§19. Limits at infinity:

There are many reasons why we might want to consider the behaviour of a function as the inputs become "very large" in magnitude.

Definition: Say $f : (a, \infty) \longrightarrow \mathbb{R}$, $a > 0$.

Then we write

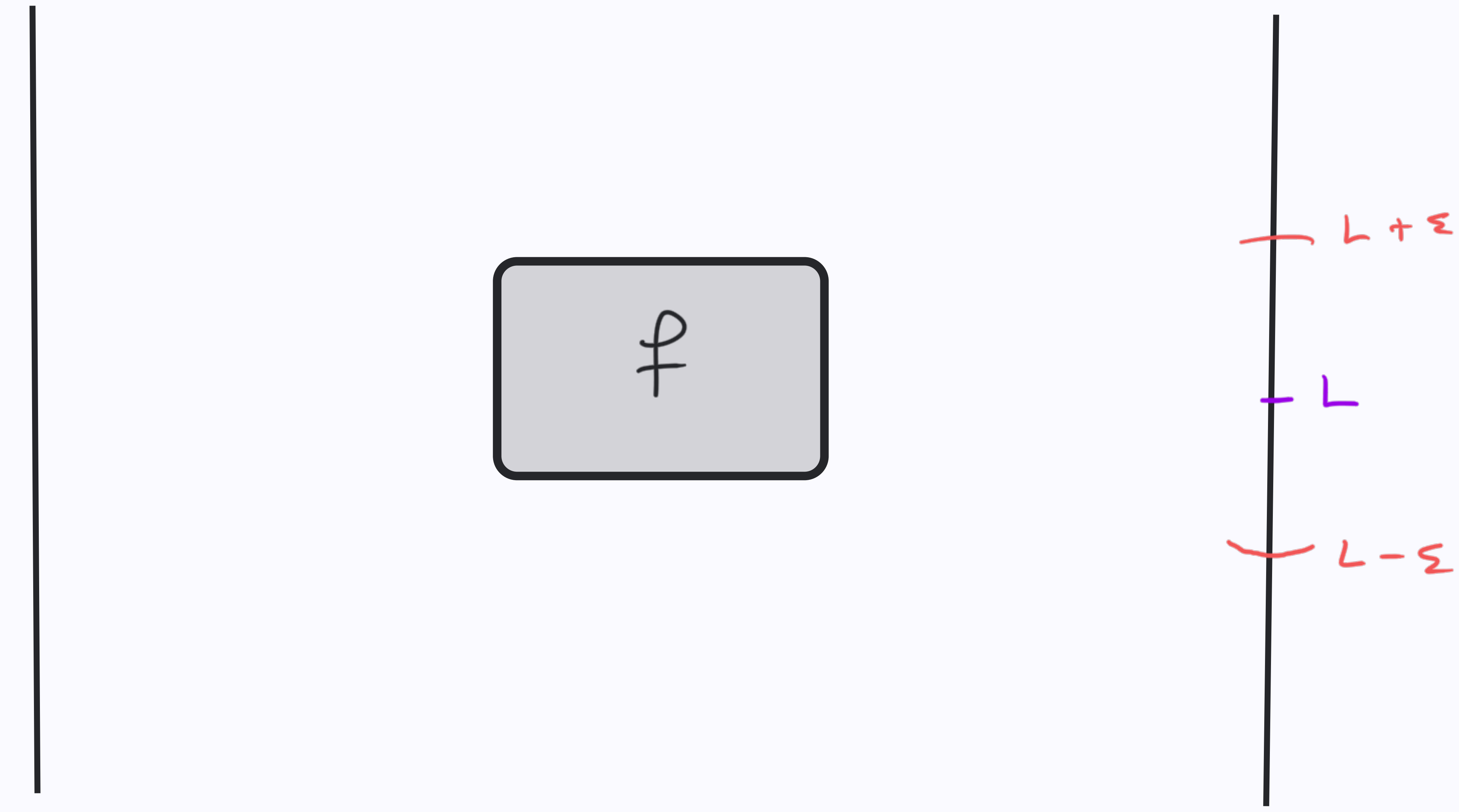
$$\boxed{\lim_{x \to \infty} f(x) = L}$$

if for all $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that for all $x \geq M$

$$|f(x) - L| < \varepsilon$$

Translation: We say $\lim_{x \to \infty} f(x) = L$ if, no matter how small of a gap we put around L, we can ensure that our outputs are inside this gap, by having our inputs larger than some M.

