§9. Derivatives of Trigonometric Functions:

Recall: Our standard trig. functions:

Function	Domain
$\sin(x)$	$\mathbb{R}$
$\cos(x)$	$\mathbb{R}$
$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$	$\mathbb{R} \setminus \{x \,;\, \cos(x) = 0\}$
$\sec(x) = \dfrac{1}{\cos(x)}$	$\mathbb{R} \setminus \{x \,;\, \cos(x) = 0\}$
$\csc(x) = \dfrac{1}{\sin(x)}$	$\mathbb{R} \setminus \{x \,;\, \sin(x) = 0\}$
$\cot(x) = \dfrac{\cos(x)}{\sin(x)}$	$\mathbb{R} \setminus \{x \,;\, \sin(x) = 0\}$

Remark: All of these functions are continuous on their domains.

Goal: To find derivatives of these functions.

We will need the following trigonometric identities:

1) $\boxed{\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)}$

2) $\boxed{\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)}$

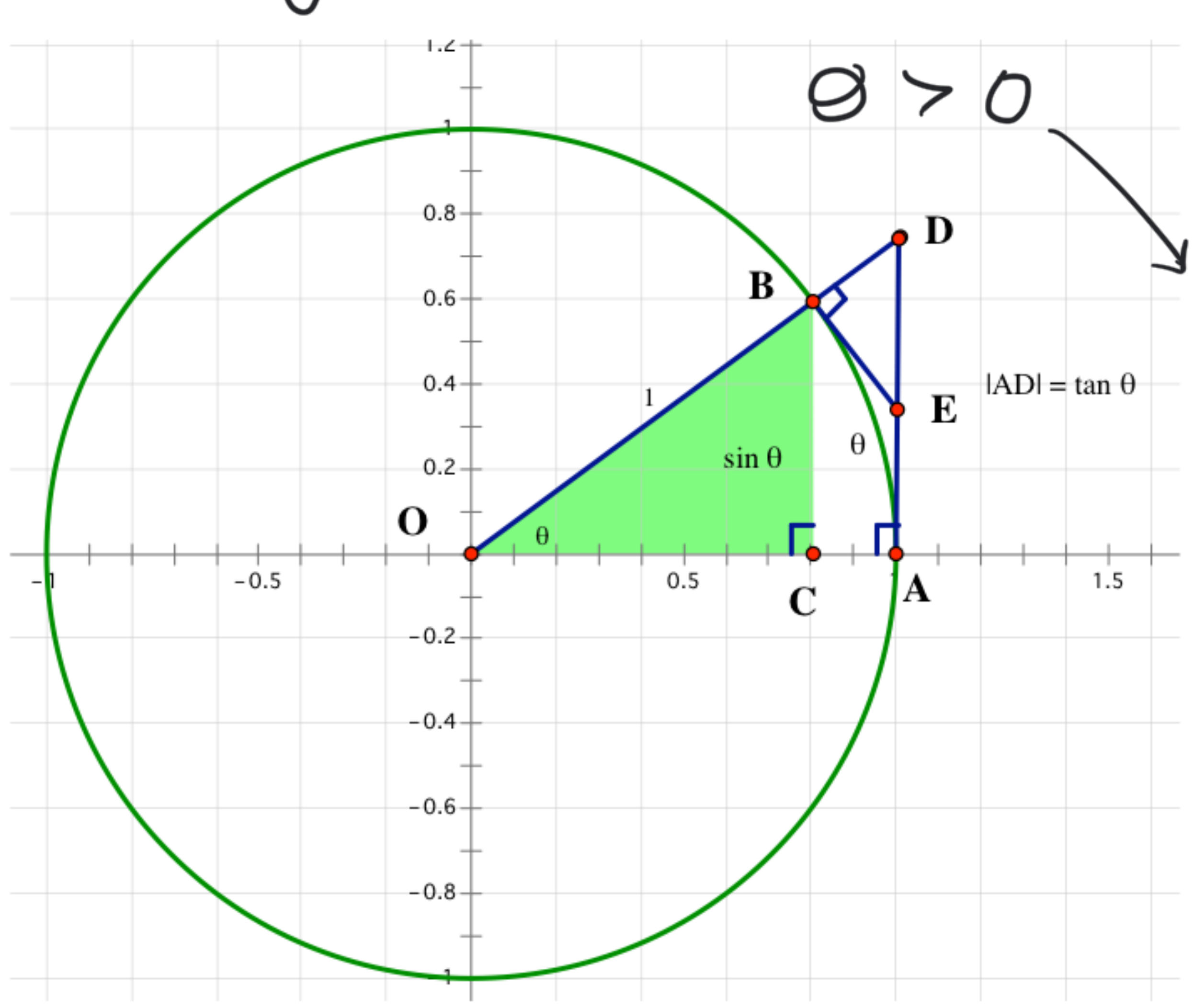
and the following limits:

3) $\boxed{\lim_{h \to 0} \frac{\sin(h)}{h} = 1}$

4) $\boxed{\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0}$

A sketch proof of 3) can be seen from the following picture:

$\theta > 0$



$\sin(\theta) < \theta < \tan(\theta)$

$\Rightarrow \frac{1}{\tan(\theta)} < \frac{1}{\theta} < \frac{1}{\sin(\theta)}$

$\Rightarrow \frac{\sin(\theta)}{\tan(\theta)} < \frac{\sin(\theta)}{\theta} < 1$

$\Rightarrow \cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$

As $\lim_{\theta \to 0} \cos(\theta) = 1$, $\lim_{\theta \to 0} 1 = 1$, we have, by the Squeeze Theorem, that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Examples: Evaluate the following limits:

1) $\lim_{x \to 0} \dfrac{\sin(7x)}{\sin(5x)}$

2) $\lim_{x \to 0} \dfrac{\sin(x^7)}{x^2}$

Now to find our desired derivatives:

Sin(x):

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h)\cos(x) + \sin(x)\left(\cos(h) - 1\right)}{h}$$

$$= \lim_{h\to 0} \left[\cos(x) \cdot \frac{\sin(h)}{h} + \sin(x) \cdot \frac{(\cos(h) - 1)}{h}\right]$$

$$= \lim_{h\to 0} \left[\cos(x) \frac{\sin(h)}{h}\right] + \lim_{h\to 0} \left[\sin(x) \left(\frac{\cos(h) - 1}{h}\right)\right]$$

$$= \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}} + \sin(x) \underbrace{\lim_{h\to 0} \frac{\cos(h) - 1}{h}}$$

$$= \cos(x)\,(1) + \sin(x)\,(0)$$

$$= \cos(x)$$

cos(x):

$$\frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x)\left(\cos(h)-1\right)-\sin(x)\sin(h)}{h}$$

$$= \lim_{h\to 0}\left[\cos(x)\cdot\frac{\left(\cos(h)-1\right)}{h}-\sin(x)\cdot\frac{\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\left[\cos(x)\cdot\frac{\cos(h)-1}{h}\right]-\lim_{h\to 0}\left[\sin(x)\cdot\frac{\sin(h)}{h}\right]$$

$$= \cos(x)\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}-\sin(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}$$

$$= \cos(x)\,(0)-\sin(x)\,(1)$$

$$= -\sin(x)$$

So we have:

1) $\boxed{\dfrac{d}{dx}\sin(x) = \cos(x)}$

2) $\boxed{\dfrac{d}{dx}\cos(x) = -\sin(x)}$

<u>tan(x):</u>

<u>Example:</u> $g(x) = \dfrac{1+\cos(x)}{x+\sin(x)}$, $x \neq 0$

$g'(x) =$

Higher Order Derivatives:

We have already seen that the derivative of $\sin(x)$ is $\cos(x)$, and the derivative of $\cos(x)$ is $-\sin(x)$. What does this say about their higher order derivatives?

