Lecture 13 :Related Rates

Please review Lecture 6; Modeling with Equations in our Algebra/Precalculus review on our web page.

In this section, we look at situations where two or more variables are related and hence their rates of change with respect to a third variable (usually time) are related. In this situation we can use information about one rate of change to calculate another.

Example An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/h, how fast is the area of the patch increasing when the patch has a radius of 100 m?

Note here that the variables A = area of patch is related to the variable r = radius of the patch. Both of the variables A and r depend on a third variable, t = time. We need to find the rate of change of area with respect to time, $\frac{dA}{dt}$, for that value of t for which $r = 100$.

Guidelines for solving Related Rate Problems

- Read the problem carefully, make a sketch to organize the given information. Identify the rates that are given and the rates that are to be determined.
- Write one or more equations to express the basic relationships between the variables.
- Introduce rates of change by differentiating the appropriate equation(s) with respect to the independent variable (usually time).
- Substitute known values and solve for the desired quantity.
- Check that units are consistent and the answer is reasonable. (For example does it have the correct sign?)

Example A 12 ft. ladder is leaning against a vertical wall. Jack pulls the base of the ladder away from the wall at a rate of 0.2 ft/s. How fast is the top of the ladder sliding down the wall when the base of the ladder is 5 ft. from the wall?

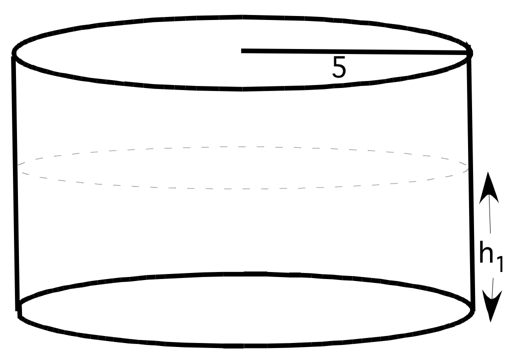
Example A person who is 5 ft. tall walks at 9 ft/s towards a street light that is 20 ft above the ground. What is the rate of change of the length of her shadow when she is 15 ft from the streetlight?

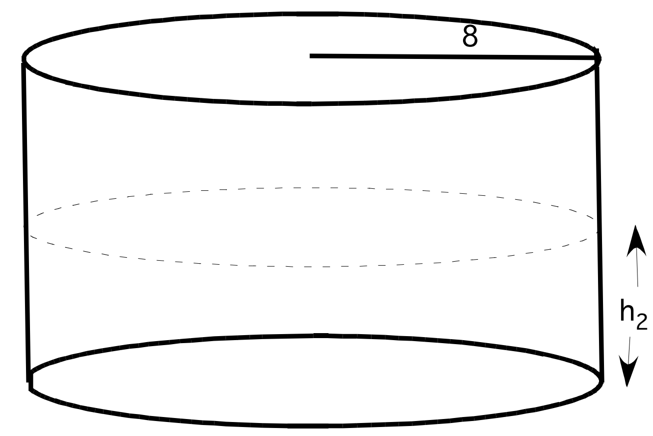
Example Two small planes approach an airport, one flying due west at 100 mi/hr and the other flying due north at 120 mi / hr. Assume that they are flying at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 mi. from the airport and the northbound plane is 200 mi. from the airport?

Example A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-co-ordinate (measured in meters) increases at a steady rate of 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m. ?

Extra problems (Please attempt before looking at the solutions)

Example Two cylindrical swimming pools are being filled simultaneously at the same rate in m^3/min. The smaller pool has a radius of 5 m. and the water level rises at a rate of 0.5 m/min. The larger pool has a radius of 8 m. How fast is the water level rising in the larger pool?





