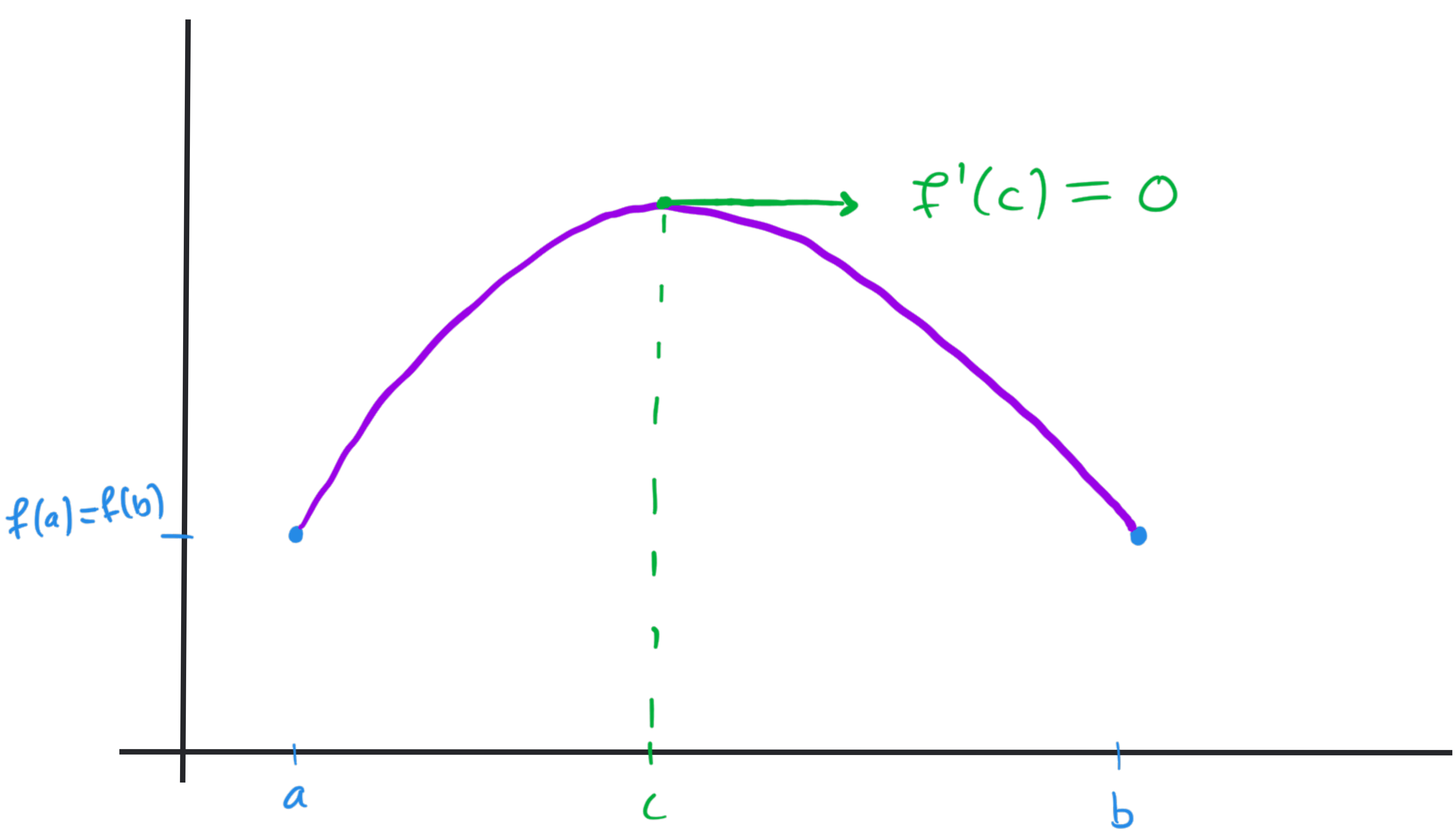
§16. The Mean Value Theorem:

Theorem: (Rolle's Theorem)

Suppose that f is a function which is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$.

Then, there is a $c \in (a,b)$ such that $f'(c) = 0$.

