§ 16. Derivatives and graphs:

Motivation: We have already seen some aspects of how the derivative of a function affects the shape of it's graph. In this lecture we will outline these relations and try to see if we can reconstruct a sketch of the graph of f from information about f' and f''.

Recall: We say f is increasing on (a,b) if $f(x_1) < f(x_2)$ for all $a < x_1 < x_2 < b$.

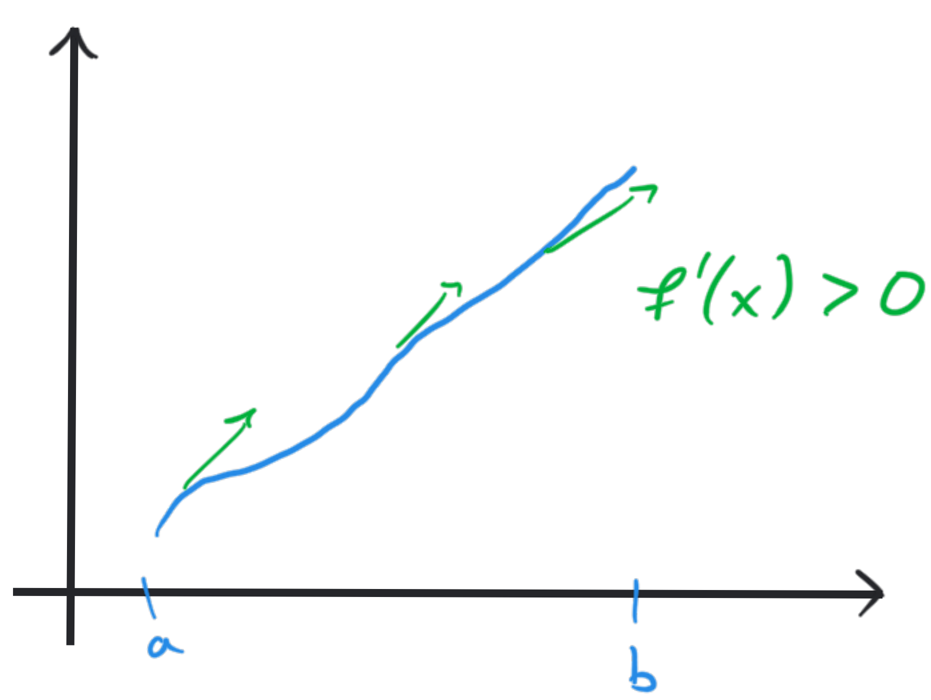
Similarly we say f is decreasing on (a,b) if $f(x_1) > f(x_2)$ for all $a < x_1 < x_2 < b$.

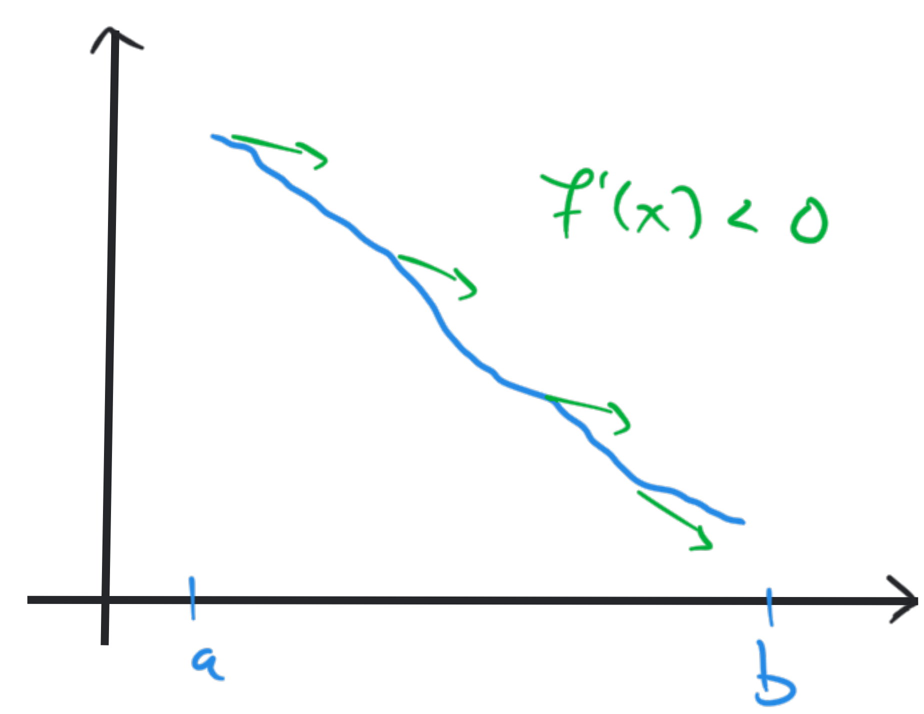
Theorem: Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then:

1) If $f'(x) > 0$ for all $x \in (a,b)$, we have that f is increasing on $[a,b]$.

2) If $f'(x) < 0$ for all $x \in (a,b)$, we have that f is decreasing on $[a,b]$.

