§7. The Derivative as a Function:

Recall: In the previous lecture we defined the derivative of a function f at a:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

when this limit exists. This gives us the slope of the tangent to the curve $y = f(x)$ at a.

Example: We saw if $f(x) = x^2 + 5x$, then $f'(a) = 2a + 5$ for any value a. Hence, $f'(1) = 7$, $f'(0) = 5$, $f'(3) = 11$, etc.

We see that for each value of a, we have corresponding value $f'(a)$. Hence, f' is a function of a. We change the variable from a to x to get a new function called the derivative of f:

$$\boxed{f' : x \longmapsto f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Remark: For x to be in the domain of f' we must have:

1) x is in the domain of f. ← Domain of f' is 'at most' the domain of f.

2) $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists.

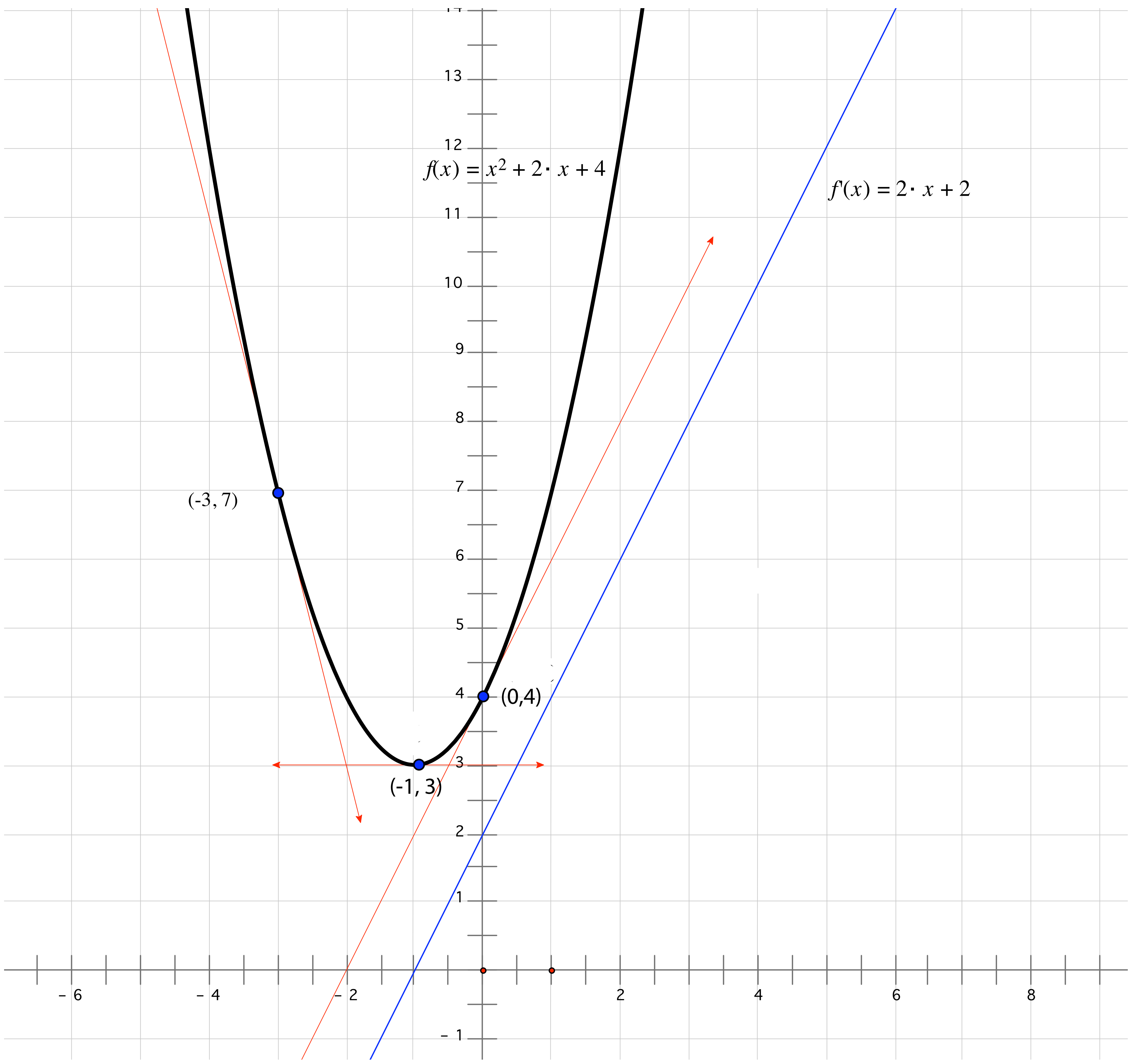
3) f must be defined in an open interval containing x.

