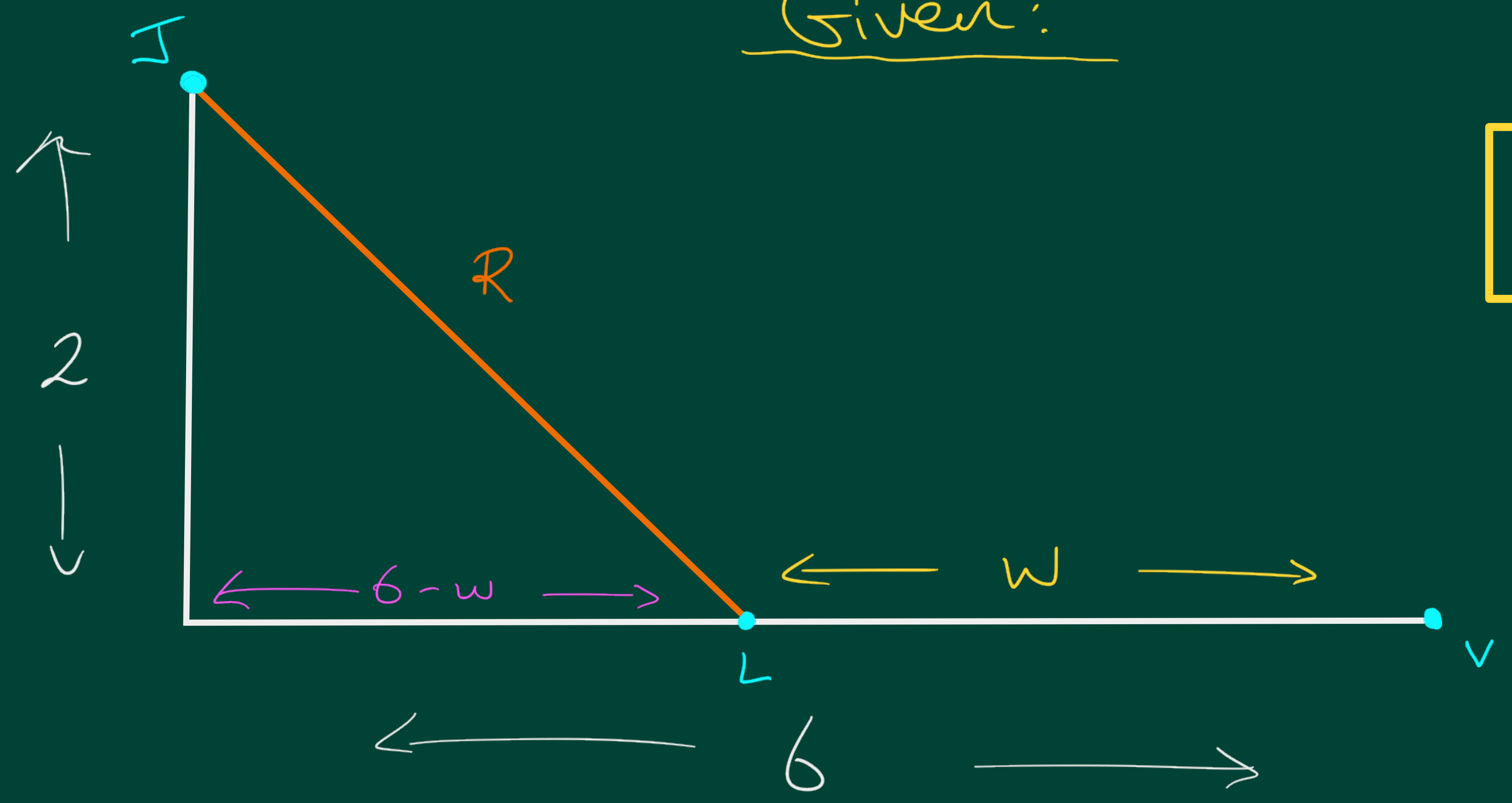
Example Jane is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down the straight shoreline from the point nearest the boat. She can row at 2 mph and can walk at 5 mph. Where should she land her boat to reach the village in the least amount of time?