$\sin(x)$:

$$\sin(x) \xrightarrow{\frac{d}{dx}} \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x) \xrightarrow{\frac{d}{dx}} -\cos(x)$$

i.e. $f(x) = \sin(x)$ $\qquad f^{(4)}(x) = \_\_\_\_\_\_$

$f'(x) = \cos(x)$ $\qquad f^{(5)}(x) = \_\_\_\_\_\_$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

Ex: What is $f^{(27)}(x)$?

Remark:

Exercise: Investigate the pattern in the derivatives of $\cos(x)$.

Summary of Derivatives:

1) $\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$

2) $\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$

3) $\boxed{\frac{d}{dx}\tan(x) = \sec^2(x)}$

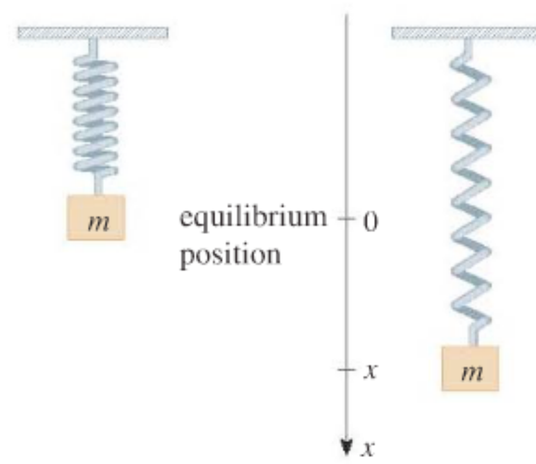
4) $\boxed{\frac{d}{dx}\sec(x) = \sec(x)\tan(x)}$

5) $\boxed{\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)}$

6) $\boxed{\frac{d}{dx}\cot(x) = -\csc^2(x)}$

Remark:

A mass on a spring released at some point other than its equilibrium position will follow a pattern of simple harmonic motion ($x(t) = A\sin(Ct + D)$ or equivalently $x(t) = A\cos(Ct + D)$), when there is no friction or other forces to dampen the effect. The values of A, C and D depend on the elasticity of the spring, the mass and the point at which the mass is released. You will be able to prove this easily later when you learn about differential equations.