Example: Let $f(x) = x^2 + 2x + 4$.

What is $f'(x)$?

What is the domain of f'?

Example Consider the function in the example above $f(x) = x^2 + 2x + 4$. The graph, $y = f(x)$ is shown below along with the graph of the new function $f'(x) = 2x + 2$. We can see how the graph of $f'(x)$ is related to the slope of the tangents to the graph of f.



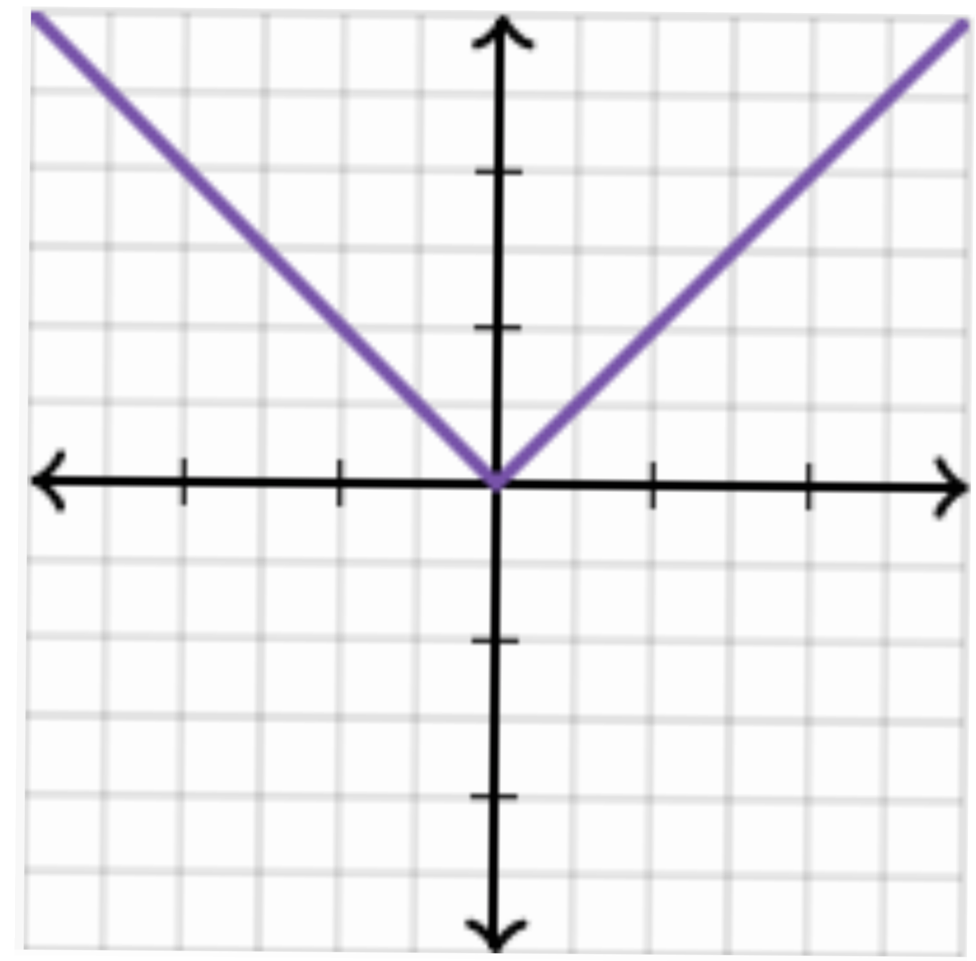
Fill in $<, >$ *or* $=$ as appropriate:

