§ 15. Maxima and Minima:

There are many obvious reasons why we might be interested in finding maxima and minima of given functions (finance, engineering, etc.).

We will see two types of maxima|minima:

1) Absolute maxima|minima
2) Local maxima|minima

can replace with $D \subseteq \mathbb{R}$

Definitions: Say $f : \mathbb{R} \to \mathbb{R}$ is a function.

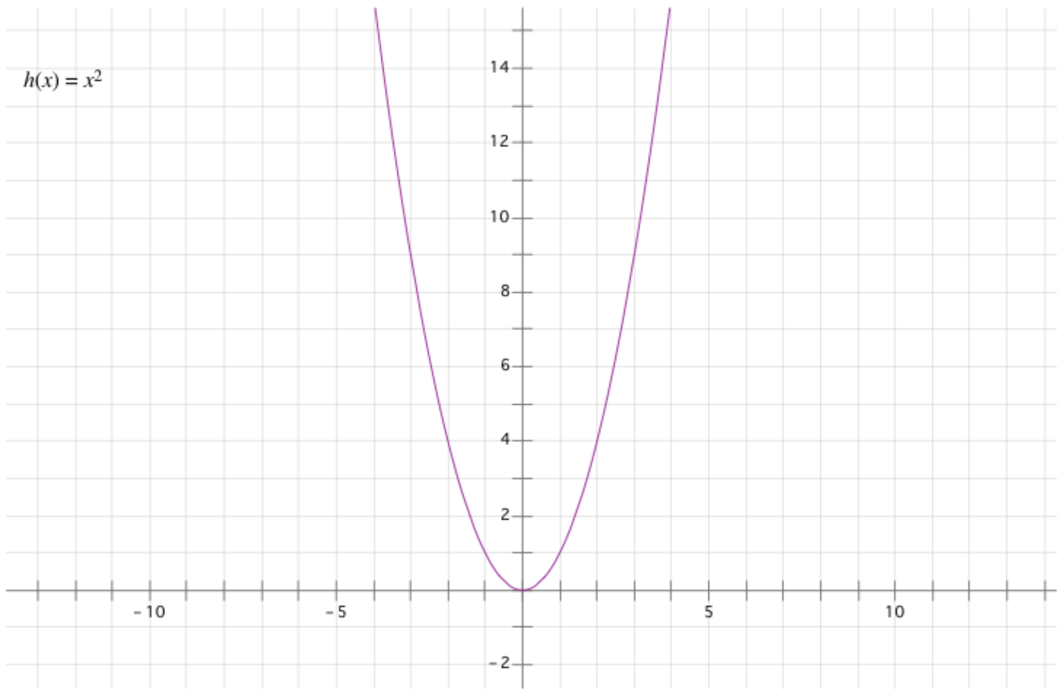
1) We say f has an absolute maximum at $a \in \mathbb{R}$ if $f(x) \leq f(a)$ for all $x \in \mathbb{R}$.

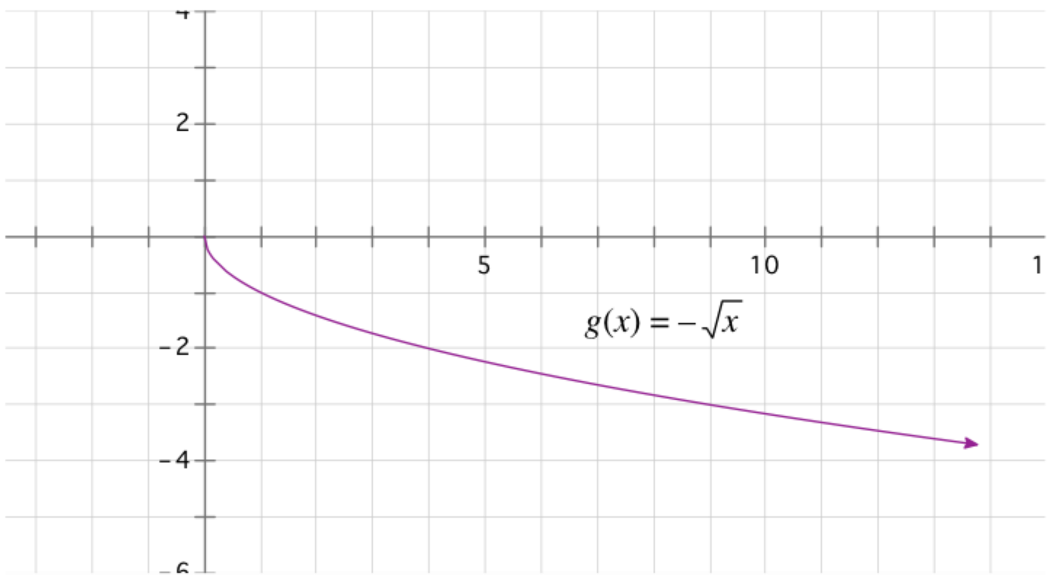
2) We say f has an absolute minimum at $b \in \mathbb{R}$ if $f(x) \geq f(b)$ for all $x \in \mathbb{R}$.