Picture:



Definition: We say $\lim_{x \to -\infty} f(x) = L$ if for all $\varepsilon > 0$, there is an $m \in \mathbb{R}$ such that, for $x < m$, we have

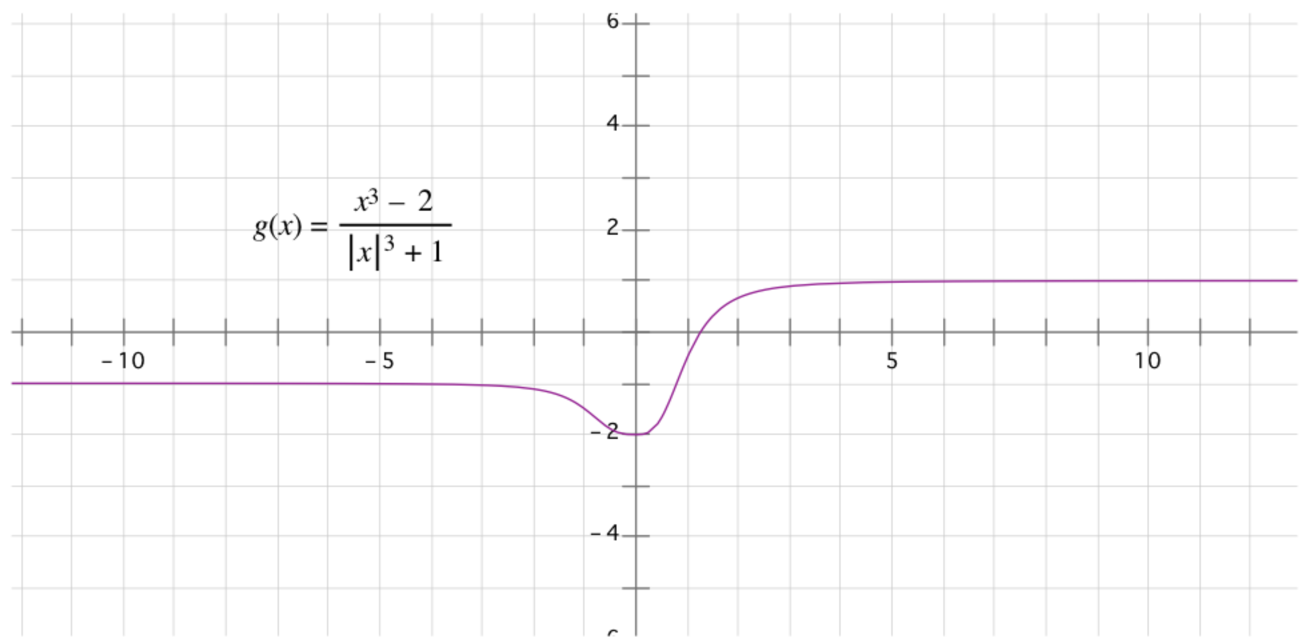
$$|f(x) - L| < \varepsilon$$

Exercise: Translate this and express it as a picture.

Note The symbol ∞ here does not represent a number, rather the symbol $\lim_{x\to\infty}$ means the limit as x becomes increasingly large.

Example Consider the graph of the function shown below. Judging from the graph, find are the limits

$$\lim_{x\to\infty} f(x) = \_\_\_\_\_\_\_\_ \qquad \lim_{x\to-\infty} f(x) = \_\_\_\_\_\_\_\_$$



We can see from the above graph that if $\lim_{x\to\infty} f(x) = L$, then the graph get closer and closer to the line $y = L$ as x approaches infinity.

Definition: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ we say the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

Example: What are the horizontal asymptotes of the graph shown below?

Definition Let f be a function defined on some interval (a, ∞). Then we say

$$\lim_{x\to\infty} f(x) = \infty$$

if the values of $f(x)$ can be made arbitrarily large by taking x sufficiently large or equivalently if for any positive integer N, there is a number M so that for all $x > M$, $f(x) > N$.
We give similar meaning to the statements

$$\lim_{x\to\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Remark: Recall the limit laws from lecture 4. These laws still apply for limits $x \longrightarrow \infty$.

$\boxed{\text{NB}}$

Strategy: When confronted with a limit as $x \to \infty$ of a rational function

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$$

divide top and bottom by x^k where k is the largest power in the denominator i.e. $k = \deg(Q)$ inside the limit and solve using one of the following:

Theorem: (i) $\displaystyle\lim_{x \to \infty} \frac{1}{x^r} = 0$ for all $r > 0$.