When $f(x)$ is decreasing the function $f'(x)$___0

When $f(x)$ is increasing the function $f'(x)$___0

At the turning point $x = -1$, $f'(x)$___0

Example Consider the function $f(x) = |x|$.



Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x > 0$?

Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x < 0$?

Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x = 0$?

What is the domain of $f'(x)$?

Alternative Notation:

Using $y = f(x)$, there are a number of notations used to denote the derivative of f:

$$f' = y' = \frac{dy}{dx} = \frac{df}{dx} = Df$$

The symbol $\frac{dy}{dx}$ comes from

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

and should not be interpreted as a quotient.

When we evaluate the derivative at a number a, we use the following notation:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiability:

Definition: Say a function f is defined in an open interval containing a. We say f is differentiable at a if $f'(a)$ exists.

If f is differentiable at every point in an open interval, we say f is differentiable on the open interval.

Example: Coming back to our example, $f(x) = |x|$, is f differentiable at 0?

Is f continuous at 0?

Remark:

Theorem: If f is differentiable at a, then f is continuous at a.

Proof: As f is differentiable at a, we know

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

To show f is continuous at a, we must show

$$\lim_{x \to a} f(x) = f(a)$$

(a has to be in domain of f for f to be differentiable at a)

or equivalently:

$$\lim_{x \to a} \left(f(x) - f(a) \right) = 0$$

Using our limit laws (Lecture 4):

$$\lim_{x \to a} \left(f(x) - f(a) \right) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{(x - a)} \cdot (x - a) \right]$$

$$\overset{(3)}{=} \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] \lim_{x \to a} (x - a)$$

$$= f'(a) \cdot (0) = 0$$

$\boxtimes$

Points where functions are not differentiable:

A function f can fail to be differentiable at a point a in a number of ways. For example:

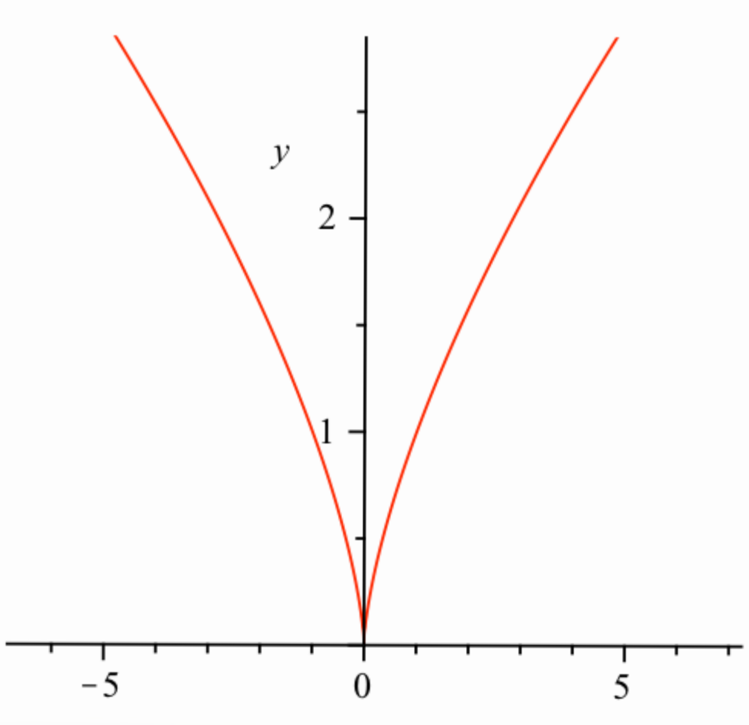
1) The function might be continuous at a, but have a sharp point or kink in the graph, like in the graph of $f(x) = |x|$ at 0.