Remark: These are also referred to as "global" maximums / minimums.

Definition: Maximum and minimum values of f in it's domain are called extreme values of f.

Example Consider the graphs of the functions shown below. What are the extreme values of the functions; $h(x) = x^2$ and $g(x) = -\sqrt{x}$?





Theorem: (Extreme value theorem)

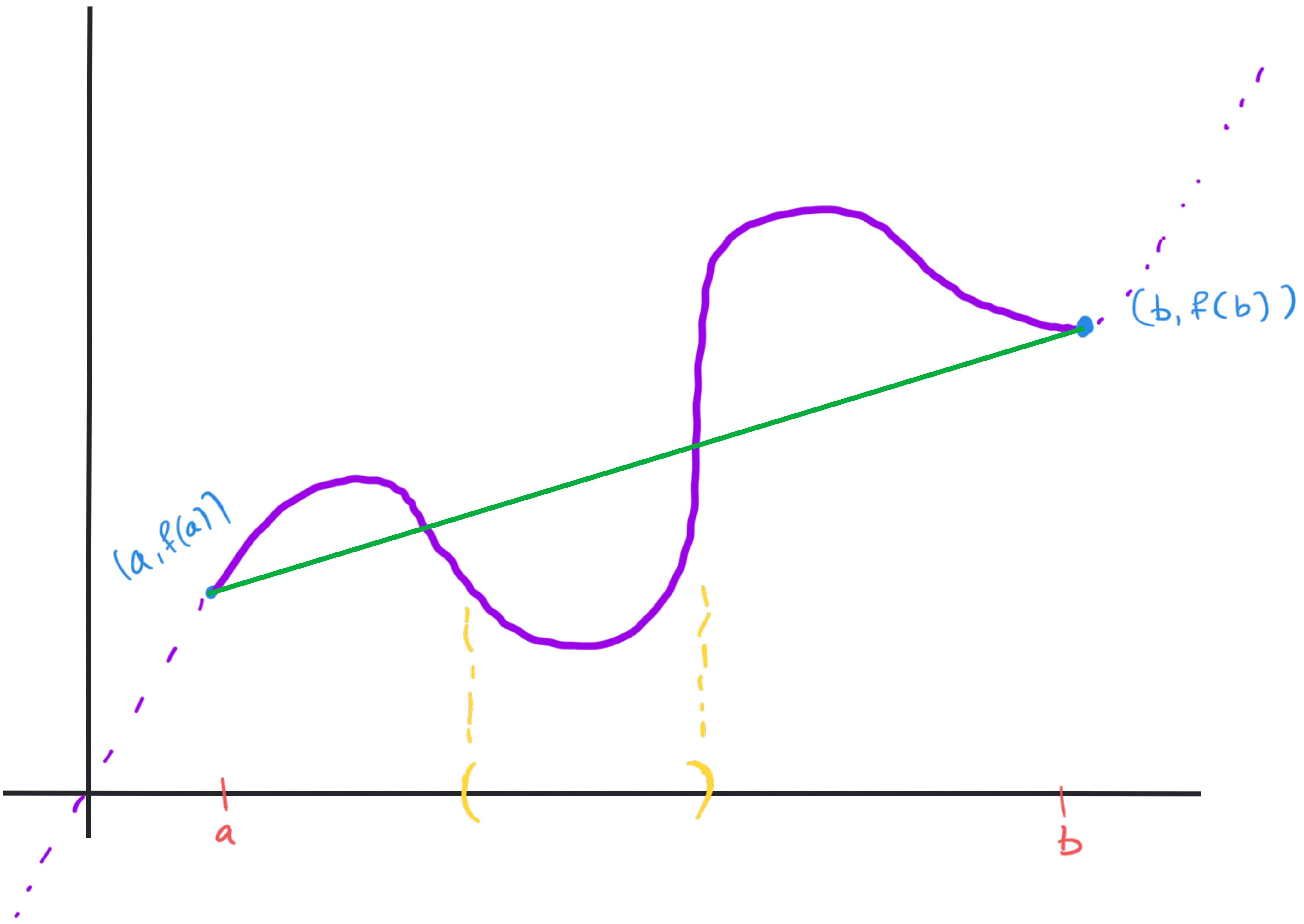
If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum M and an absolute minimum m on $[a,b]$.