Picture:



Proof of Rolle's Theorem:

By the extreme value theorem, f attains a maximum and a minimum on $[a,b]$.

If either of these happen at $c \in (a,b)$, it means they are a local maximum or minimum.

Fermat's theorem then tells us that $f'(c) = 0$ or $f'(c)$ doesn't exist.

But we have f is differentiable on (a,b).

Hence we would have to have $f'(c) = 0$ and we would be done.

If instead f attains it's max and min at a and b we have:

$f(a) = f(b) = K$ such that $K \leq f(x) \leq K$ for all $x \in (a,b)$. Hence $f(x) = K$ for all $x \in (a,b)$ and we have $f'(x) = 0$ for all $x \in (a,b)$.

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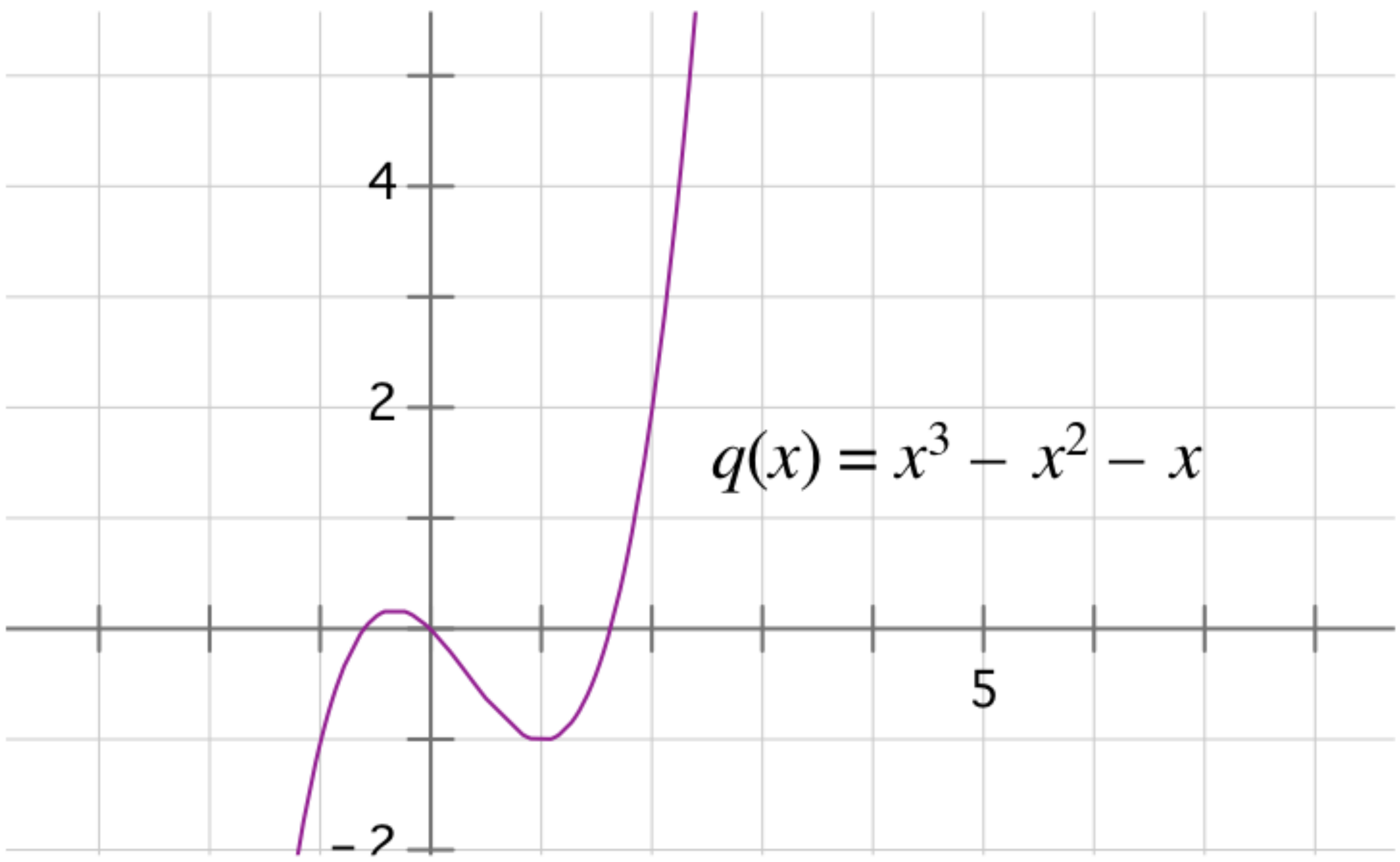
Examples:

1)



$f(x) = x^3 - x^2 - x$

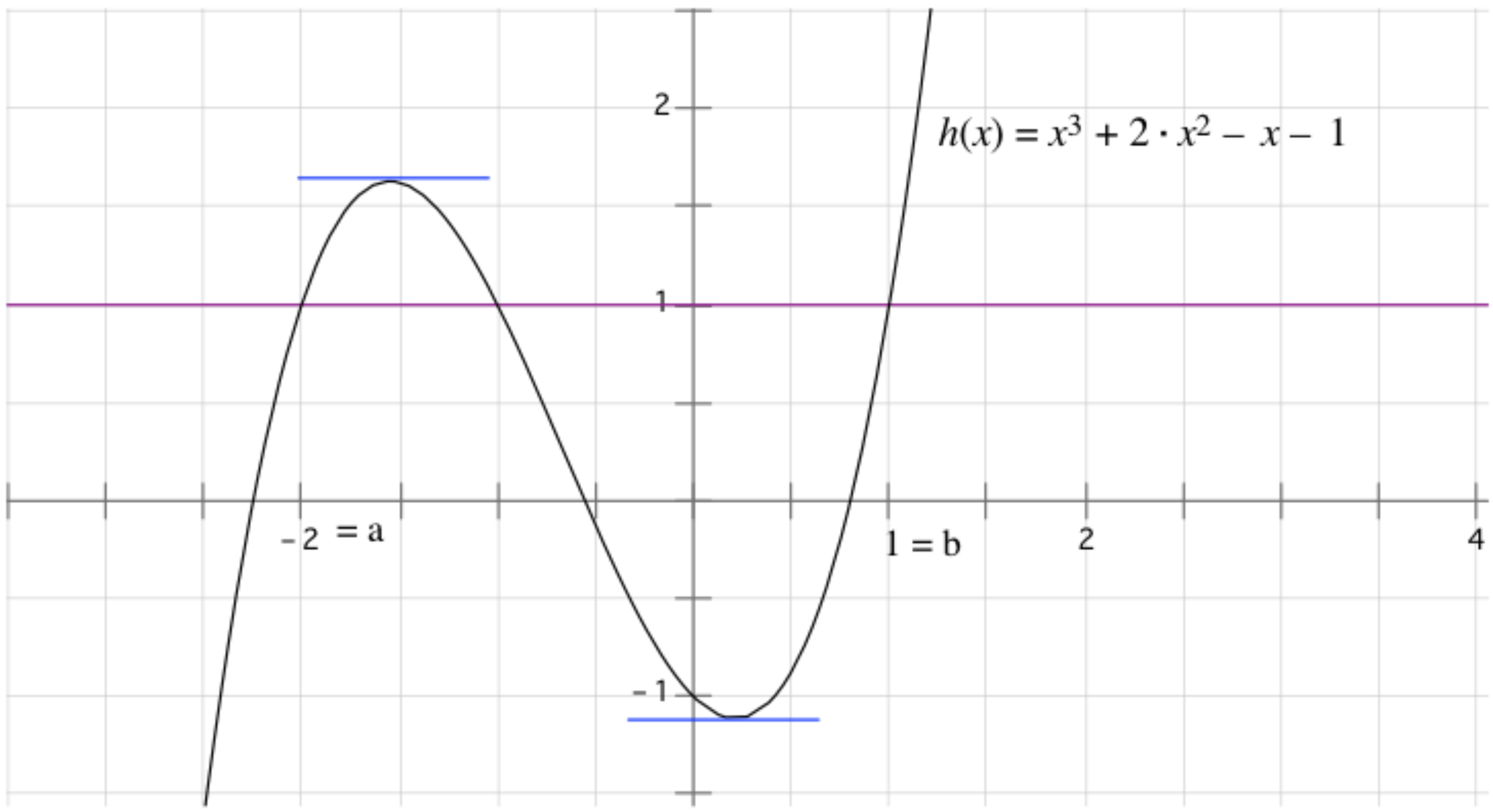
$f(-1) = -1 - 1 + 1 = -1$

$f(1) = 1 - 1 - 1 = -1$

f is a polynomial, so it is continuous and differentiable on all of $\mathbb{R}$.

Hence, by Rolle's theorem, there is a $c \in (-1, 1)$ such that $f'(c) = 0$