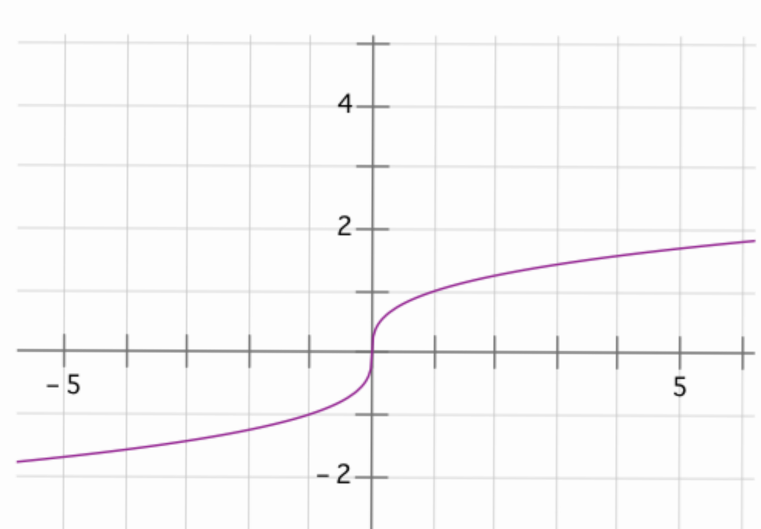
2) The function might not be continuous or might be undefined at a.

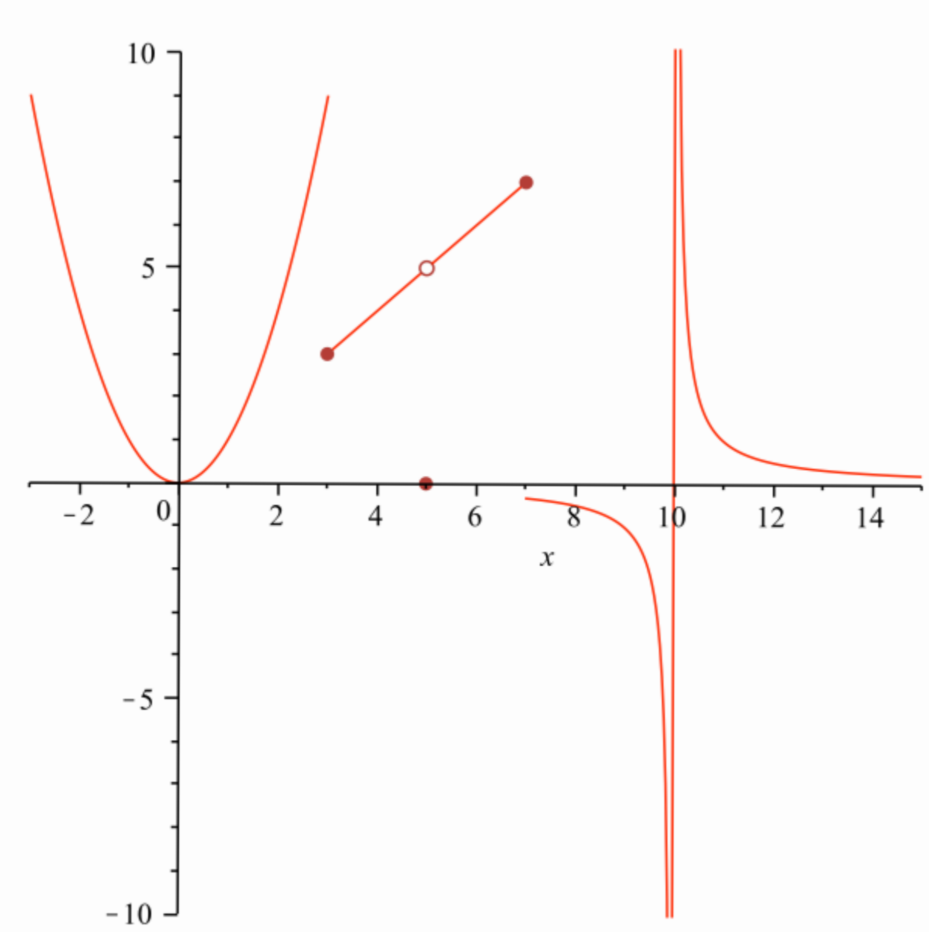
3) The function might be continuous, but the tangent line might be vertical.