i.e. there is a $c \in [a,b]$ and a $d \in [a,b]$;

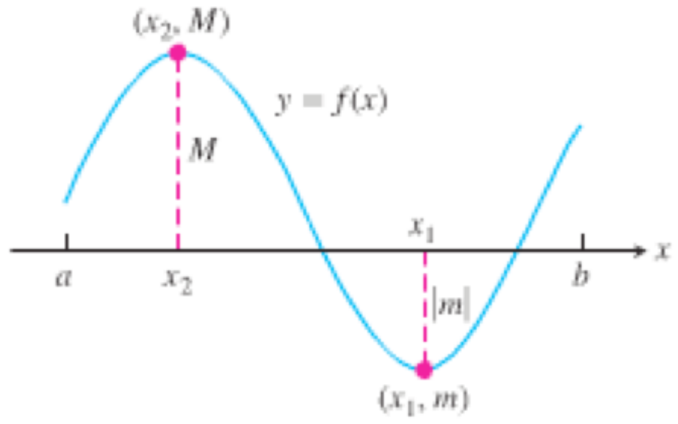
$$f(x) \leq f(c) = M \quad \text{for all } x \in [a,b]$$

and $f(x) \geq f(d) = m$ for all $x \in [a,b]$.

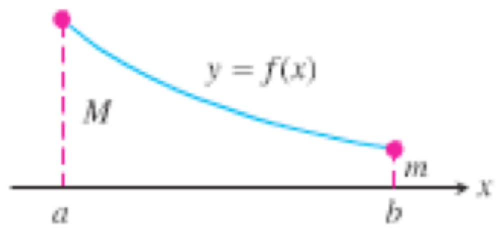
Picture:



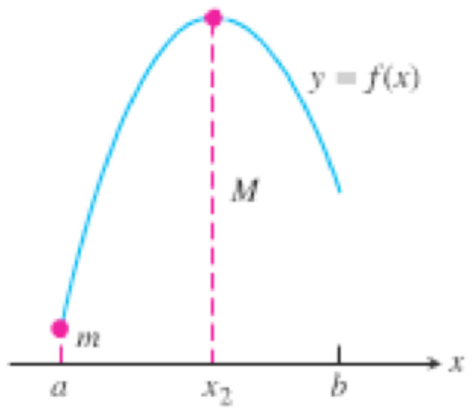
This can happen in a variety of ways. We can see some of the possibilities in the picture below.