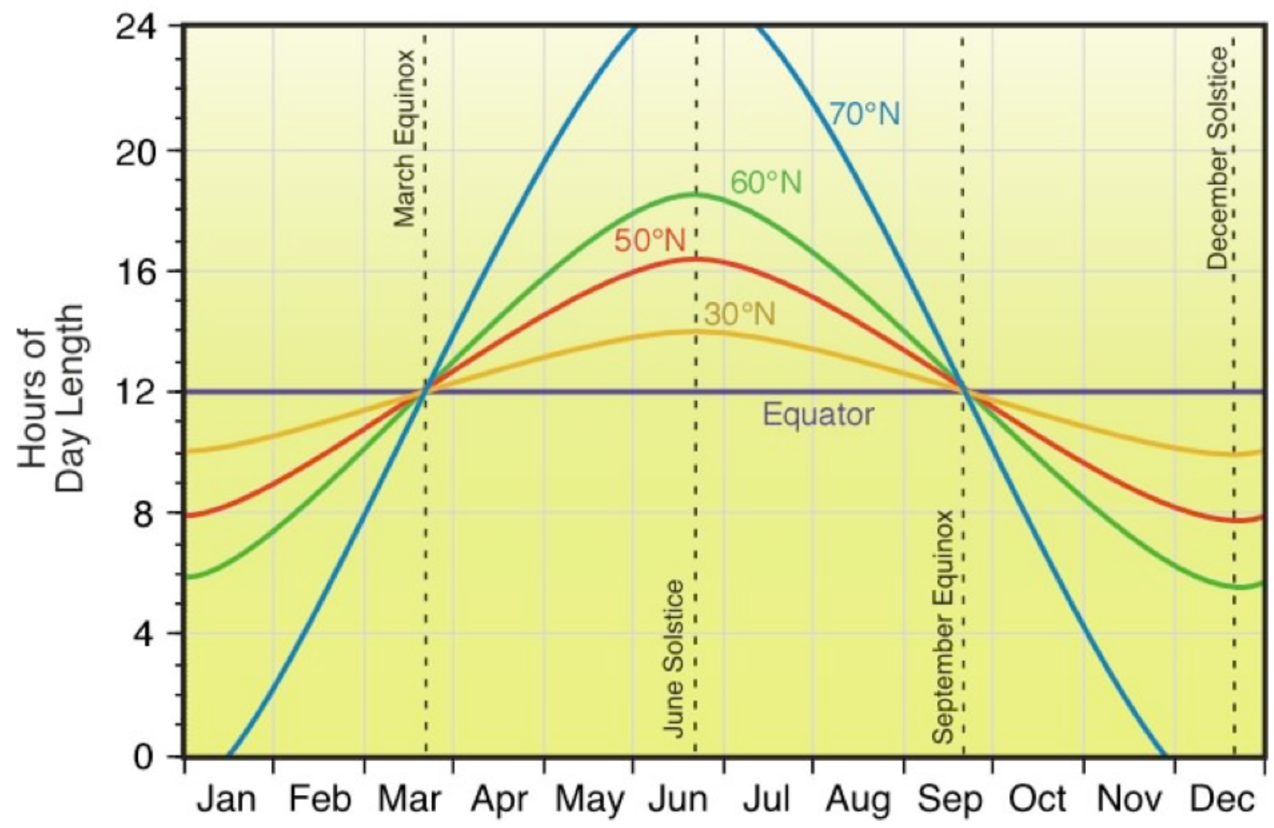
Example An object at the end of a vertical spring is stretched 5cm beyond its rest position and released at time $t = 0$. Its position at time t is given by $x(t)$ with the positive direction as shown in a downward direction, where

$$x(t) = 5\cos(t).$$

(a) Find the velocity and acceleration at time t.

(b) Find the position, velocity and acceleration of the mass at time $t = \frac{\pi}{4}$. In which direction is it moving at that time?

Example The graph below shows the variations in day length for various degrees of Lattitude.

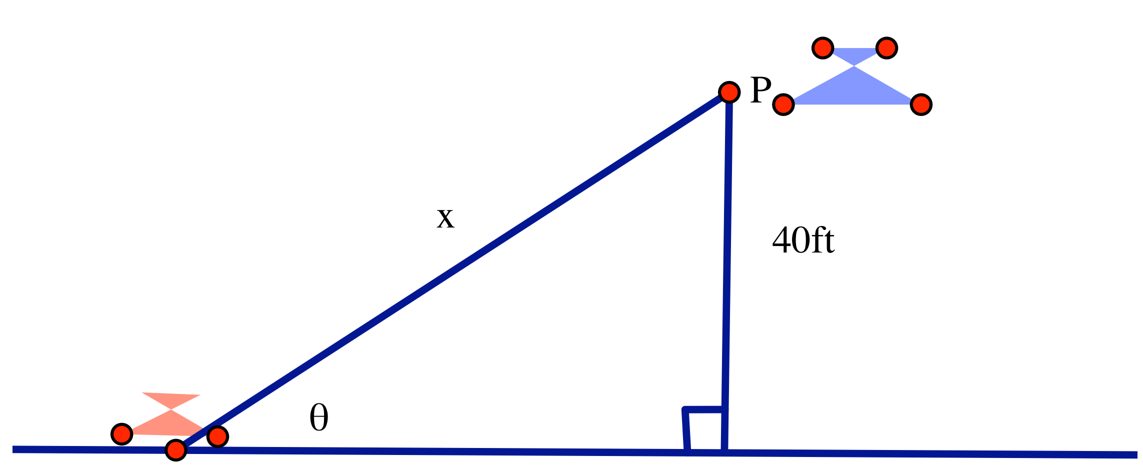


i-2: Annual variations in day length for locations at the equator, 30, 50, 60, and 70° North latitude.

