§ 20. Summary of curve sketching:

Definition: If $\lim_{x \to \pm\infty} [f(x) - (ax+b)] = 0$

for some $a, b \in \mathbb{R}$, we say $y = ax + b$

is a slant asymptote of $y = f(x)$.

Remark: If $a = 0$, we see $y = f(x)$ has

a horizontal asymptote : $y = b$.

Remark / Strategy: In the case of rational functions,

$\frac{P(x)}{Q(x)}$, we get a slant asymptote when

$\boxed{\deg(P) = \deg(Q)}$ (horizontal) or $\boxed{\deg(P) = \deg(Q) + 1}$.

To find the equation of the asymptote,

divide $Q(x)$ into $P(x)$.

You will be left with the equation of

the horizontal asymptote + a remainder which

goes to zero in the limit.

Examples: Determine if the following functions have a horizontal / slant asymptote or neither.

(i) $g(x) = \dfrac{1 - x^4}{2x+3}$

Sol^n: The degree of the numerator is 3 higher than the degree of the denominator. Hence $y = g(x)$ does not have a horizontal or slant asymptote.

(ii) $h(x) = \dfrac{10x^3 + x^2 + 1}{55x^3 + 23}$

Sol^n: The degree of the numerator = the degree of the denominator. Hence $y = h(x)$ has a horizontal asymptote.

Exercise: Show $\displaystyle\lim_{x \to \pm\infty} h(x) = \frac{2}{11}$.

(iii) $f(x) = \dfrac{x^2 - 3}{2x - 4}$

<u>Sol$^{\underline{n}}$:</u> The degree of the numerator is 1 higher than the degree of the denominator. Hence, f has a slant asymptote:

$$\begin{array}{r|l} & \overbrace{\tfrac{1}{2}x + 1} + \left\{\frac{1}{2x-4}\right\} \\ \hline 2x-4 & x^2 - 3 \\ & \overset{-}{x^2}\ \overset{+}{-}\ 2x \\ \hline & \qquad 2x - 3 \\ & \qquad \overset{-}{2x}\ \overset{+}{-}\ 4 \\ \hline & \qquad\qquad 1 \end{array}$$

$ax + b$: $a = 1/2$
$b = 1$

$\longrightarrow 0$ as $x \longrightarrow \pm\infty$

$$\lim_{x \to \pm\infty} \left[f(x) - \left(\tfrac{1}{2}x + 1 \right) \right] = \lim_{x \to \pm\infty} \left[\frac{x^2 - 3}{2x - 4} - \left(\tfrac{1}{2}x + 1 \right) \right]$$

$$= \lim_{x \to \pm\infty} \left[\left(\tfrac{1}{2}x + 1 + \frac{1}{2x-4} \right) - \left(\tfrac{1}{2}x + 1 \right) \right]$$

$$= \lim_{x \to \pm\infty} \frac{1}{2x - 4} = 0$$

Hence $y = \frac{1}{2}x + 1$ is a slant asymptote of $y = f(x)$.

Summary of Curve Sketching

In this section we use the tools developed in the previous sections to sketch the graph of a function. The following gives a check list for sketching the graph of $y = f(x)$.

Domain of f The set of values of x for which $f(x)$ is defined. (We should pay particular attention to isolated points which are not in the domain of f, these may be points where there removable discontinuities or vertical asymptotes. The first and second derivative may also switch signs at these points.)

x and y-intercepts

- The x-intercepts are the points where the graph of $y = f(x)$ crosses the x-axis. They occur at the values of x which give solutions to the equation $f(x) = 0$.
- The y-intercept is the point where the graph of $y = f(x)$ crosses the y-axis. The y-value is given by $y = f(0)$.

Symmetry and Periodicity

- A function is <u>even</u> if $f(-x) = f(x)$ for all x in the domain. In this case the function has mirror symmetry in the y axis. For example $f(x) = x^2$. In this case it is enough to graph the function for $x > 0$ and the other half of the graph can be determined using symmetry.
- A function is <u>odd</u> if $f(-x) = -f(x)$ for all x in the domain of f. In this case the function has central symmetry through the origin. For example $f(x) = x^3$. In this case it is enough to graph the function for $x > 0$ and the other half of the graph can be determined using symmetry.
- A function is said to have period p if p is the smallest number such that $f(x+p) = f(x)$ for all x in the domain of f. In this case it is enough to draw the graph of $f(x)$ on an interval of length p. This graph then repeats itself on adjacent intervals of length p in the domain of f. For example $\tan(x) = tan(x + \pi)$ for all x in the domain of $\tan x$.