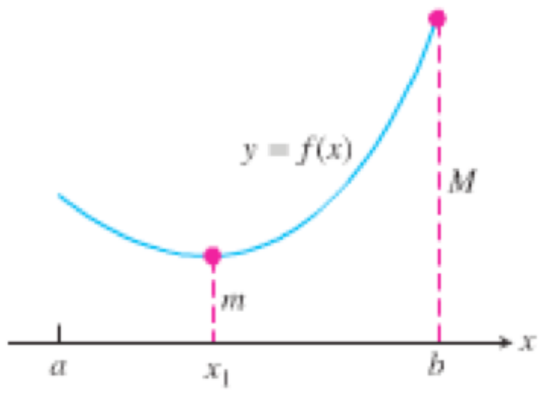
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

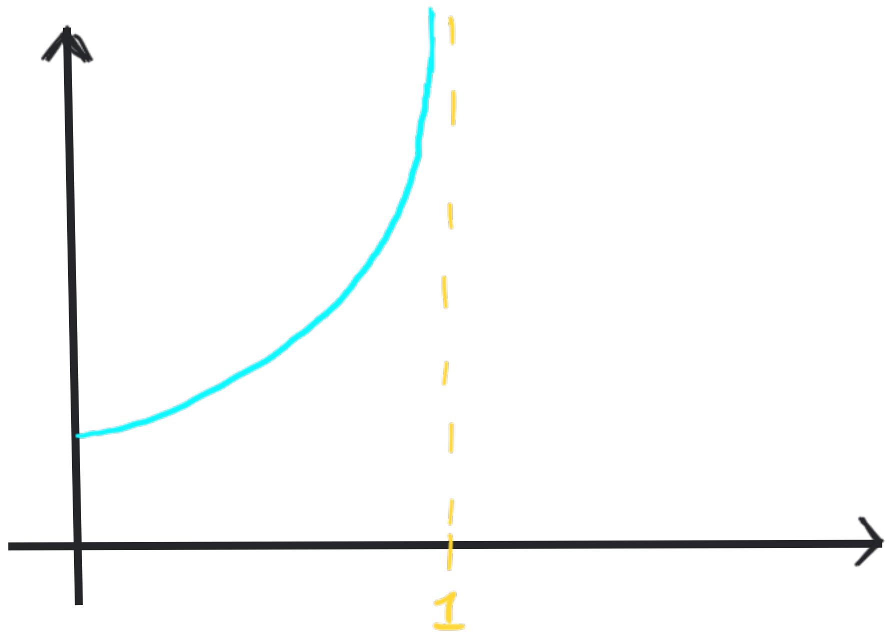
Example: If $f(x) = \sin(x)$, what is the absolute maximum / minimum value of f on $[0, 2\pi]$?

Absolute Max @ $\pi/2$: (1)

Absolute Min @ $\frac{3\pi}{2}$: (-1)

Non-example: Continuity and closed interval are necessary:

1) $f(x) = \frac{-1}{x-1}$ on $[0,1)$



2) $$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{on } [-1,1]$$

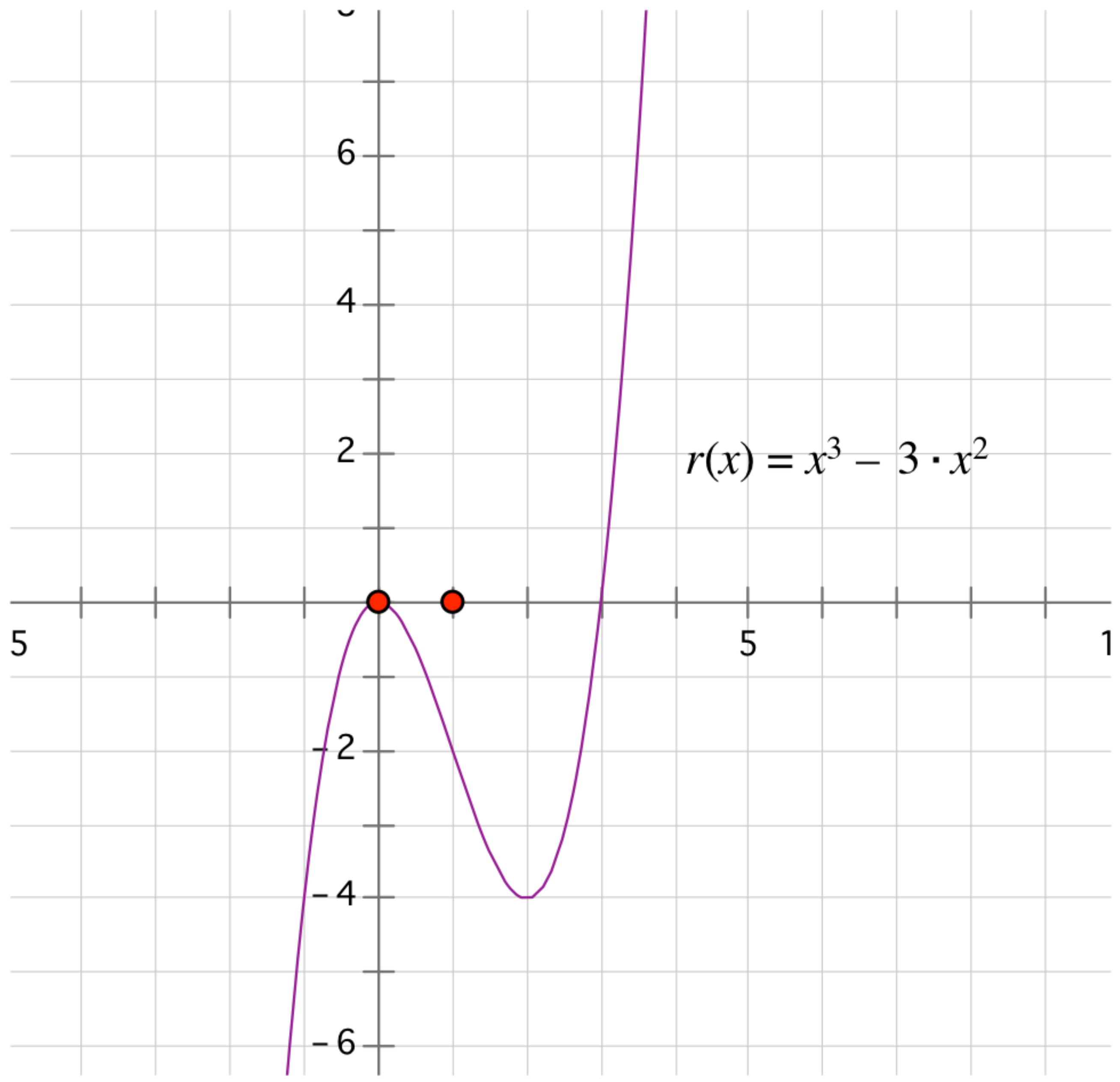
Remark: We have seen that some points are maxima/minima in a neighbourhood, but are not absolute maxima/minima. This motivates the following definitions:

Definitions: Say $f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a function.

1) We say f has a local maximum at a point $c \in D$ if $f(c) \geq f(x)$ for all x in some open interval around c.

2) We say f has a local minimum at a point $d \in D$ if $f(d) \leq f(x)$ for all x in some open interval around d.

Example The graph of $r(x) = x^3 - 3x^2$ is shown below. Find the points where the function has local maxima and minima.



Theorem: (Fermat's Theorem)

If f has a local maximum or minimum at a and $f'(a)$ exists, then $f'(a) = 0$.

Proof:

- Suppose f has a local maximum at a.

  So $f(a) \geq f(x)$ for x close to a.

  As f is differentiable at a, we have

$$f'(a) = \lim_{h \to 0^+} \frac{\overset{\swarrow\ -}{f(a+h) - f(a)}}{\underset{\nwarrow\ +}{h}} \leq 0$$

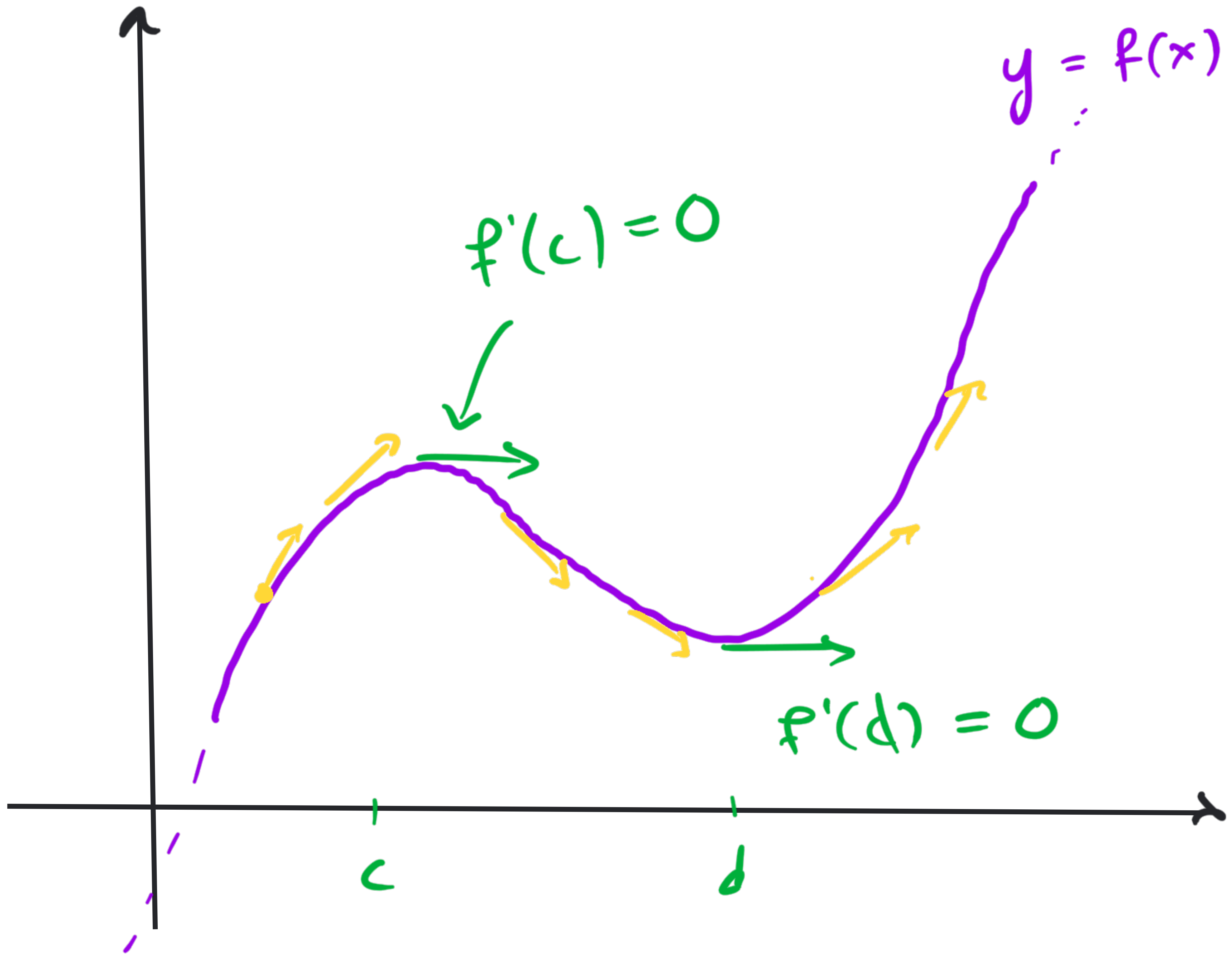
$$f'(a) = \lim_{h \to 0^-} \frac{\overset{\swarrow\ -}{f(a+h) - f(a)}}{\underset{\nwarrow\ -}{h}} \geq 0$$

Hence $f'(a) = 0$.

Similarly for local minimums.

$\boxtimes$

Picture:



Example: Reconsider $r(x) = x^3 - 3x^2$.

Verify that $r'(0) = r'(2) = 0$.

$$r'(x) = 3x^2 - 6x$$

$$r'(0) = 3(0)^2 - 6(0) = 0$$

$$r'(2) = 3(2)^2 - 6(2) = 12 - 12 = 0$$

Remarks:

1) $f'(c) = 0$ does not mean c is a local max. or min.

   e.g. $f(x) = x^3$, $c = 0$.

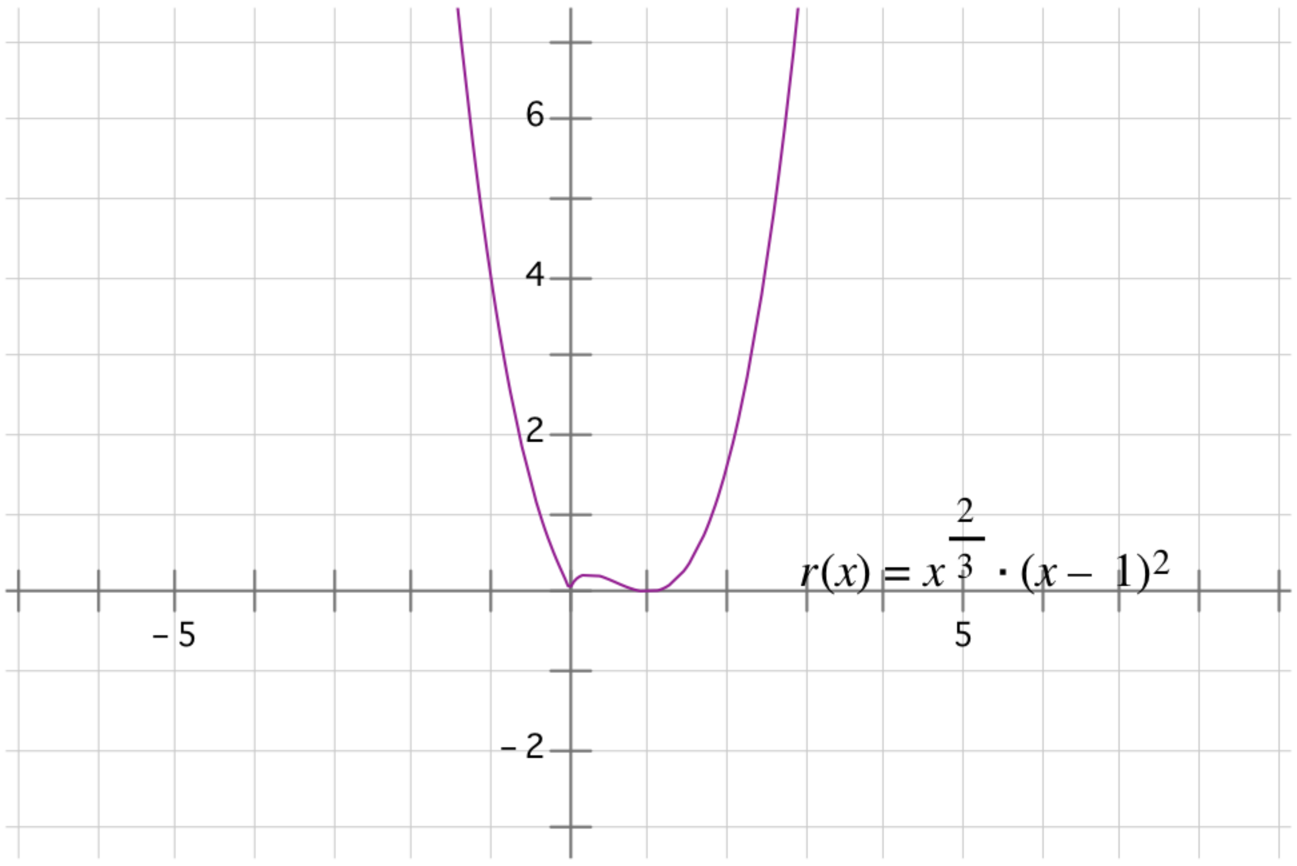
2) A function may have a local max. or min. where it is not differentiable.

   e.g. $f(x) = |x|$, $c = 0$.

Nevertheless, identifying points where $f'(x) = 0$ helps us to find local max. and min.

Definition: A critical point/number of a function f is a $c \in \mathbb{R}$ such that $f'(c) = 0$ or $f'(c)$ is not defined.

Example Find the critical numbers of the function $r(x) = x^{2/3}(x-1)^2$.



$$r'(x) = \frac{2}{3} x^{-1/3}(x-1)^2 + x^{2/3} \cdot 2(x-1) \Bigg\}$$

$$= (x-1)\left[\frac{2(x-1)}{3\sqrt[3]{x}} + 2\sqrt[3]{x^2} \right]$$

$r'(1) = 0$ $\circledast$

$r'(x) = 0$ for $x \neq 1 \Rightarrow \dfrac{2(x-1)}{3\sqrt[3]{x}} + 2\sqrt[3]{x^2} = 0$

$\circledast$

$r'(0)$ is undefined.

Exercise : ~~Finish!~~

$$\frac{2(x-1)}{3\sqrt[3]{x}} + 2\sqrt[3]{x^2} = 0$$

$\big)\, 3\sqrt[3]{x}$

$$2(x-1) + 6x = 0$$

$$2x - 2 + 6x = 0$$

$$8x = 2$$

$$x = 1/4$$ ⊛

Critical pts: $0, 1/4, 1$

Finding the absolute maximum and minimum of a continuous function on a closed interval $[a, b]$.

To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$;

1. Find all of the critical points of f in the interval $[a, b]$.
2. Evaluate f at all of the critical numbers in the interval $[a, b]$.
3. Evaluate f at the endpoints of the interval, (calculate $f(a)$ and $f(b)$.)
4. The largest of the values from steps 2 and 3 is the absolute maximum of the function on the interval $[a, b]$ and the smallest of the values from steps 2 and 3 is the absolute maximum of the function on the interval $[a, b]$.

Example Find the absolute maximum and minimum of the function $r(x) = x^{2/3}(x-1)^2$ on the interval $[-1, 1]$.

Exercise!

$r(-1) =$

$r(0) =$

$r(1/4) =$

$r(1) =$

Find biggest / smallest.

Note Sometimes the absolute maximum can occur at more than one point c. The same is true for the absolute minimum.

Example Find the absolute maximum and minimum of the function $f(x) = x^3 - 3x^2$ for $1 \leq x \leq 4$.

$f(1) = -2$, $f(4) = 16$ } check endpoints

$f'(x) = 3x^2 - 6x \Rightarrow$ Critical pts where :

1) $3x^2 - 6x = 0 \Longleftrightarrow x^2 - 2x = 0 \Longleftrightarrow x = 0$ or $x = 2$

2) $f'(x)$ is defined everywhere

$f(0) = 0$, $f(2) = -4$

Absolute Max : $c = 4$, $f(c) = 16$

Absolute Min : $d = 2$, $f(2) = -4$

Exercise : Try same problem on $[-1,4]$

Example The profit function for my company depends (partly) on the number of widgets I produce. The relationship between x = the number of widgets I produce and my profits (all other variables remaining constant) is given by

$$P(x) = 4 + 0.03x^2 - 0.001x^3.$$

Find the production level for widgets that will maximize this function if I have the capacity to produce at most 50 widgets.

Since production is limited to $0 \leq x \leq 50$, we must maximize the profit function $P(x) = 4 + 0.03x^2 - 0.001x^3$ on the interval $[0, 50]$. $P(x)$ is continuous on this interval since it is a polynomial, therefore by the Extreme value theorem $P(x)$ has an absolute maximum on the interval. Following our 3 step proceedure:

1. **Critical Points** $P'(x) = 0.06x - 0.003x^2$. All values of x in the interval $[0, 50]$ are in the domain of P and in the domain of P', so the critical points occur where $P'(x) = 0$.

$$P'(x) = 0.06x - 0.003x^2 = 0.003x(20 - x) = 0$$

if $x = 0$ or $x = 20$.

$$\boxed{\text{Critical points} \quad x = 0 \text{ and } \quad x = 20}$$

2. **Evaluate at critical points** $P(0) = 4$, $P(20) = 4 + 0.03(20) - 0.003(20^2) = 8$.

3. **Evaluate at end points** $P(0) = 4$, $P(50) = 4 + 0.03(50) - 0.003(50^2) = -46$.

4. **Choose the largest value** Absolute maximum at $x = 20$. $P(20 = 8$ is the absolute maximum profit in this production range.