Picture:





Proof: Follows from the mean value theorem.

Very similar proof to consequence of MVT on page 14 of notes from lecture 16.

—— o ——

Finding intervals where f is Increasing/Decreasing

To find the intervals where a function f is increasing or decreasing we must identify the intervals where f' is positive and negative. We first make a list of all points where the derivative (when it exists) might switch sign.

1. **Find the domain of f.** Make a list of the endpoints of the connected intervals in the domain. (For rational functions this is just a list of isolated points not in the domain)

2. **First find the critical points of f.** Recall that these are the points in the domain of f where $f'(x) = 0$ or $f'(x)$ does not exist.

3. **Check if f' is continuous on the intervals between the points** on the list from 1 and 2. Note that if f' is continuous on (a, b), with no zeros in the interval, then by the Intermediate Value theorem applied to f', we must have that f' is positive everywhere on (a, b) or negative everywhere on (a, b).

4. **Check the sign of f' on the above intervals** by checking the value of f' at a point in the interval or by checking the sign of the factors of f' if it is a rational function.

5. **f is increasing on the intervals where $f' > 0$ and decreasing on the intervals where $f' < 0$.**

Example: Find where $f(x) = 2x^3 - 3x^2 - 12x + 4$ is increasing and where it is decreasing.

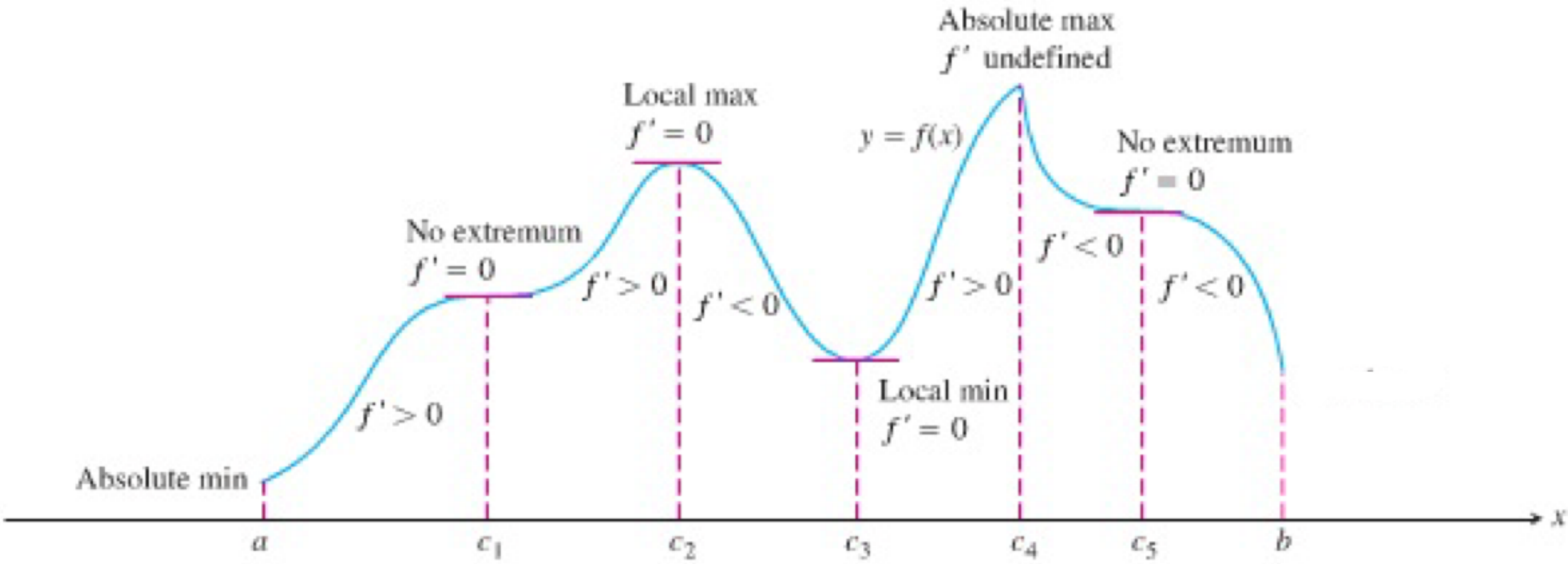
First derivative test for local extrema:

Suppose c is a critical point of a function f and f is differentiable in a neighbourhood of c, but not necessarily at c. Then, from left to right:

1) If $f'(x)$ changes from negative to positive, then f has a local minimum at c.

2) If $f'(x)$ changes from positive to negative, then f has a local maximum at c.

3) If $f'(x)$ does not change sign, then f has no local extrema at c.



Example Let $f(x) = 2x^3 - 3x^2 - 12x + 4$ as in the previous example. Classify the critical points as either local maxima, local minima or neither.