At 60^o North, at what times of the year is the length of the day changing most rapidly?

Extras

Example (Preparation for Related Rates) A police car is parked 40 feet from the road at the point P in the diagram below. Your vehicle is approaching on the road as in the diagram below and the police are pointing a radar gun at your car. Let x denote the distance from your car to the police car and let θ be the angle between the line of sight of the radar gun and the road. How fast is x changing with respect to θ when $\theta = \frac{\pi}{4}$? (Please attempt this problem before looking at the solution on the following page.)



Solution We have that the variables x and θ are related in the following way:

$$\frac{40}{x} = \sin(\theta).$$

Therefore

$$\frac{40}{\sin(\theta)} = x$$

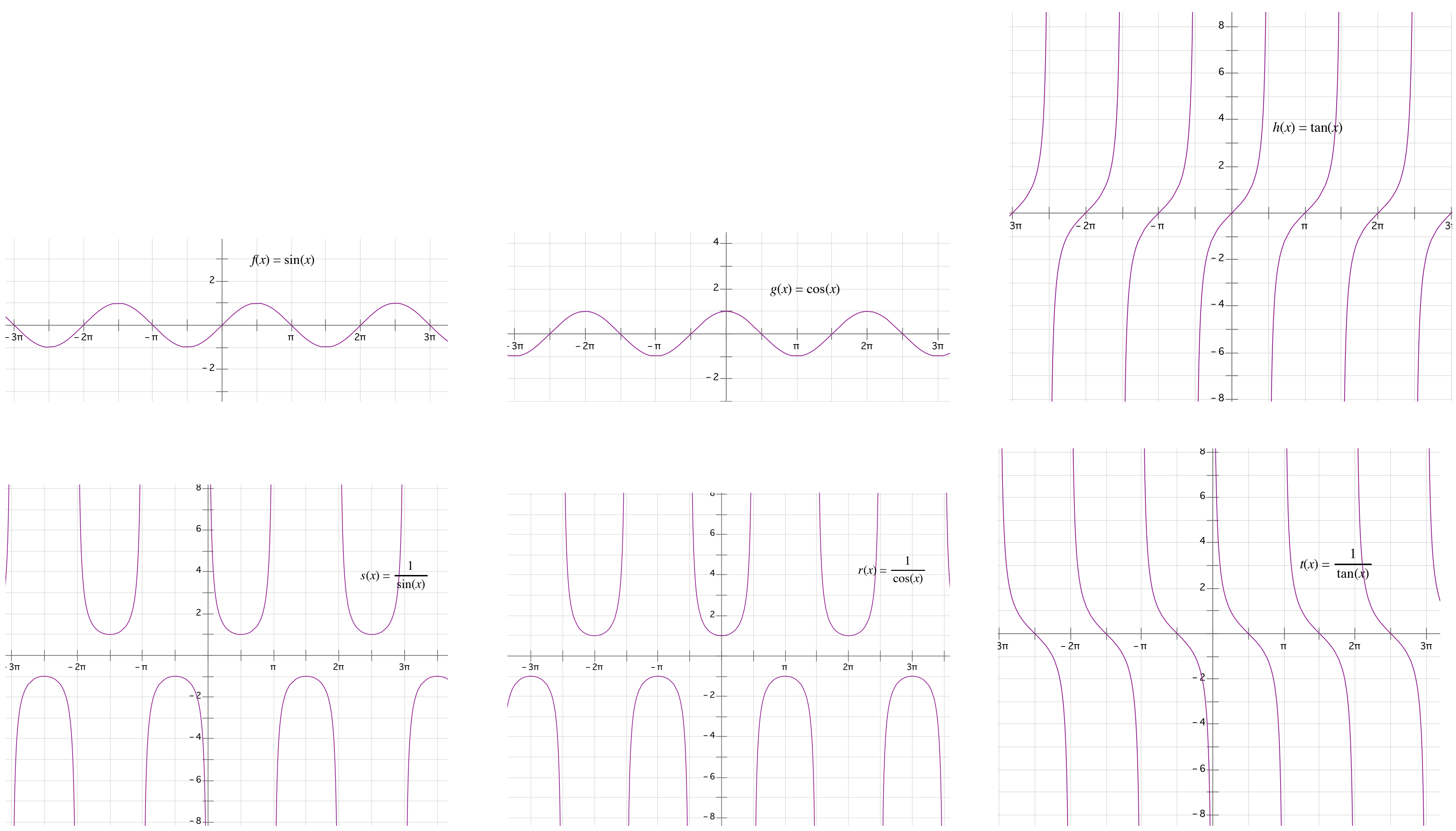
and

$$\frac{dx}{d\theta} = 40\Big[\frac{-\cos(\theta)}{\sin^2(\theta)}\Big].$$

When $\theta = \frac{\pi}{4}$,

$$\frac{dx}{d\theta}\Big|_{\theta=\frac{\pi}{4}} = 40\Big[\frac{-\cos(\frac{\pi}{4})}{\sin^2(\frac{\pi}{4})}\Big] = 40\frac{-1/\sqrt{2}}{1/2} = -40\sqrt{2} \quad \text{feet per radian.}$$

Graphs of Trigonometric functions



Extra Problems

1. Calculate
$$\lim_{x \to o} \frac{\sin(x^3)}{x}.$$

2. Calculate
$$\lim_{x \to 0} 7x \cot(3x).$$

3. If $g(x) = \cos(x)$, what is $g^{(42)}(x)$?

4. Find $f'(x)$ if $f(x) = x^2 \cos(x) \sin(x)$.