   i.e. $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$.

Example Identify the points in the graphs below where the functions are not differentiable.







Higher Derivatives:

We have seen how, from a function f, we can establish a new function, f'.

We can continue this process:

$$f \rightsquigarrow f' \rightsquigarrow f'' \rightsquigarrow f''' \rightsquigarrow f^{(4)} \rightsquigarrow \dots$$

We call f'' the second derivative of f.

We call f''' the third derivative of f.

⋮

We call $f^{(n)}$ the nth derivative of f.

Example: We saw that the derivative of $f(x) = x^2 + 2x + 4$ was $f'(x) = 2x + 2$. Find $f''(x)$.

Remark: Recall that we considered $f'(x)$ to be the rate of change of f at x.

Hence, $f''(x)$ is the rate of change of the rate of change of f at x.

In terms of our intuition from physics:

•) velocity at time t = rate of change of position at time t

$$v(t) = s'(t)$$

•) ________________ = rate of change of velocity at time t

________ $= v'(t) = s''(t)$

Notation

The second derivative is also denoted by

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$

The third derivative of f is the derivative of the second derivative, denoted

$$\frac{d}{dx}f''(x) = f'''(x) = y''' = y^{(3)} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher derivative are denoted

$$f^{(4)}(x) = y^{(4)} = \frac{d^4y}{dx^4}, \qquad f^{(5)}(x) = y^{(5)} = \frac{d^5y}{dx^5}, \; etc\ldots$$

Example If $f(x) = x^2 + 2x + 4$, find $f^{(4)}(x)$ and $f^{(5)}(x)$.

12.

Old Exam Questions

1. Find the derivative of the function

$$f(x) = \frac{x}{x-5}$$

using the **limit** definition of the derivative.

2. Which of the statements given below is false?

(a) If f is differentiable at $x = a$, then $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ must equal $f(a)$.

(b) If f is differentiable at $x = a$, then a must be in the domain of f.

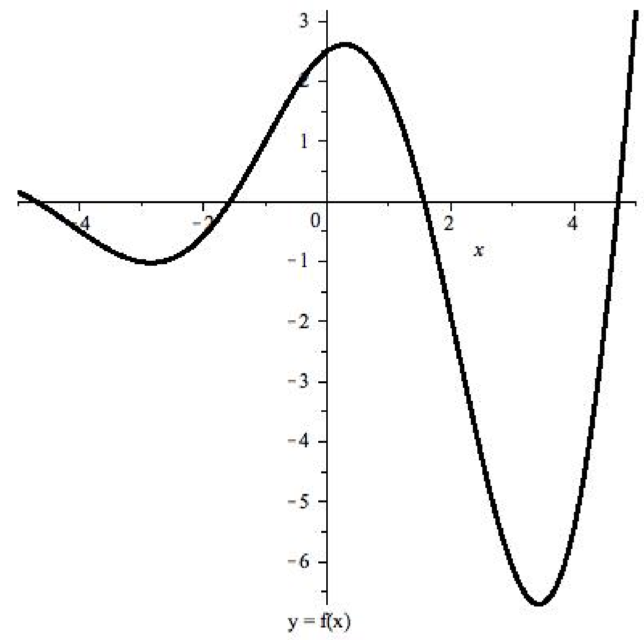
(c) If f is differentiable at $x = a$, then $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ must exist.