(ii) $\displaystyle\lim_{x \to -\infty} \frac{1}{x^r} = 0$ for all $r > 0$ such that x^r is defined.

OR:

$$\lim_{x\to\infty} x^n = \infty, \quad \lim_{x\to-\infty} x^{2n} = \infty \quad \lim_{x\to-\infty} x^{2n+1} = -\infty$$

for all positive integers n. Using this and law 10 above, we get that for all positive integers m, n

$$\lim_{x\to\infty} x^{\frac{n}{m}} = \infty, \quad \lim_{x\to-\infty} x^{\frac{2n}{2m+1}} = \infty \quad \lim_{x\to-\infty} x^{\frac{2n+1}{2m+1}} = -\infty$$

Example Evaluate

$$\lim_{x\to\infty} \frac{5x^3+x+1}{x^2-1}, \qquad \lim_{x\to-\infty} \frac{5x^3+x+1}{x^2-1}, \qquad \lim_{x\to\infty} \frac{5x+1}{x^2-4}, \qquad \lim_{x\to-\infty} \frac{5x+1}{x^2-4} \qquad \lim_{x\to-\infty} \frac{5x^5+1}{|x|^5-4}$$

Example Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+3}}{2x+5}, \qquad \lim_{x\to-\infty} \frac{\sqrt{3x^2+3}}{2x+5}, \qquad \lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-2x}\right)$$

Example: If you are confronted with the following limit:

$$\lim_{x\to\infty} \frac{x^9 + x^2 + 7}{x^6 + 3}$$

what should you do?

Ans:

Example: Find the following limits:

(i) $\lim_{x\to\infty} \dfrac{2x+1}{x-5}$

Solution:

(i) $$\lim_{x\to\infty} \frac{2x+1}{x-5} = \lim_{x\to\infty}\left(\frac{2x+1}{x-5} \cdot \underbrace{\frac{1/x}{1/x}}_{=1}\right)$$

as $\frac{1}{x} \neq 0$ for $x \in \mathbb{R}$

$$= \lim_{x\to\infty} \frac{2 + 1/x}{1 - 5/x}$$

$$= \frac{\lim_{x\to\infty}(2 + 1/x)}{\lim_{x\to\infty}(1 - 5/x)}$$

$$= \frac{2}{1} = 2$$

(ii) $\lim_{x \to -\infty} \dfrac{2x + 1}{x - 5}$

(iii) $\lim_{x \to \infty} \dfrac{3x^{17} + x^3 + 1}{x^7 + 4x^{17}}$

(iv) $\lim_{x \to -\infty} \dfrac{x^{21} + 3x + 1}{x^{37} + 2}$

(v) $$\lim_{x \to \infty} \frac{2x^2 + x + 1}{3x^2 - 1}$$

(vi) $$\lim_{x \to -\infty} \frac{2x^2 + x + 1}{3x^2 - 1}$$

(vii) What are the vertical and horizontal asymptotes of $g(x) = \dfrac{2x^2 + x + 1}{3x^2 - 1}$?

Note we can also use the squeeze theorem when calculating limits at ∞.

Example Find

$$\lim_{x\to\infty} \cos x, \quad \lim_{x\to\infty} \frac{\cos x}{x} \quad \lim_{x\to\infty} \sin\left(\frac{1}{x}\right), \quad \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right), \quad \lim_{x\to-\infty} \frac{5x+1}{x^2+\sin x-4}.$$

if they exist.

Limits of Polynomials at Infinity and minus infinity

Let

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

be a polynomial function. Then the behavior of $P(x)$ at $\pm\infty$ is the same as that of its highest term. That is

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} a_nx^n \quad \text{and} \quad \lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} a_nx^n.$$

(To prove this consider the limit $\lim_{x\to\pm\infty} \frac{P(x)}{a_nx^n}$.)

Example Find

$$\lim_{x\to\infty} x^4+2x+1, \quad \lim_{x\to-\infty} 2x^3+x^2+1, \quad \lim_{x\to\infty} -3x^5+10x^2+4562x+1, \quad \lim_{x\to\infty} (x-2)^3(x+1)^2(x-1)^5$$

Note that we can use the following short cut for calculating limits of rational functions as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} \frac{ax^n + \text{lin. comb. of lower powers}}{bx^m + \text{lin. comb. of lower powers}} = \lim_{x\to\pm\infty} \frac{ax^n}{bx^m}$$

where m and n are positive integers.