2)



$h(x) = x^3 + 2x^2 - x - 1$

Find a $c \in (-2, 1)$ such that $h'(c) = 0$.

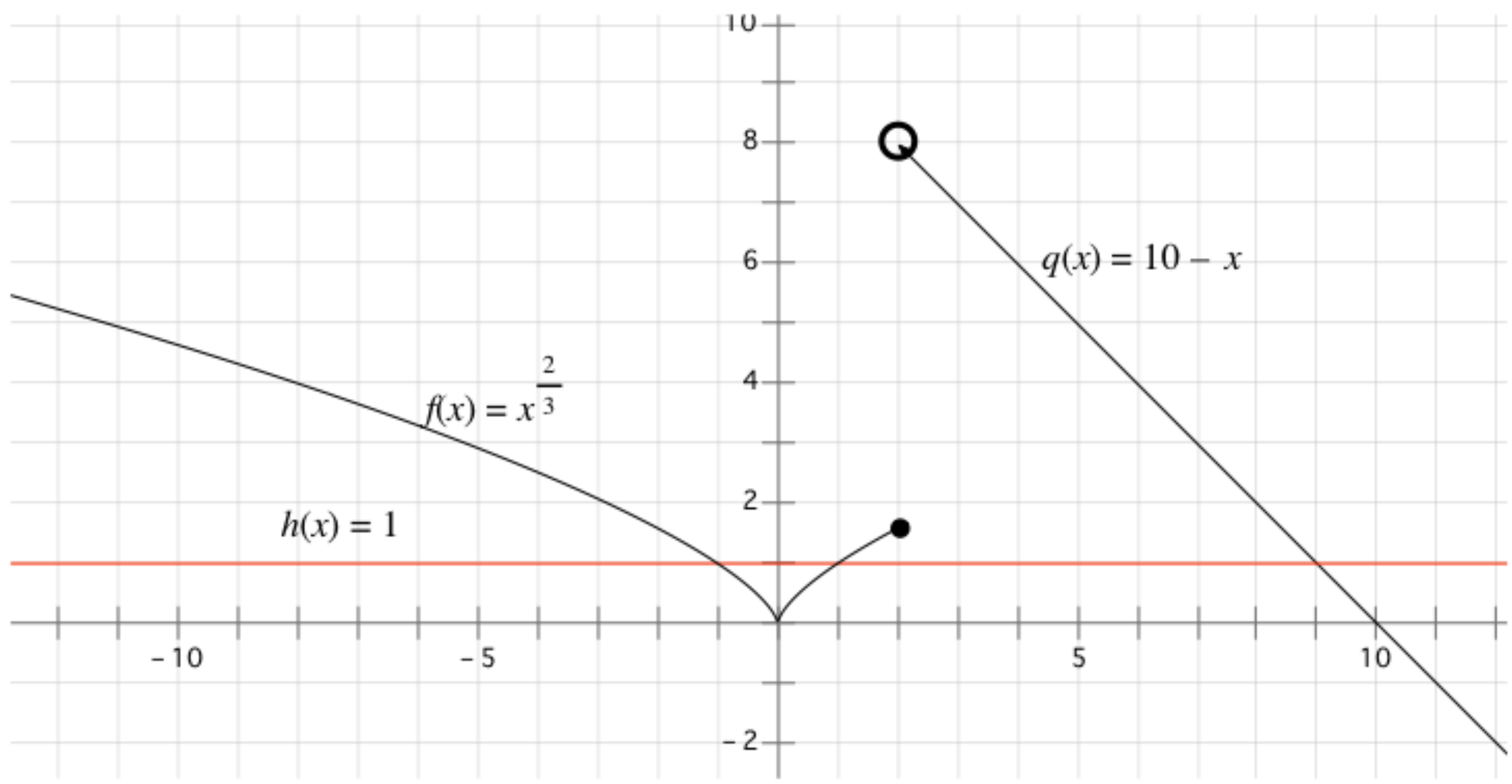
Solution: As $h(-2) = h(1) = 1$, and h is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$, Rolle's Theorem guarantees that such a c exists.

