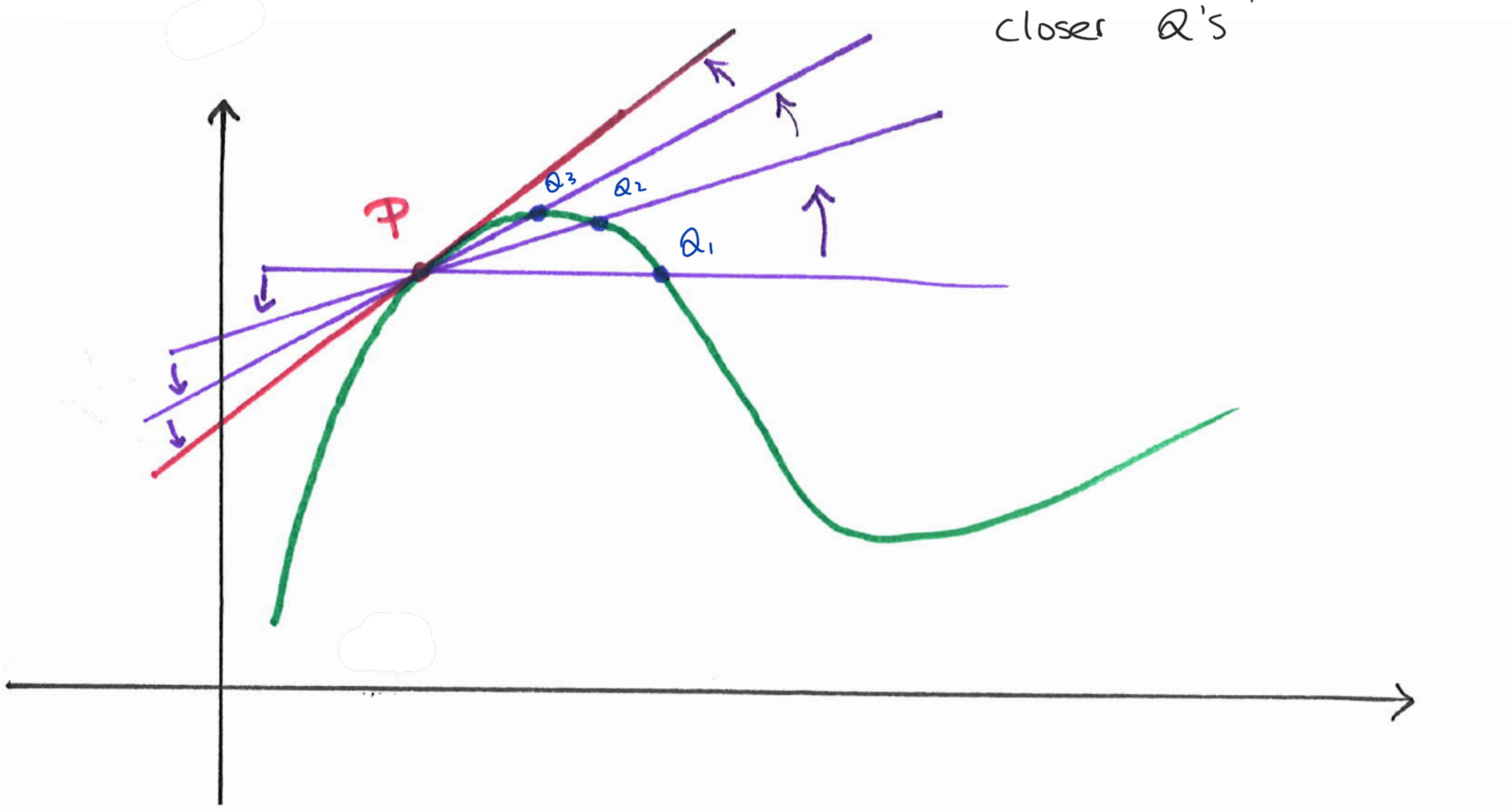
§6 Derivatives and Rates of Change

Goal: To improve our methods for finding tangent lines and instantaneous rates of change, using our new knowledge of limits.

Tangents:

Recall our method for calculating the slope of the tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$:

"Take closer and closer Q's"



If $Q = (x, f(x))$: $\quad m_{PQ} = \dfrac{f(x) - f(a)}{x - a}$

So we gathered that the slope of our secant lines connecting $P = (a, f(a))$ to $Q = (x, f(x))$ approached our desired slope for the tangent line as Q got closer to P (i.e. as x got closer to a). Hence, we have:

Definition: When $f(x)$ is defined on an open interval containing a, the Tangent Line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope:

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided that the limit exists.

Example: Find the equation of the tangent line to the curve $y = \sqrt{x}$ at $P = (1, 1)$.