(d) If f is differentiable at $x = a$, then f must be continuous at $x = a$.

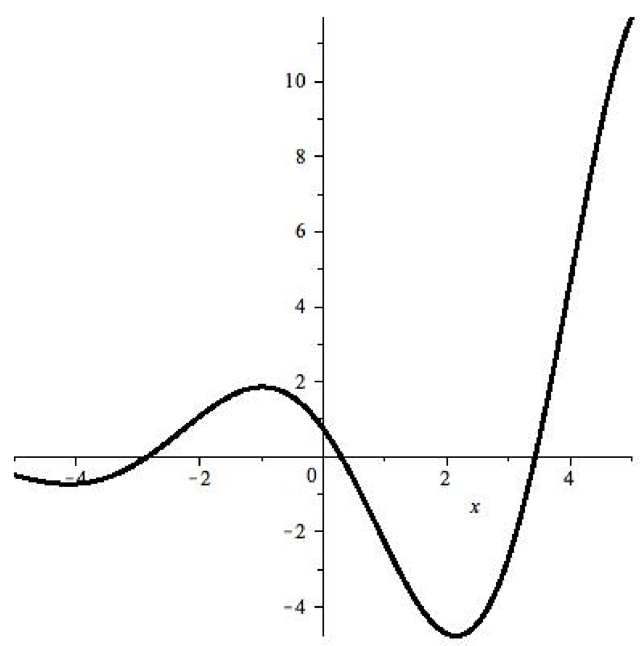
(e) If f is differentiable at $x = a$, then $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

13.

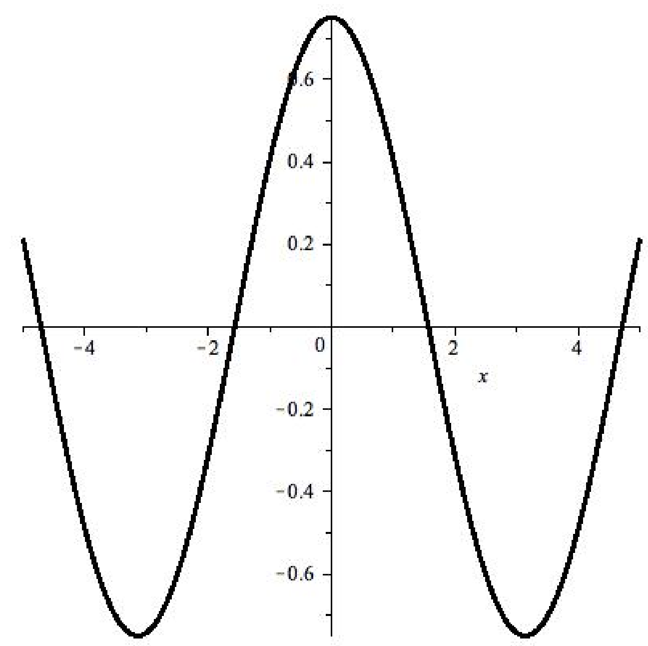
1. The graph of the function $f(x)$ is shown below:



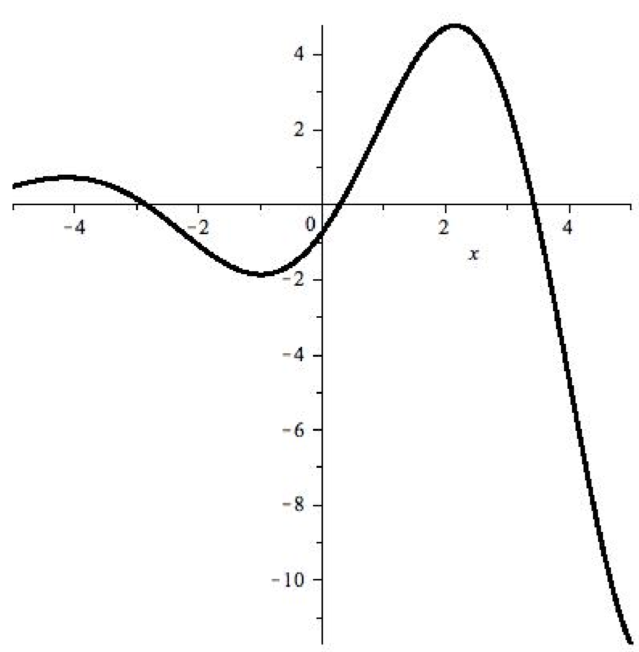
Which of the following gives the graph of $f'(x)$?