$h'(x) = 3x^2 + 4x - 1$

$h'(d) = 0 \Rightarrow d = \dfrac{-4 \pm \sqrt{(4)^2 - 4(3)(-1)}}{2(3)}$

So $c = \dfrac{-4 \pm \sqrt{28}}{6} = \dfrac{-2 \pm \sqrt{7}}{3} \simeq -1.55, \ 0.215$

Non - example:



Here we have a function:

$$g(x) = \begin{cases} x^{2/3} & \text{if} \quad x \le 2 \\ 10 - x & \text{if} \quad x > 2 \end{cases}$$

$$g(-1) = g(1) = g(9) = 1$$

but $g'(x) \neq 0$ for any $x \in (-1,1) \cup (1,9)$.

Why does Rolle's Theorem fail in this example?

Ans: Because the function is not differentiable at $0 \in (-1,1)$ or at $2 \in (1,9)$.

Example: (Particle in motion)

If $s(t)$ is the position function for a particle and we have $s(t_1) = s(t_2)$ for two different times, what can we conclude from Rolle's theorem?

Does this coincide with our intuition?

Ans: There is a $t_* \in (t_1, t_2)$ such that $s'(t_*) = 0$.

i.e. $v(t_*) = 0$

This coincides with "staying still" or "turning around".

Remark: Unless otherwise stated, we assume motion in real life to be smooth

(continuous, differentiable, twice differentiable, ...)

Using Rolle's Theorem with IVT :

Example: Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Sol^n:

Define $f(x) = x^3 + 3x + 1$.

f is continuous and differentiable on $\mathbb{R}$ as it is a polynomial.

$f(-1) = -3$

$f(0) = 1$

Hence, by the Intermediate Value Theorem, there is an $a \in (-1, 0)$ such that $f(a) = 0$.

So there is at least one root.

Question: Can there be another root?

Answer : No.

Reason : Say there is another root : b.

So $f(a) = f(b) = 0$.

Then, by Rolle's theorem, there is a point c between a and b such that $f'(c) = 0$.

But $f'(x) = 3x^2 + 3 \geq 3(x^2+1) \geq 3 > 0$

Hence, such a c cannot exist.

Hence, such a b cannot exist.

So $x^3 + 3x + 1 = 0$ has exactly one solution.

Concept: A common method of proof in mathematics is proof by contradiction:

- Assume statement X is false.
- Arrive at a contradiction.
- Hence, statement X must be true.

Example:

Claim: There is no biggest number.

Proof: Assume there is a number, $K \in \mathbb{R}$, such that $K \geq x$ for all $x \in \mathbb{R}$.

Let $x = K+1 \in \mathbb{R}$

$$(*) \Longrightarrow K+1 \leq K \Rightarrow 1 \leq 0$$

This is obviously nonsense. So we made a false assertion along the way.

i.e. $(*)$ is false and there is no biggest number.