(This is the problem we solved in Lecture 2)

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} =$$

Alternate Definition: If instead we write $Q = (a+h, f(a+h))$, we have

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Remark: The slope of the tangent line to a curve at a point is sometimes referred to as the "slope of the curve at the point".
y-values (or heights) on the curve near the point are close to the y-values (heights) on the tangent line near the point.
We will talk about this in more detail when we see Linear Approximation.

Example: Find the equation of the tangent line to the curve $f(x) = x^2 + 5x$ at the point $(1, 6)$.

Definition: When $f(x)$ is defined on an open interval containing a, the derivative of the function f at a is:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

if the limit exists.

Remark: The slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$.

Example: Let $f(x) = x^2 + 5x$. Find $f'(a)$, $f'(2)$ and $f'(-1)$.

Equation of the Tangent Line:

The Equation to the Tangent Line to the graph $y = f(x)$ at the point $(a, f(a))$ is given by:

$$\boxed{y - f(a) = f'(a)(x - a)}$$

Example: Find the equation of the tangent line to the graph $f(x) = x^2 + 5x$ at:

(i) $x = 2$:

(ii) $x = -1$:

Remark: When $f'(a)$ is positive, the function is increasing and when it is negative, the fuction is decreasing.

When the absolute value of $f'(a)$ is small, the function is changing slowly at a.

When the absolute value (size) of $f'(a)$ is large, the function is changing rapidly at a.

This captures how the derivative is (related to) the rate of change of the function.

Some limits are easy to calculate when we recognize them as derivatives:

Example The following limits represent the derivative of a function f at a number a. In each case, what is $f(x)$ and a?

$$(a) \quad \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x - \pi/4} \qquad\qquad (b) \quad \lim_{h \to 0} \frac{(1+h)^4 + (1+h) - 2}{h}$$

(a) $\lim_{x \to \frac{\pi}{4}} \dfrac{\sin(x) - 1/\sqrt{2}}{x - \pi/4} = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$: $f(x) =$ ____ $a =$ ____

(b)

Velocity:

Remark: speed = |velocity|

$$\boxed{\text{Average Speed} = \frac{\text{Distance Covered}}{\text{Time Taken}}}$$

$$\boxed{\text{Average Velocity} = \frac{\text{Overall Displacement}}{\text{Time Taken}}}$$

Recall that we estimated the instantaneous velocity at a time $t = a$, by finding the average velocity over finer and finer time intervals.

Definition: If $s = f(t)$ is a position function which gives the displacement of an object at time t, the velocity of object at time t is given by:

$$\boxed{v(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)}$$

Thus the velocity at time $t = a$ is the slope of the tangent line to the curve $y = s = f(t)$ at the point where $t = a$.

Example The position function of a stone thrown from a bridge is given by $s(t) = 10t - 16t^2$ feet (below the bridge) after t seconds.

(a) What is the average velocity of the stone between $t_1 = 1$ and $t_2 = 5$ seconds?

(b) What is the instantaneous velocity of the stone at $t = 1$ second. (Note that speed $= |\text{Velocity}|$).

Alternative Notation: If $y = f(x)$, and $P = (a, f(a))$ is a point on the corresponding curve we may write:

$$\Delta y = f(x) - f(a) \quad \leftarrow \text{Change in 'height'}$$

$$\Delta x = x - a \quad \leftarrow \text{Change in 'width'}$$

Then:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

In economics, the instantaneous rate of change of the cost function (revenue function) is called the **Marginal Cost** (Marginal Revenue).

Example The cost (in dollars) of producing x units of a certain commodity is $C(x) = 50 + \sqrt{x}$.

(a) Find the average rate of change of C with respect to x when the production level is changed from $x = 100$ to $x = 169$.

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$ (Marginal cost when $x = 100$, usually explained as the cost of producing an extra unit when your production level is 100).

Example The cost (in dollars) of producing x units of a certain commodity is $C(x) = 50 + \sqrt{x}$.

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Solution The average rate of change of C is the average cost per unit when we increase production from $x_1 = 100$ tp $x_2 = 169$ units. It is given by

$$\frac{\Delta x}{\Delta y} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{50 + \sqrt{169} - (50 + \sqrt{100})}{169 - 100} = \frac{13 - 10}{69} = \frac{3}{69} = .04347.$$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$ (Marginal cost when $x = 100$, usually explained as the cost of producing an extra unit when your production level is 100).

Solution The instantaneous rate of change of C when $x = 100$ It is given by

$$\lim_{x \to 100} \frac{\Delta x}{\Delta y} = \lim_{x \to 100} \frac{f(x) - f(100)}{x - 100} = \lim_{x \to 100} \frac{50 + \sqrt{x} - (50 + \sqrt{100})}{x - 100} = \lim_{x \to 100} \frac{\sqrt{x} - 10}{x - 100}$$

$$= \lim_{x \to 100} \frac{(\sqrt{x} - 10)}{(\sqrt{x} - 10)(\sqrt{x} + 10)} == \lim_{x \to 100} \frac{1}{(\sqrt{x} + 10)} = \frac{1}{20} = .05$$