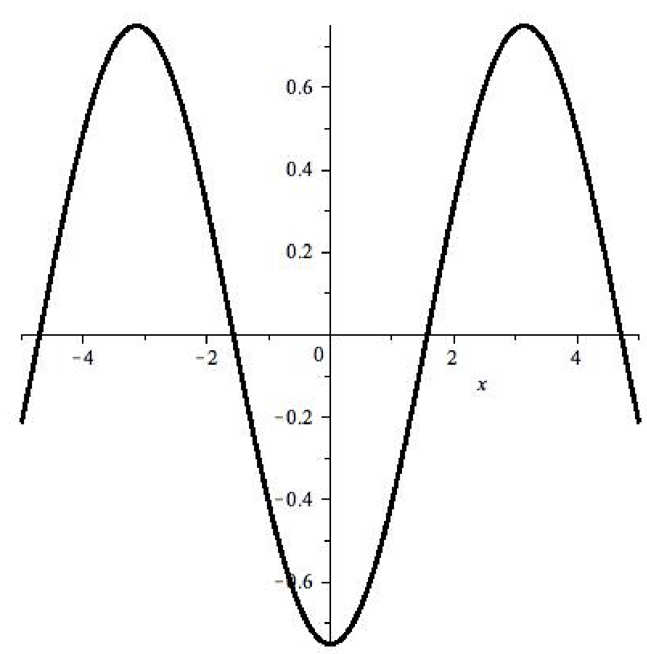
(a)



(b)



(c)



(d)

(e) None of the above

Old Exam Question , Sample Solution

1. Find the derivative of the function

$$f(x) = \frac{x}{x-5}$$

using the **limit** definition of the derivative.

Note the format of the solution below. It is important to carry the limits and show all calculations in order to recieve full credit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-5} - \frac{x}{x-5}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x-5) - x(x+h-5)}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + hx - 5x - 5h - x^2 - xh + 5x}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x}^2 + \cancel{h}x - \cancel{5}x - 5h - \cancel{x}^2 - \cancel{x}h + \cancel{5}x}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-5\,\cancel{h}}{(x+h-5)(x-5)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-5}{(x+h-5)(x-5)}$$

$$= \frac{-5}{(x-5)(x-5)}$$

$$= \frac{-5}{(x-5)^2}$$

2. Which of the statements given below is false?

If f is differentiable at a,

1. a must be in the domain of f.
2. $\lim_{h\to 0} \frac{f(a+h)-f(x)}{h}$ must exist at a.
3. f must be defined in an open interval containing a.

(a) If f is differentiable at $x = a$, then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ must equal $f(a)$. false, it is not required that this limit is $f(a)$. For example consider $f(x) = x^2 + 2x + 4$ from the notes. $f'(x) = 2x + 2$. $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 4 \neq f(1) = 7$.

(b) If f is differentiable at $x = a$, then a must be in the domain of f. True see 1 above.

(c) If f is differentiable at $x = a$, then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ must exist. True see 2 above.

(d) If f is differentiable at $x = a$, then f must be continuous at $x = a$. True by the theorem given in notes.

(e) If f is differentiable at $x = a$, then $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ True since the limit exists only if the laft and right hand limits exist and are equal.

3. The derivative must be positive when $f(x)$ is increasing and negative when it is decreasing. In particular $f'(x) > 0$ for all values of x bigger than 4 in this instance. Therefore the answer is (a).