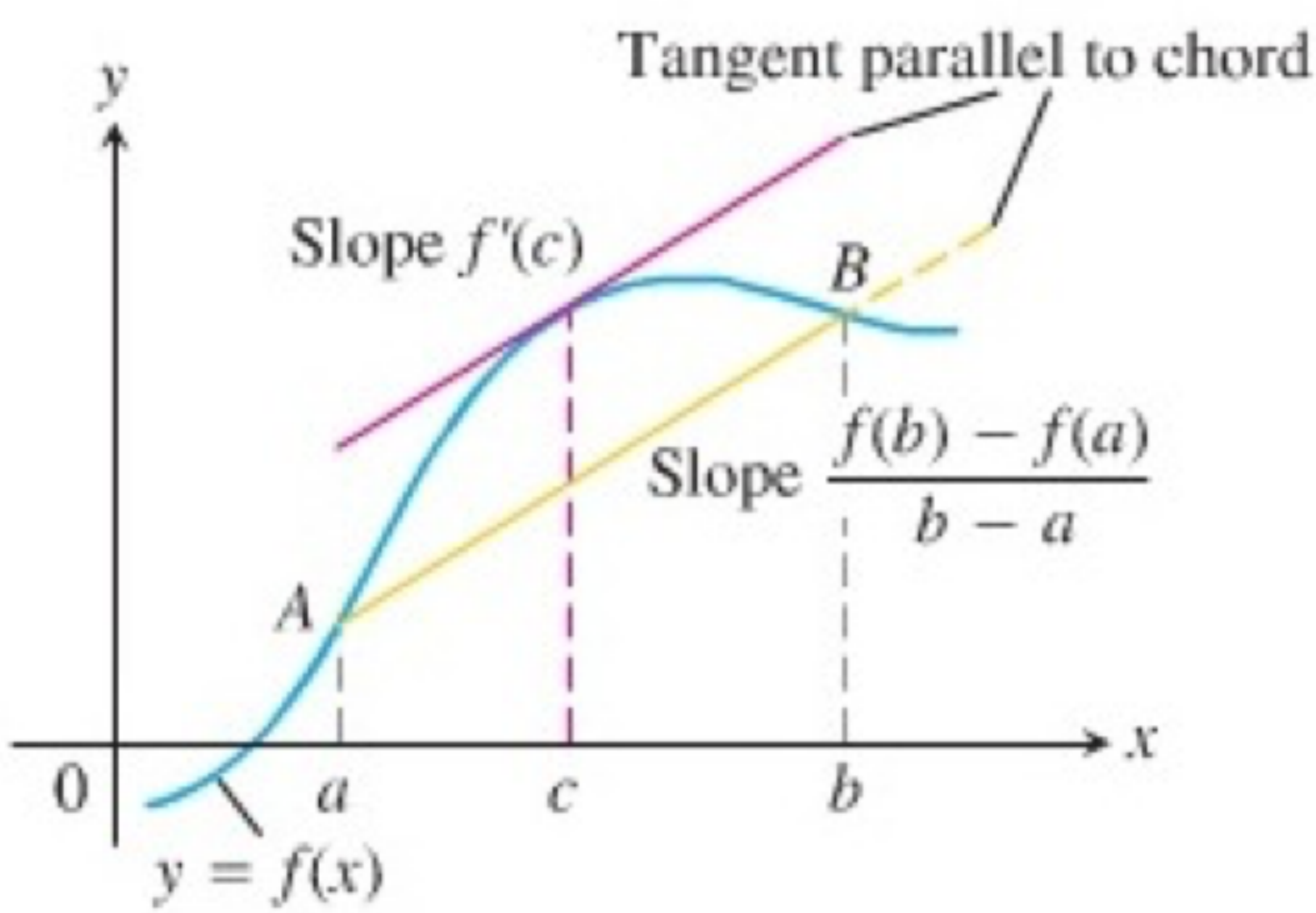
Theorem: (The Mean Value Theorem)

Suppose that a function f is continuous on an interval $[a,b]$ and differentiable on (a,b).

Then there is at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Picture:



Proof: The chord AB corresponds with the line $g(x) = \dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)$

Consider $h(x) = f(x) - g(x)$.

