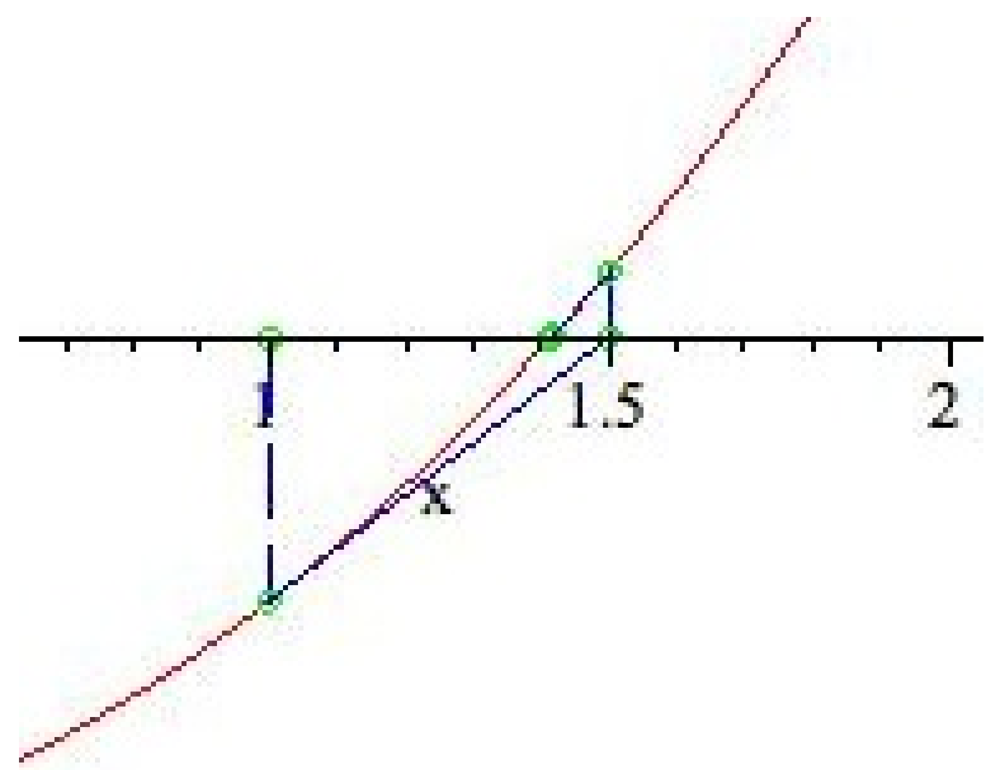
If so estimate the value of x at which they meet up to 3 decimal places.

		$f(x) =$	$f'(x) =$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1				
2				
3				
4				

We stop when successive approximations are equal up to 5 decimal places.

Example 2 Solution From calculator ($\sqrt{2} = 1.41421$). To use Newton's method, we look for a positive root of $f(x) = x^2 - 2$. We stop when successive approximations are equal up to 5 decimal places.

		$f(x) = x^2 - 2$	$f'(x) = 2x$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	2	1.5
2	1.5	.25	3	1.416666666
3	1.416666666	.006944444	2.833333333	1.414215686
4	1.414215686	.000006005	2.828431372	1.414213563



Example 3 Solution To use Newton's method, $f(x) = 3x - \cos x$. We use $x_1 = 0$ stop when successive approximations are equal up to 3 decimal places.

		$f(x) = 3x - \cos x$	$f'(x) = 3 + \sin x$	
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	0	-1	3	$1/3 = .333333333333$
2	$1/3$	.05504305	3.3271946	.316789952
3	.316789952	.0001295555	3.31517854	.316750829