Extra Problems : Solutions

1. Calculate

$$\lim_{x\to o} \frac{\sin(x^3)}{x}.$$

$\lim_{x\to o} \frac{\sin(x^3)}{x} = \lim_{x\to 0} \frac{\sin(x^3)}{x\cdot x^2} \cdot x^2 = \lim_{x\to 0} \frac{\sin(x^3)}{x^3} \cdot \lim_{x\to 0} x^2 = 1 \cdot 0 = 0.$

2. Calculate

$$\lim_{x\to 0} 7x\cot(3x).$$

$$\lim_{x\to 0} 7x\cot(3x) = \lim_{x\to 0} 7x\frac{\cos(3x)}{\sin(3x)} = \lim_{x\to 0} 7x \cdot \frac{3x}{3x} \cdot \frac{\cos(3x)}{\sin(3x)} =$$

$$= \lim_{x\to 0} \underbrace{\frac{3x}{\sin(3x)}}_{1} \cdot \underbrace{\frac{7x}{3x}}_{7/3} \cdot \underbrace{\cos(3x)}_{1} = \frac{7}{3}$$

3. If $g(x) = \cos(x)$, what is $g^{(42)}(x)$?

$$g'(x) = -\sin x, \quad g''(x) = -\cos x, \quad g^{(3)}(x) = \sin x, \quad g^{(4)}(x) = \cos x, \ldots$$

Therefore $g^{(40)}(x) = \cos x$ and $g^{(42)}(x) = -\cos x$.

4. Find $f'(x)$ if $f(x) = x^2\cos(x)\sin(x)$.

Using the product rule, we get

$$f'(x) = (\cos x \cdot \sin x)2x + x^2\frac{d}{dx}(\cos x \cdot \sin x)$$

using the quotient rule a second time, we get

$$f'(x) = 2x(\cos x \cdot \sin x) + x^2(\sin x(-\sin x) + \cos x\cos x) = 2x(\cos x \cdot \sin x) + x^2(\cos^2 x - \sin^2 x).$$

In fact if we know our trig formulas very well, we see that

$$f'(x) = x\sin(2x) + x^2(\cos(2x)).$$

as $2\sin(x)\cos(x) = \sin(2x)$, and

$\cos^2 x - \sin^2 x = \cos(2x)$

↑ these follow from formulas 1 and 2 on page 2.