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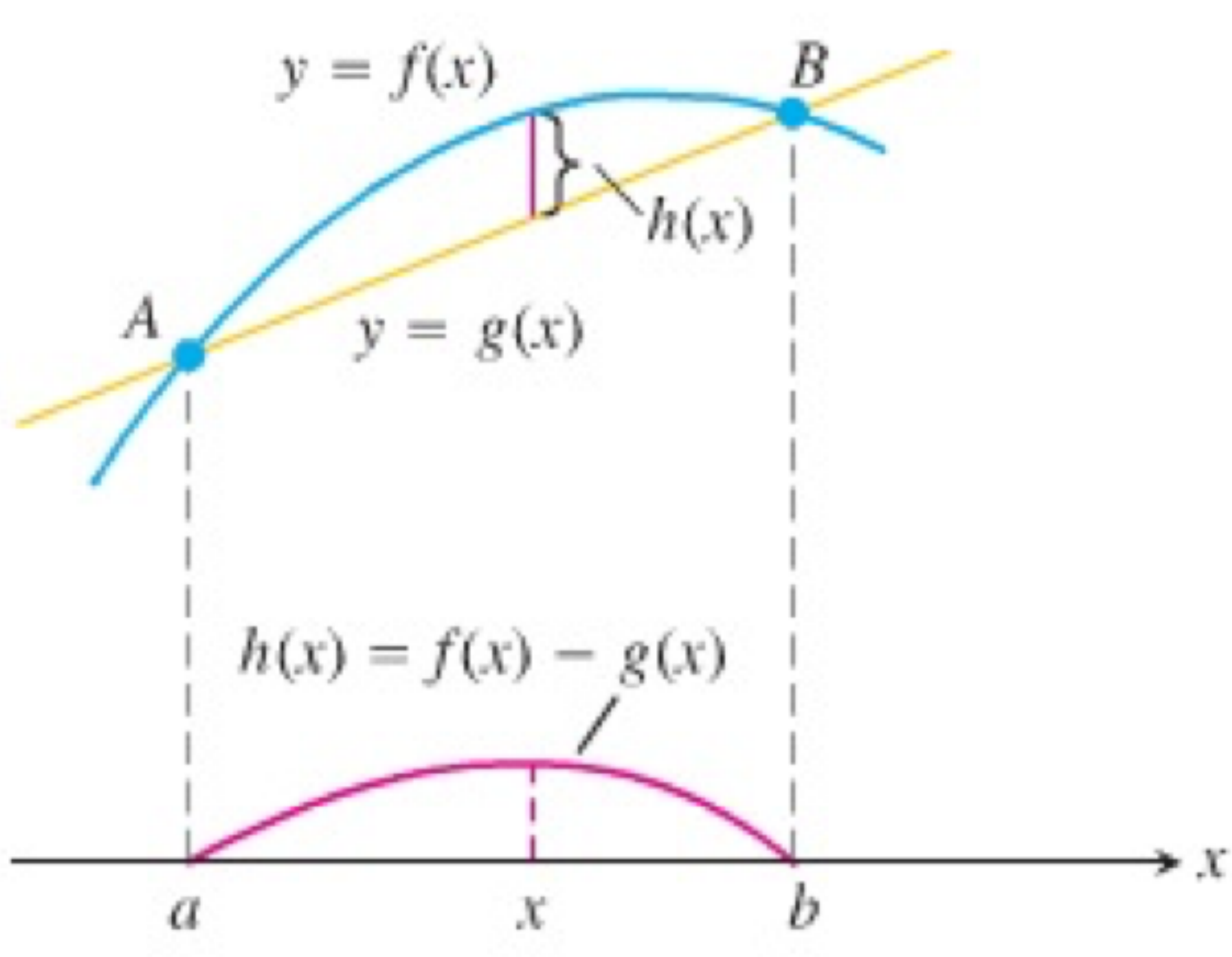
$h(a) = f(a) - g(a) = 0$

$h(b) = f(b) - g(b) = f(b) - f(b) = 0$

Hence, by Rolle's theorem, there is a $c \in (a,b)$ such that $h'(c) = 0$

$\Rightarrow \quad f'(c) - g'(c) = 0$

$\Rightarrow \quad f'(c) = g'(c) = \dfrac{f(b)-f(a)}{b-a}$



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Physical Interpretation Recall $\frac{f(b)-f(a)}{b-a}$ is the average rate of change of the function f on the interval $[a,b]$ and $f'(c)$ is the instantaneous rate of change at the point c. *The Mean Value Theorem says that at some point in the interval $[a,b]$ the instantaneous rate of change is equal to the average rate of change over the interval (as long as the function is continuous on $[a,b]$ and differentiable on (a,b).)*

Sometimes we can find a value of c that satisfies the conditions of the mean Value Theorem.