Given:



$$\boxed{T = \frac{R}{2} + \frac{W}{5}}$$

$$\boxed{w \in [0, 6]}$$

Want: Find point L which minimises T.

Sol 1: Need relationship between R and W:

$$R^2 = 4 + (6 - w)^2$$

$$R = \sqrt{4 + (6-w)^2} \longleftarrow \text{always positive!}$$

$$T(w) = \frac{\sqrt{4 + (6-w)^2}}{2} + \frac{w}{5}$$

$$\frac{dT}{dw} = \frac{\cancel{\tfrac{1}{2}}\left(4 + (6-w)^2\right)^{-1/2} \cancel{2}(6-w)(-1)}{2} + \frac{1}{5}$$

$$= \frac{w-6}{2\sqrt{4 + (6-w)^2}} + \frac{1}{5}$$

Find critical pts:

$$0 = \frac{w-6}{2\sqrt{4 + (6-w)^2}} + \frac{1}{5}$$

multiply by $10\sqrt{4+(6-w)^2}$

$$\Rightarrow \quad 0 = 5(w-6) + 2\sqrt{4 + (6-w)^2}$$

$$\Rightarrow \quad -5(w-6) = 2\sqrt{4+(6-w)^2}$$

square both sides

$\&$

$(6-w)^2 = (w-6)^2$

$$\Rightarrow \quad 25(w-6)^2 = 4\left(4+(w-6)^2\right)$$

$$\Rightarrow \quad 25(w-6)^2 = 16 + 4(w-6)^2$$

$$\Rightarrow \quad 21(w-6)^2 = 16$$

$$(w-6)^2 = \frac{16}{21}$$

$$w - 6 = \pm\sqrt{\frac{16}{21}} = \pm\frac{4}{\sqrt{21}}$$

$$w = 6 \pm \frac{4}{\sqrt{21}}$$

Ignore '+' answer as $w \leq 6$.

$$w^* = 6 - \frac{4}{\sqrt{21}}$$

Check critical points and end points: $(w \in [0,6])$

$$T(w) = \frac{\sqrt{4 + (6-w)^2}}{2} + \frac{w}{5}$$

•) $T(0) = \frac{\sqrt{4 + (6)^2}}{2} = \frac{\sqrt{40}}{2} = \sqrt{10} \approx 3.16$ hours