Example Find the critical points of

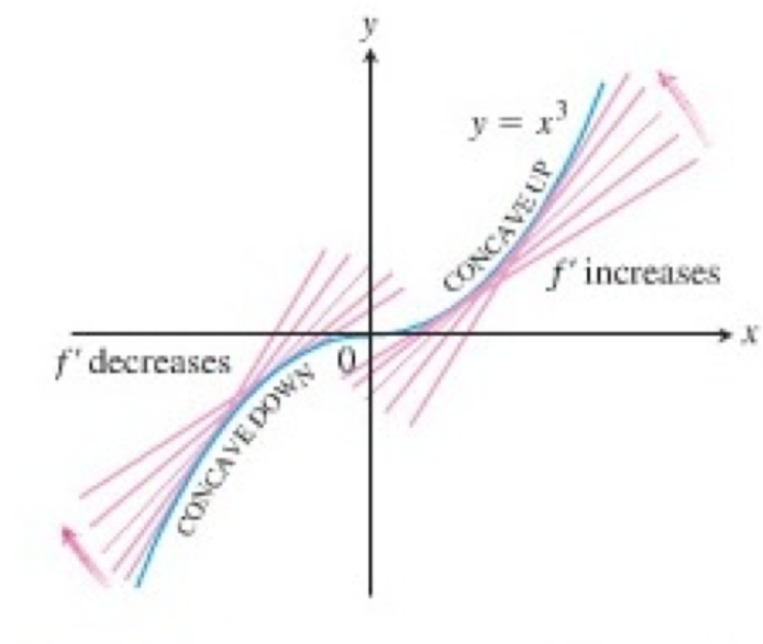
$$f(x) = x^{1/3}(x^2 - 4).$$

Identify the intervals on which f is increasing and decreasing. Find the function's local maxima and minima. Draw a rough sketch of the graph of the function.

Example Find the local maxima and minima of the function $g(x) = 1 - \cos^2 x$ on the interval $[-3\pi/4, 3\pi/4]$.

The Second Derivative and the graph of a function.

Definition (Concavity) If the graph of f lies above all of its tangents on an interval I, we say that the graph of f is **Concave up** on I. If the graph of f lies below all of its tangents on an interval I, we say that the graph of f is **Concave down** on I.



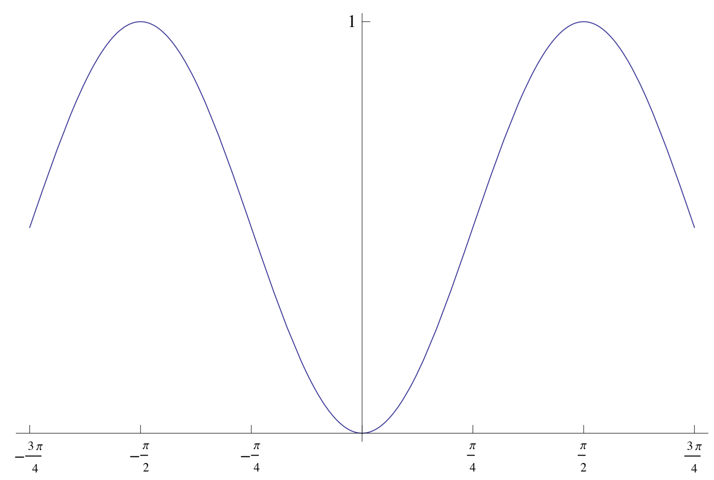
We see that $f(x) = x^3$ is concave up on the interval $(0, \infty)$ and concave down on the interval $(-\infty, 0)$.

We also see that a function f is concave up if the derivative f' is increasing and concave down if the derivative f' is decreasing. this gives us the concavity test using the second derivative:

Concavity Test

1. If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

Example On which intervals is the function $g(x) = 1 - \cos^2 x$ for $-3\pi/4 \le x \le 3\pi/4$ concave up and concave down (see graph below) ?



$g'(x) = \sin(2x)$, $g''(x) =$

$g''(x) > 0$ if

$g''(x) < 0$ if

Graph is concave up on

Graph is concave down on

Note that there are some points on the curve above where the graph switches from being concave up to being concave down and vice versa.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at that point.

Example Find the points of inflection on the curve $y = 1 - \cos^2 x$ on the interval $[-3\pi/4, 3\pi/4]$.

Second Derivative Test for local maxima and minima

We can also use the second derivative to classify the local extrema:

Second Derivative test Suppose that $f''(x)$ is continuous on an open interval containing c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

Example Consider the function $f(x) = x^4 - 4x^3 + 10$.

(a) Identify the critical points.

(b) Use the second derivative test to check for local extrema.

(c) Where is the curve Increasing/Decreasing?

(d) Where is the curve concave up/concave down? Where are the points of inflection?

(e) Draw a rough sketch of the graph below

Example Sketch a smooth connected curve $y = f(x)$ with;

$$f(-2) = 8, \quad f(0) = 4, \quad f(2) = 0.$$

$$f'(x) > 0 \text{ for } |x| > 2, \quad f'(2) = f'(-2) = 0, \quad f'(x) < 0 \text{ for } |x| < 2$$
$$f''(x) < 0 \text{ for } x < 0, \quad f''(x) > 0 \text{ for } x > 0.$$