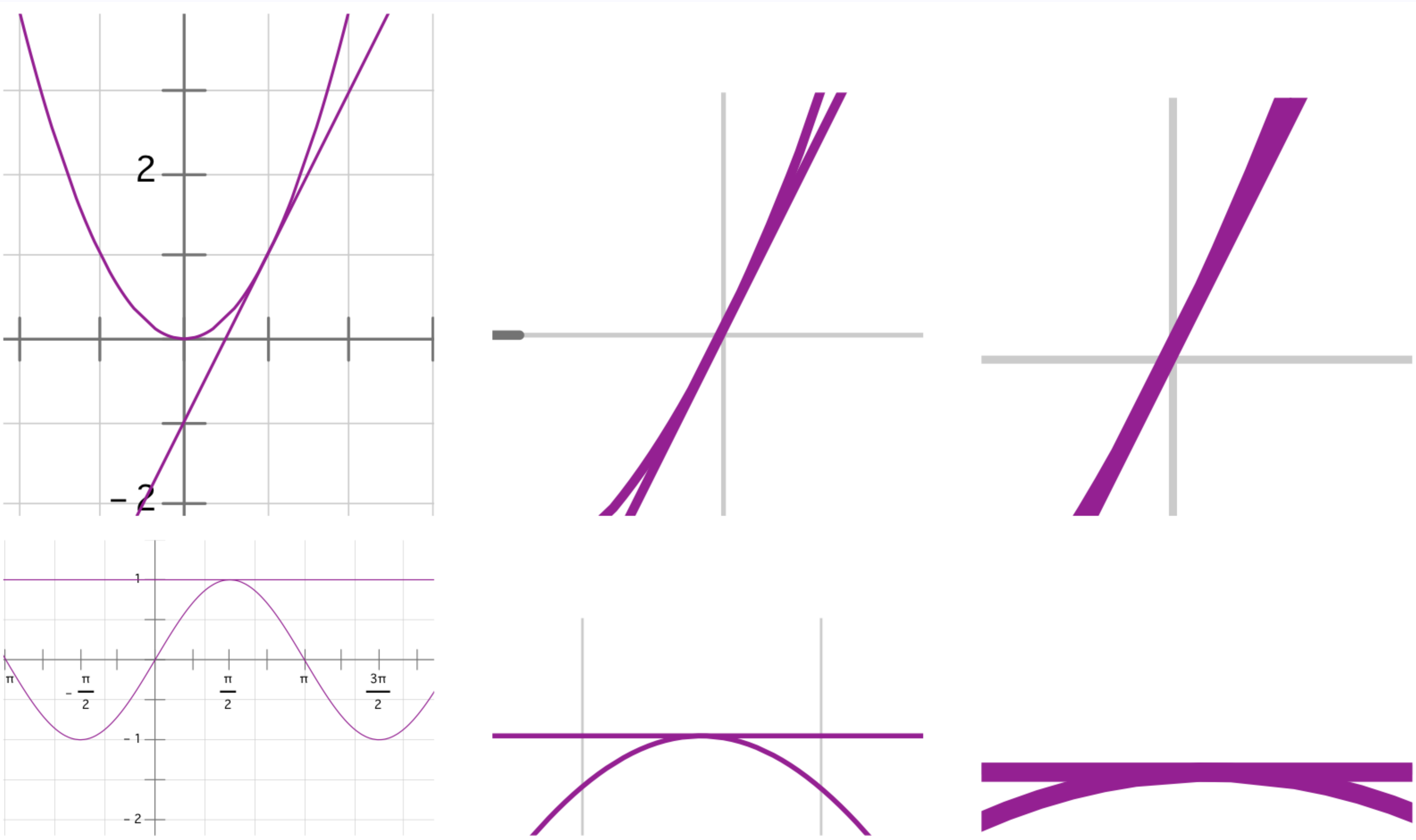
§14. Linear approximations and differentials:

Set up: Say we have a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that f is differentiable at a.

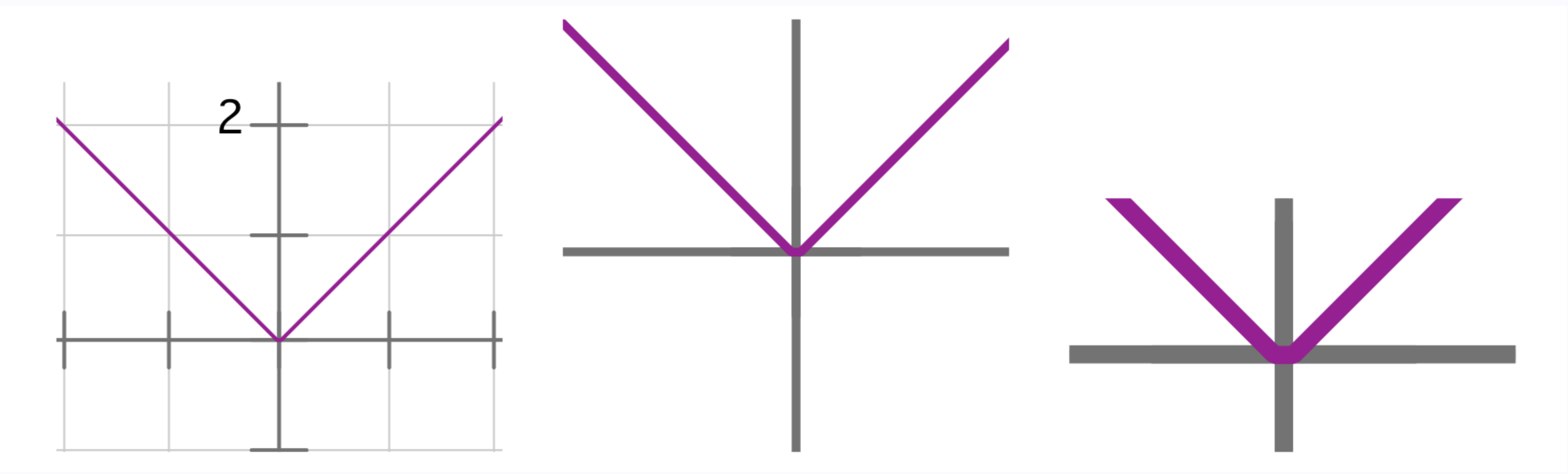
We have noted before that, if we zoom in on the point $(a, f(a))$, the "heights" of the points on the graph of the function are very close to the "heights" of the corresponding points on the tangent line.

See the curves $y = x^2$ at $(1,1)$ and $y = \sin(x)$ at $(\pi/2, 1)$.



On the other hand, if we examine a point on a graph of a function where it is not differentiable, we can't find an analogous idea.

Example: $y = |x|$ at $(0,0)$.



Recall: The equation of the tangent line to f at a is

$$L(x) = f'(a)(x-a) + f(a)$$

Conclusion: If f is differentiable at a then:

$$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$

NB

This is called the linear approximation, or tangent line approximation to f at a.

$L : \mathbb{R} \to \mathbb{R}$ is called the linearization of f at a.

Error in approximation: We can find bounds on $|f(x) - L(x)|$, the "error". We will see this later.

Motivation: If f is differentiable at a, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

so, for x 'close to' a, we have

$$\frac{f(x) - f(a)}{x - a} \approx f'(a)$$

$$\Rightarrow \quad f(x) - f(a) \approx f'(a)(x-a)$$

$$\Rightarrow \quad f(x) \approx f'(a)(x-a) + f(a)$$

$$\Rightarrow \quad f(x) \approx L(x)$$

Example (a) Find the linearization of the function $f(x) = \sqrt[3]{x}$ at $a = 27$.

$f(x)$	$f'(x)$	a	$f(a)$	$f'(a)$

$L(x) = f(a) + f'(a)(x - a) =$

(b) Use the linearization above to approximate the numbers $\sqrt[3]{27.01}$ and $\sqrt[3]{26.99}$.

(c) We can get an approximation for $\sqrt[3]{x}$ from the linearization of the function, $L(x)$, above, for any x in the interval $26 \leq x \leq 28$. We can see, from the table below, that the closer the value of x gets to 27, the better the approximation to the actual value of $\sqrt[3]{x}$.

$f(x)$	x	From $L(x)$	Actual Value $\sqrt[3]{x}$
$\sqrt[3]{26.5}$	26.5	2.9814815	2.9813650
$\sqrt[3]{26.9}$	26.9	2.996296	2.996292
$\sqrt[3]{26.99}$	26.99	2.9996296	2.9996296
$\sqrt[3]{27}$	27	3	3
$\sqrt[3]{27.01}$	27.01	3.0003704	3.0003703
$\sqrt[3]{27.1}$	27.1	3.0037037	3.0036991
$\sqrt[3]{27.5}$	27.5	3.0185185	3.0184054

Error of approximation In fact by zooming in on the graph of $f(x) = \sqrt[3]{x}$, you will see that

$$|\sqrt[3]{x} - L(x)| < 0.001 \quad \text{or} \quad -0.001 < \sqrt[3]{x} - L(x) < 0.001$$

when x is in the interval $26.5 \leq x \leq 27.5$.

Such bounds on the error are useful when using approximations. We will be able to derive such estimates later when we study Newton's method.

Example (a) Find the linearization of the function $f(x) = \sqrt{x+9}$ at $a = 7$.

$f(x)$	$f'(x)$	a	$f(a)$	$f'(a)$

$L(x) =$

(b) Use the linearization above to approximate the numbers $\sqrt{16.03}$ and $\sqrt{15.98}$.

Example (commonly used linearizations) (a) Find the Linearization of the functions $\sin\theta$ and $\cos\theta$ at $\theta = 0$.

$f(\theta)$	$f'(\theta)$	a	$f(a)$	$f'(a)$

$\cos(\theta) \approx$

$\sin(\theta) \approx$

(b) Estimate the value of $\sin(\frac{\pi}{100})$, $\cos\frac{\pi}{95}$ and $\sin 2^o = \sin\frac{\pi}{90}$.

Notation: Δy

Suppose f is differentiable at a.

Linear approximation says

$$f(x) \approx h(x) = f'(a)(x-a) + f(a)$$

Equivalently: $f(x) - f(a) \approx f'(a)(x-a)$ (*)

We denote a small change in width (on the curve) by $\boxed{\Delta x = x - a}$

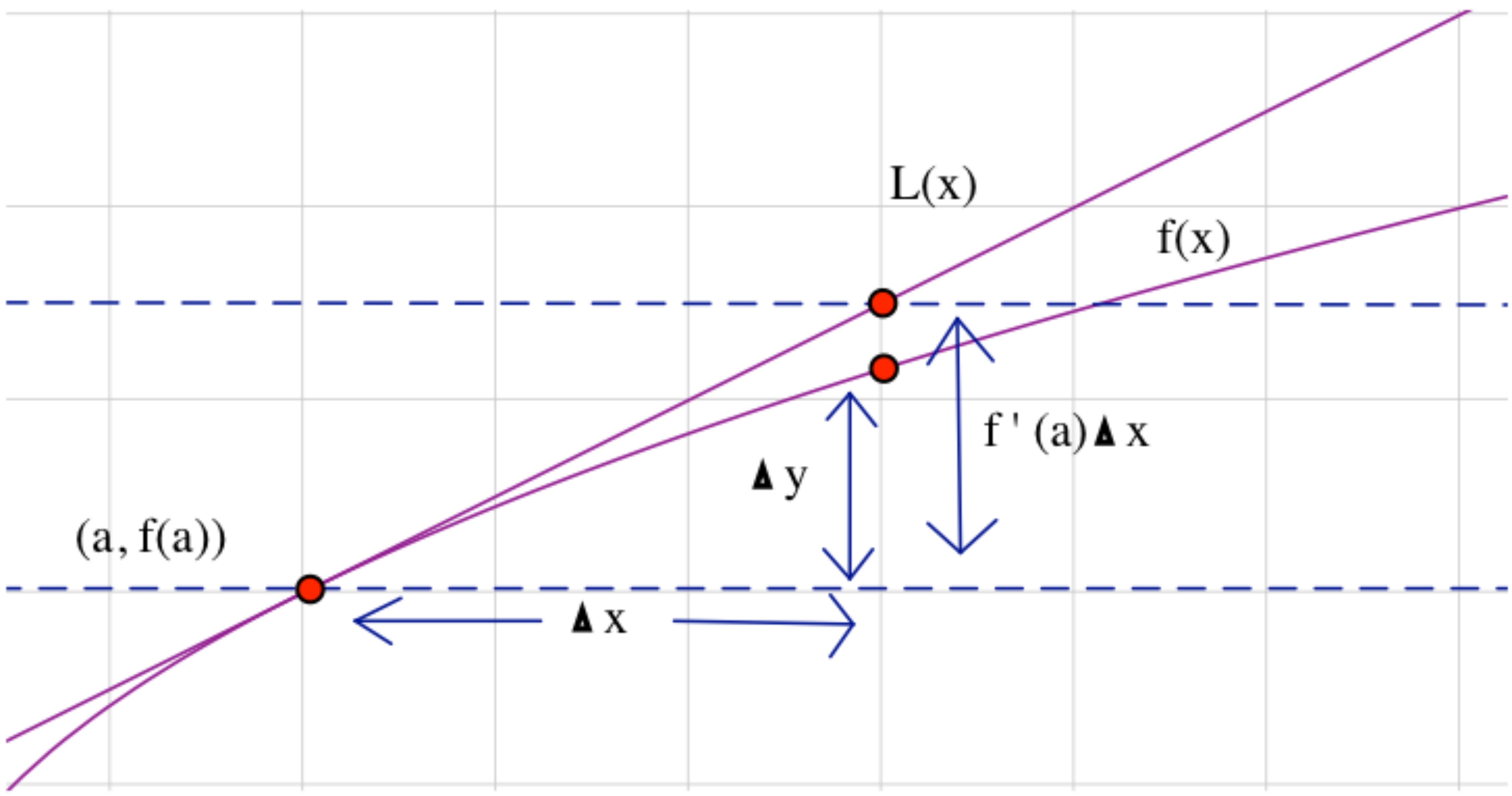
We denote a small change in height on the curve by

$$\boxed{\Delta y = f(x) - f(a)}$$

(*) says:

$$\boxed{\Delta y \approx f'(a)\,\Delta x}$$

NB



Example Approximate the change in the surface area of a spherical hot air balloon when the radius changes from 4 to 3.9 meters. (The surface area of a sphere of radius r is given by $S = 4\pi r^2$.)

$$\Delta S \approx S'(a)\Delta r$$

$$S(r) = 4\pi r^2, \qquad S'(r) = 8\pi r.$$

$$\Delta S \approx S'(4)\Delta r = 32\pi(-0.1) = -3.2\pi.$$

Differentials: dy

Suppose f is differentiable at a, with tangent line $L(x) = f'(a)(x-a) + f(a)$.

We denote a small change in width (on the tangent line) by $dx = x - a$

We denote a small change in height on the tangent line by

$$dy = L(x) - L(a) = L(a+\Delta x) - L(a)$$

i.e.

$$\begin{aligned} dy &= [f'(a)(x-a) + f(a)] - [f'(a)(a-a) + f(a)] \\ &= f'(a)(x-a) \\ &= f'(a)\,dx \end{aligned}$$

Hence,

$$dy = f'(a)\,dx$$

NB

Remarks:

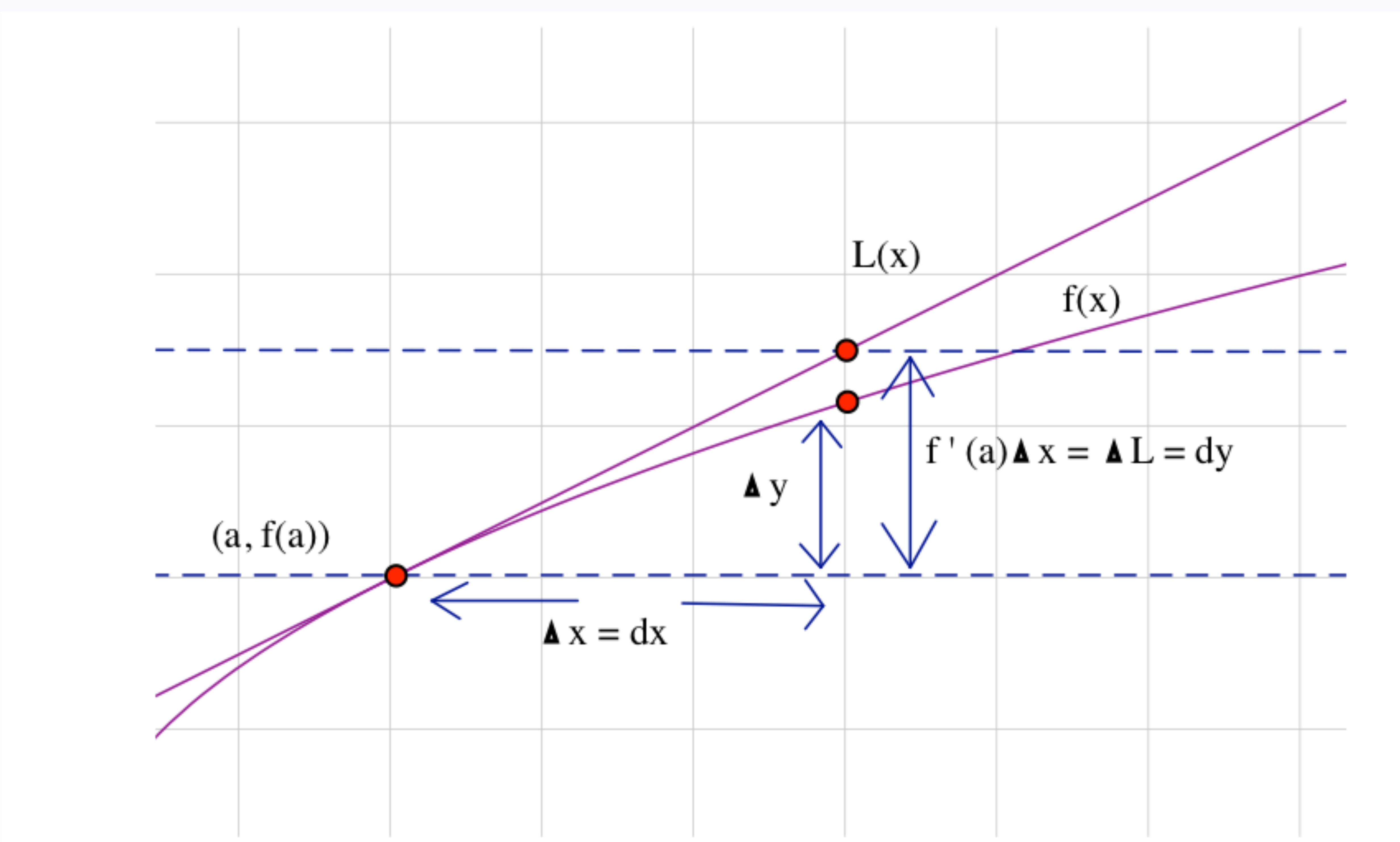
1) $dx = \Delta x$

2) $dy = f'(a)\,dx = f'(a)\,\Delta x \approx \Delta y$

$$dy \approx \Delta y$$

NB

This is equivalent to "$f(x) \approx L(x)$".



Example: Compare the values of Δy and dy as x changes from 2 to 2.04 if $y = 3x^4 + 2x + 1$.

Solution: Recall : $\Delta x = dx = x - a$

$a = 2$ ← base point

$x = 2.04$ ← near by point

So, $dx = \Delta x = 2.04 - 2 = 0.04 = \frac{4}{100}$

$$\Delta y = f(x) - f(a) = f(2.04) - f(2)$$

$$= \left(3(2.04)^4 + 2(2.04) + 1\right) - \left(3(2)^4 + 2(2) + 1\right)$$

$$= 57.0367 - 53$$

$$= 4.0367$$

$f'(x) = 12x^3 + 2 \Rightarrow f'(2) = 12(2)^3 + 2 = 98$

$dy = f'(x)\,dx = 98\left(\frac{4}{100}\right) = 3.92$

Example:

(a) Find the differential for the function

$$y = 3\cos^2(x).$$

Solution: $f'(x) = -6\cos(x)\sin(x)$

$$dy = f'(x)\,dx = -6\cos(x)\sin(x)\,dx$$

(b) Use the differential to approximate the change in values of the function $f(x) = 3\cos^2(x)$ when we have a small change in x at $x = \pi/4$.

Solution: $\Delta y \approx dy = f'(\frac{\pi}{4})\,dx$

$$f'(\pi/4) = -6\cos(\pi/4)\sin(\pi/4) = -6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -3$$

$$\Rightarrow \quad \Delta y \approx -3\,dx = -3\Delta x$$

(c) Use differentials to estimate $3\cos^2(44°) = 3\cos^2\left(\frac{\pi}{4} - \frac{\pi}{180}\right)$

Solution: $dx = \left(\pi/4 - \frac{\pi}{180}\right) - \left(\pi/4\right) = -\frac{\pi}{180}$

$$\Delta y \approx -3\left(\frac{-\pi}{180}\right) = \frac{\pi}{60}$$

$$\Delta y = 3\cos^2(44^\circ) - 3\cos^2(\pi/4)$$

Hence, $$3\cos^2(44^\circ) = 3\cos^2(\pi/4) + \Delta y$$

$$\approx \frac{3}{2} + \frac{\pi}{60}$$

Commonly Used Linear Approximations

Note that if $x \approx 0$, we get the following approximations for some commonly used functions using Linear approximation:

1. $\sin x \approx x$ if $x \approx 0$
2. $\cos x \approx 1$ if $x \approx 0$
3. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ if $x \approx 0$
4. $(1+x)^r \approx 1 + rx$ if $x \approx 0$.

Recall for $x \approx 0$, $f(x) \approx f(0) + f'(0)x$.

The above results come from the following table which you should verify:

$f(x)$	$f(0)$	$f'(x)$	$f'(0)$
$\sin x$	0	$\cos x$	1
$\cos x$	1	$-\sin x$	0
$(1+x)^r$	1	$r(1+x)^{r-1}$	r

Estimating Error

Note that Δy measures the resulting error in our value for the variable y if we make a mistake in our calculations of the variable x of size Δx, $\Delta y = f(x + \Delta x) - f(x)$. We saw above that $dy \approx \Delta y$ and we can use differentials to approximate the maximum error in our calculations for y when we have some bound on our error for the variable x.

Example The radius of a spherical hot air balloon was estimated to be 4 meters with a possible error of at most 0.5 meters. What is the maximum error you can make in calculating the surface area of the balloon using the estimate of 4 meters?

$$S = 4\pi r^2 \ m^2$$

We need bounds for ΔS here, but we will instead use the linear approximation $dS \approx \Delta S$ to approximate the error.

$$dS = 8\pi r dr \quad = \quad \frac{dS}{dr} dr$$

When $r = 4$,

$$dS = 32\pi dr$$

If $-0.5 \leq dr \leq 0.5$, then

$$-0.5(32\pi) \leq dS \leq 0.5(32\pi)$$

or

$$-50.26 \leq dS \leq 50.26.$$

We can interpret this result as saying that if our estimate of 4 meters for the radius of the balloon is off by at most 0.5 meters, then our estimate of the surface area of the balloon is off by (approximately) at most 50.26 meters squared in absolute value.

We can also find bounds for the **Relative Error** $= \frac{\Delta S}{S}$ and the **Percentage Error** $= \frac{\Delta S}{S} \cdot 100\%$.

$$\text{relative error} = \frac{\Delta S}{S} \approx \frac{dS}{S}$$

When $r = 4$, $S = 4\pi(16) = 201.06$. From our calculations above the relative error is at most $\frac{50.26}{201.06} = .25$. The percentage error is at most 25%.