•) $T(w^*) = \frac{\sqrt{4 + (6-w^*)^2}}{2} + \frac{w^*}{5}$

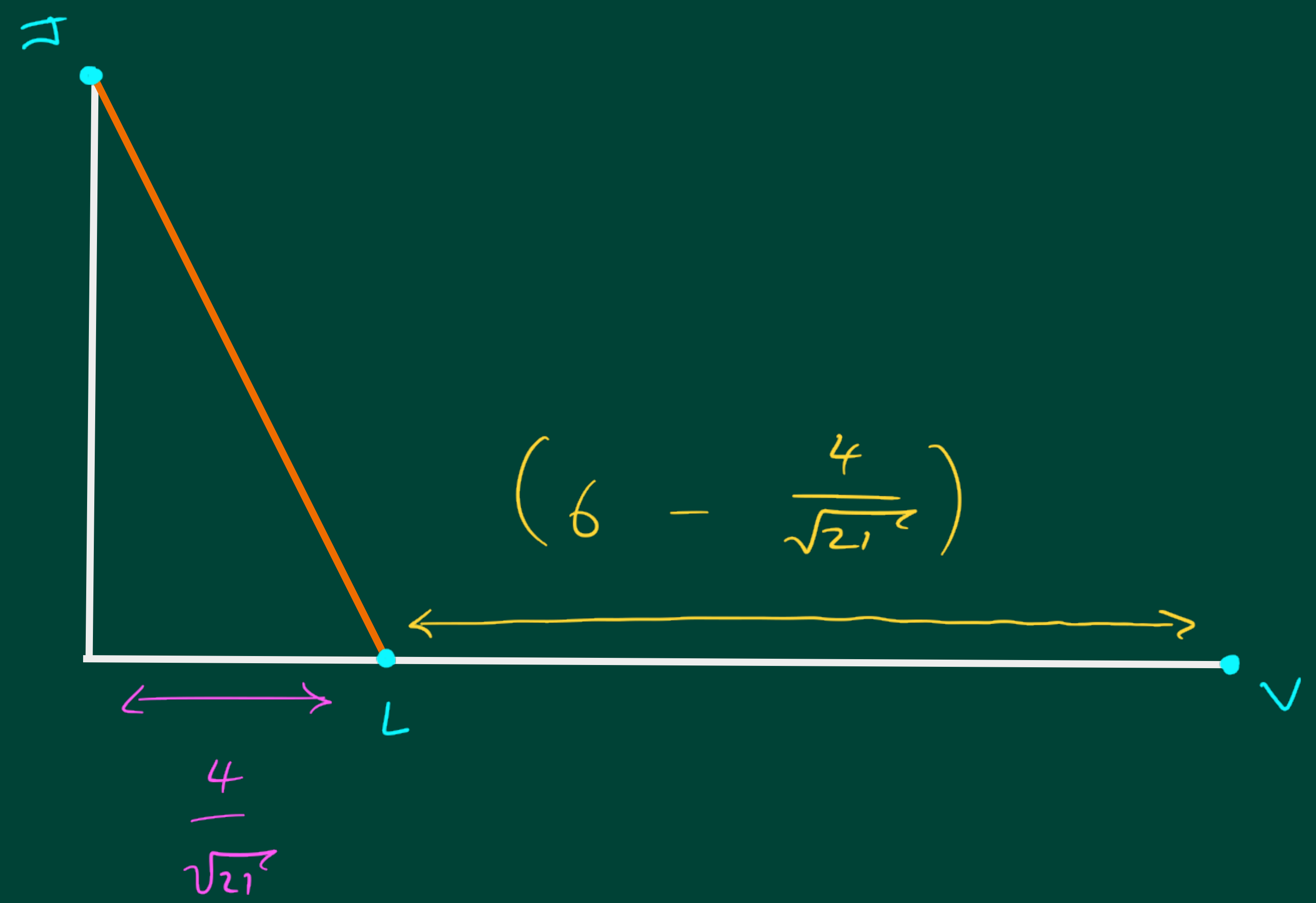
$$= \frac{\sqrt{4 + \left(\frac{4}{\sqrt{21}}\right)^2}}{2} + \frac{\left(6 - \frac{4}{\sqrt{21}}\right)}{5}$$

$$= \sqrt{1 + 1/21} + \frac{\left(6 - 4/\sqrt{21}\right)}{5} = \frac{5\sqrt{22} + 6\sqrt{21} - 4}{5\sqrt{21}} \approx 2.05 \text{ hours}$$

•) $T(6) = \frac{\sqrt{4}}{2} + \frac{6}{5} = 1 + 6/5 = 2.2$ hours

Hence, minimum time is achieved for $w = 6 - \frac{4}{\sqrt{21}}$

$\Rightarrow$ Land $\frac{4}{\sqrt{21}}$ miles down shore:



Alternatively:

$$\boxed{T = \frac{R}{2} + \frac{6-L}{5}}$$

$$R^2 = 2^2 + L^2$$

$$R = \sqrt{4 + L^2}$$

$$\boxed{L \in [0, 6]}$$

$$T = \frac{\sqrt{4 + L^2}}{2} + \frac{6 - L}{5}$$ ← Algebraically nicer to deal with.

$$\frac{dT}{dL} = \frac{L}{2\sqrt{4 + L^2}} - \frac{1}{5}$$

Critical pts: $0 = 5L - 2\sqrt{4 + L^2}$

$$5L = 2\sqrt{4 + L^2}$$

$$25L^2 = 4(4+L^2) = 16 + 4L^2$$

$$\Rightarrow 21L^2 = 16$$

$$\Rightarrow \quad L = \pm\sqrt{\frac{16}{21}} = \pm\frac{4}{\sqrt{21}}$$

ignore '$-$' as $L \geq 0$.

$$\Rightarrow \quad L^* = \frac{4}{\sqrt{21}} \quad \text{is a critical point.}$$

Test $L = 0, L^*, 6$ as in previous method.