Example Let $f(x) = x^3 + 2x^2 - x - 1$, find all numbers c that satisfy the conditions of the Mean Value Theorem in the interval $[-1, 2]$.

Aim: To find all $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{13 - 1}{3} = 4$$

Soln: $f'(x) = 3x^2 + 4x - 1$

So we want $3c^2 + 4c - 1 = 4$

Remark: The Mean Value Theorem guarantees there is an answer.

$$3c^2 + 4c - 5 = 0 \Rightarrow c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{2(3)}$$

$$= \frac{-2 \pm \sqrt{19}}{3}$$

$$\approx -2.12 \text{ or } 0.786$$

$(-1, 2)$

So the c we were looking for is $c = 0.786$.

Example A car passes a camera at a point A on the toll road with speed 50 mph. Sixty minutes later the same car passes a camera at a point B, located 100 miles down the road from camera A, traveling at 50 mph. Can we prove that the car was breaking the speed limit (75 m.p.h.) at some point along the road?

$$\text{Average velocity} = \frac{\text{Change in Position}}{\text{Time taken}}$$

$$= \frac{100 \text{ miles}}{1 \text{ hour}}$$

$$= 100\text{mph}$$

The mean value theorem tells us (as we assume physical motion to be smooth) that there must have been a time t_* where

$$s'(t_*) = v(t_*) = \frac{s(1) - s(0)}{1 - 0} = 100\text{mph}$$

i.e. the car must have broke the speed limit.

We can also use this theorem to make inferences about the growth of a function from knowledge about its derivative:

Example If $f(0) = 1$, $f'(x)$ exists for all values of x and $f'(x) \leq 4$ for all x, how large can $f(2)$ possibly be?

By MVT : There is a $c \in (0,2)$ such that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \implies \frac{f(2) - 1}{2} \leq 4 \Rightarrow f(2) \leq 9$$

Example If $f(0) = 5$, $f'(x)$ exists for all x and $-1 \leq f'(x) \leq 3$ for all x, show that

$$-5 \leq f(10) \leq 35$$

By MVT there is a $c \in (0,10)$ such that

$$\frac{f(10) - f(0)}{10 - 0} = f'(c)$$

But

$$-1 \leq f'(c) \leq 3$$

Hence

$$-1 \leq \frac{f(10) - f(0)}{10 - 0} \leq 3$$

$$\implies \quad -10 \leq f(10) - 5 \leq 30$$

$$\implies \quad -5 \leq f(10) \leq 35$$

Consequences of MVT:

We now have answers to the following questions:

1) If $f'(x) = 0$ for all $x \in (a,b)$, does that mean f is constant on (a,b)?

Answer: Yes.

Proof: Let $a < x_1 < x_2 < b$.

As f is differentiable on (a,b), we know it is continuous on (a,b). Hence the MVT applies. Hence, there is a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $c \in (a,b)$. So $f'(c) = 0$

$$\Rightarrow \quad f(x_2) - f(x_1) = 0(x_2 - x_1) = 0$$

i.e. $f(x_2) = f(x_1)$.

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2) If $f'(x) = g'(x)$ on (a,b), what can be said about the relationship between f and g?

Answer: $f(x) = g(x) + C$ where C is a constant.

Proof: Apply previous result to $f-g$.

Example: Find a function with $f'(x) = 3x^2$ and $f(0) = 1$.

Sol^n: A first guess would be $g(x) = x^3$

But $g(0) = 0 \neq 1$.

So we want to find f such that

$f'(x) = g'(x)$ with $f(0) = 1$.

Consequence 2 tells us that we must have $f(x) = g(x) + c_1$ for some constant c_1.

Then we want:

$$1 = f(0) = g(0) + c_1 = 0 + c_1 = c_1$$

Hence, we must have:

$$f(x) = g(x) + 1 = x^3 + 1.$$