Asymptotes

- If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then the line $x = a$ is a <u>vertical asymptote</u> to the graph $y = f(x)$.
- If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a <u>horizontal asymptote</u> to the graph $y = f(x)$.
- If $\lim_{x\to\infty}[f(x) - (ax + b)] = 0$ or $\lim_{x\to-\infty}[f(x) - (ax + b)] = 0$, then the line $y = ax + b$ is a <u>slant asymptote</u> to the graph $y = f(x)$.

Intervals of Increase or Decrease By computing the sign of $f'(x)$, we can determine the intervals on which the graph of $f(x)$ is increasing and decreasing. The graph of f is increasing on intervals where $f'(x) > 0$ and decreasing on intervals where $f'(x) < 0$.

Local Minima/Maxima To locate the local maxima/minima, we find the critical points of f. These are the values of x in the domain of f for which $f'(x)$ does not exist, or $f'(x) = 0$. If c is a critical point we can classify c as a local maximum, local minimum or neither using the first derivative test:

- If f' switches from positive to negative at c, as we move from left to right along the graph, then f has a local maximum at $x = c$.

- If f' switches from negative to positive at c, as we move from left to right along the graph, then f has a local minimum at $x = c$.
- If f' does not switch sign at c, as we move from left to right along the graph, then f has neither a local maximum nor a local minimum at $x = c$.

If c is a critical point sometimes we can determine whether the graph has a local maximum or minimum at $x = c$ using the second derivative test:

- If $f''(c) > 0$, then the graph of f has a local minimum at $x = c$.
- If $f''(c) < 0$, then the graph of f has a local maximum at $x = c$.

Concave up/Concave down and points of inflection We can determine the intervals on which the graph of $f(x)$ is concave up and concave down by computing the sign of $f''(x)$. The graph of f is concave up on intervals where $f''(x) > 0$ and concave down on intervals where $f''(x) < 0$.
Sketching the curve With the above information, we should draw the asymptotes, plot the x and y intercepts, local maxima, local minima and points of inflection. We draw the curve through these points, increasing, decreasing, concave up, concave down and approaching the asymptotes as appropriate.

Example Sketch the graph of the function:

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

Example Sketch the graph of

$$g(x) = \frac{1}{1 + \sin x}.$$

Example Sketch the graph of

$$g(x) = \frac{1}{1+\sin x}.$$

- Domain of $g = \{x|1+\sin x \neq 0\}$ = the set of all values of x except $\frac{3\pi}{2} + 2n\pi$, where n is an integer.
- y-intercept: $g(0) = \frac{1}{1+\sin 0} = 1$, gives y-intercept at $(0,\ 1)$.
  No x-intercept $g(x) = 0$ has no solution.
- We have $g(x+2\pi) = g(x)$ for all x, therefore it is enough to draw the graph on the interval $[0, 2\pi]$ and repeat.
- Vertical asymptote where $\sin x = -1$ or at $x = \frac{3\pi}{2}$.
  $\lim_{x\to\pm\infty} g(x)$ D.N.E., since the sin function oscillate back and forth between -1 and 1. Hence we have no horizontal asymptotes or slant asymptotes.
- Critical Points:
  $f'(x) = \frac{-\cos x}{(1+\sin x)^2}$
  Critical points where $\cos x = 0$, i.e. at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. $x = \frac{3\pi}{2}$ is not a critical point since it is not in Dom f. Hence we have 1 critical point : $x = \pi/2$. $f' < 0$ on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$, $f' > 0$ on $[\pi/2, 3\pi/2]$.
  We have a local minimum at $x = \pi/2$. $f(\pi/2) = 1/2$. Hence we have a local minimum at $(\pi/2, 1/2)$.
- $f''(x) = \cdots = \frac{2-\sin x}{(1+\sin x)^2} > 0$ for asll x since $2 > \sin x$ for all x.
  Therefore the graph is concave up everywhere.
- Putting it all together, we get the graph below: