Optimization Problems

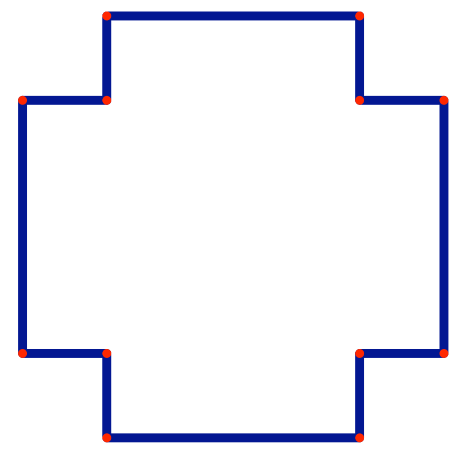
In this section we will apply our skills in identifying maxima and minima to problems in optimization.

Example I have 400 meters of fence with which I will fence off a rectangular plot of land. What should the dimensions of the plot be in order to maximize the area of the plot?

General principles for solving optimization problems

1. *Read the problem carefully* so that you fully understand what you have to solve for and what information you are given.
2. *Draw a diagram* and identify what information you are given and what you need to find.
3. *Introduce notation* If possible express the unknown quantity to be maximized/minimized as a function of one or two variables. Write down all equations relating these variables.
   Try to express the unknown quantity to be maximized/minimized as a function of one variable.
4. *Identify the critical points of the function* and use these and the values at endpoints to find the absolute maximum/ minimum of the function as appropriate.

Example An open top box is to be made by cutting small congruent squares from the corners of a 12-in by 12-in sheet of cardboard and bending up the sides. Find the largest possible volume of the box.



Example Jane is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down the straight shoreline from the point nearest the boat. She can row at 2 mph and can walk at 5 mph. Where should she land her boat to reach the village in the least amount of time?

To find absolute extreme values for a function $f(x)$ on an interval (perhaps $(0, \infty)$ or $(-\infty, \infty)$), we can sometimes use the following variant of the first derivative test, which agrees with our common sense:

Modified First Derivative Test Suppose that c is a critical number of a continuous function f on an interval I. Then

1. If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
2. If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$.

Example The revenue from selling x widgets is \$$6x$ where x is given in millions. The cost of producing x items is given by $c(x) = x^2 + 8$. Is there a production level that maximizes profit, $p(x) = r(x) - c(x)$?

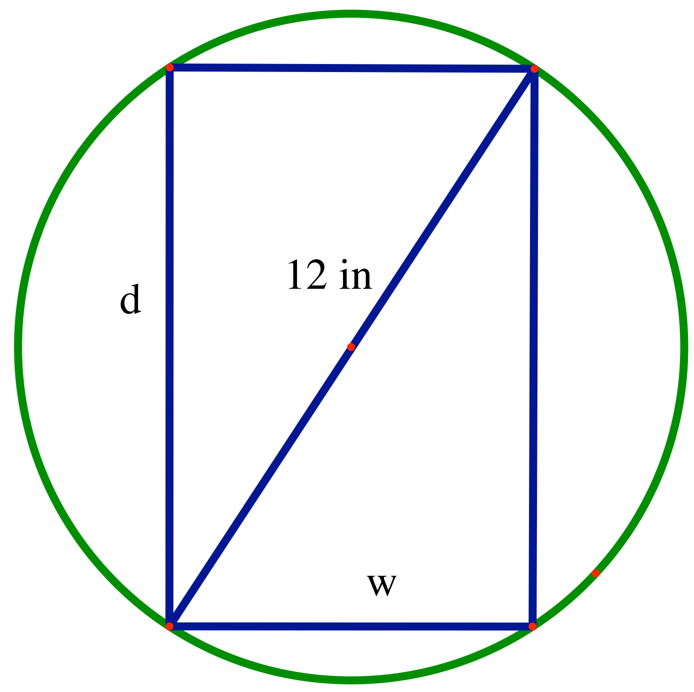
$$p(x) = r(x) - c(x) = 6x - [x^2 + 8] = -x^2 + 6x - 8 = -[x^2 - 6x + 8].$$

We must maximize $p(x)$ on the interval $[0, \infty)$.

Critical Points of $p(x)$:
$p'(x) = -2x + 6 = -2(x - 3)$. $p(x) = 0$ if $\boxed{x = 3}$.
We see that $p'(x) < 0$ if $x > 3$ and $p'(x) > 0$ if $x < 3$.
Therefore at $x = 3$ the profit function reaches a local maximum, by the first derivative test.
Since $p(x)$ is increasing on the interval $(0, 3)$ and decreasing on the interval $(3, \infty)$, we can conclude that there is a global maximum at $x = 3$.

The profit at $x = 3$(million widgets) is $P(3) = -9 + 18 - 8 = 1$ million dollars.

Extra Example(answer on next page) The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a 12 inch diameter cylindrical log.



It is given that $S = kwd^2$ where k is a constant. If we find the value of w that gives us the maximum of the function $S_1 = wd^2$, then the same value of w maximizes S.
So we want to maximize $S_1 = wd^2$ for $0 \leq w \leq 12$.
We also have $d^2 + w^2 = 144$, giving us that $d^2 = (144 - w^2)$.
This gives us that $S_1 = w(144 - w^2) = 144w - w^3$.

Critical Points:

$$\frac{dS_1}{dw} = 144 - 3w^2.$$

$$\frac{dS_1}{dw} = 0 \text{ if } 144 = 3w^2 \quad \text{or} \quad w = \sqrt{48},$$

since w must be greater than 0.

We know that S_1 has an absolute maximum at $w = \sqrt{48}$ from checking the value of $S_1 = w(144 - w^2)$ at the endpoints of the interval $[0, 12]$ and at the critical point $w = \sqrt{48}$.

$S_1(0) = 0, \quad S_1(\sqrt{48}) = \sqrt{48}(144 - 48) = 96\sqrt{48}, \quad S_1(12) = 12(144 - 144) = 0$

Example: (Heron's Problem)

A person is beside a river bank with a bucket. They want to get water from a nearby (perfectly straight) river and bring it to an unmovable container. What point on the river should they fetch the water from to minimise the distance of their journey?

Challenge: Solve this with and without calculus. i.e. give two solutions.