V_1 = vol. of water in small pool
h_1 = water level in small pool
$V_1 = f(h_1) =$

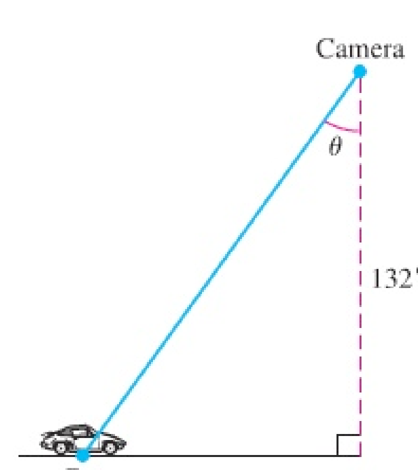
V_2 = vol. of water in large pool
h_2 = water level in large pool
$V_2 = g(h_2) =$

(a) What is the relationship between $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$?

(b) What rates are you given in the statement of the problem and what rates are you asked to find?

(c) Solve.

Example: Videotaping a moving car You are videotaping a race from a stand 132 ft. from the track, following a car that is moving at 180 mi/h (264 ft/sec) as shown below. How fast will your camera angle θ be changing when the car is right in front of you?



Solutions to extras

Example Two cylindrical swimming pools are being filled simultaneously at the same rate in m^3/min. The smaller pool has a radius of 5 m. and the water level rises at a rate of 0.5 m/min. The larger pool has a radius of 8 m. How fast is the water level rising in the larger pool?

V_1 = vol. of water in small pool
h_1 = water level in small pool
$V_1 = f(h_1) = \pi r_1^2 h_1 = \pi 25 h_1$

V_2 = vol. of water in large pool
h_2 = water level in large pool
$V_2 = g(h_2) = \pi r_2^2 h_2 = \pi 64 h_2$

(a) What is the relationship between $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$?

Since both pools are being filled at the same rate, we have

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}.$$

(b) What rates are you given in the statement of the problem and what rates are you asked to find?

Given : $\frac{dh_1}{dt} = 0.5$ m/min.
Asked to find $\frac{dh_2}{dt} = ?$

(c) Solve.

$$\frac{dV_1}{dt} = 25\pi \frac{dh_1}{dt} = 25\pi(0.5)$$

$$25\pi(0.5) = \frac{dV_1}{dt} = \frac{dV_2}{dt} = 64\pi \frac{dh_2}{dt}$$

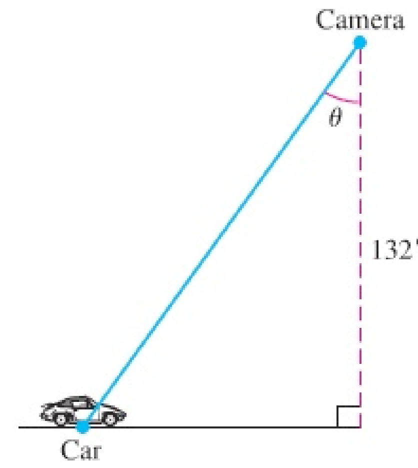
Therefore

$$25(0.5) = 64 \frac{dh_2}{dt}$$

and

$$\frac{dh_2}{dt} = \frac{25(0.5)}{64} = 0.1953\ m/min.$$

Example: Videotaping a moving car You are videotaping a race from a stand 132 ft. from the track, following a car that is moving at 180 mi/h (264 ft/sec) as shown below. How fast will your camera angle θ be changing when the car is right in front of you?



Let x be the distance from the car to the base of the stand (the base of the triangle above).
The height of the stand is given in feet, so we will use feet as our unit of measurement for distance.
The information given about the speed of the car translates to

$$\frac{dx}{dt} = -264 \quad ft/sec.$$

x and θ are related by

$$\tan\theta = \frac{x}{132}$$

Differentiating both sides with respect to t, we get

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{132}\frac{dx}{dt}.$$

Therefore

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{132}\frac{dx}{dt}.$$

When the car is directly in front of the stand, $x = 0$ and $\theta = 0$. At this point, we also have $\cos\theta = 1$.
Therefore when $x = 0$, we have

$$\frac{d\theta}{dt} = \frac{1}{132}\frac{dx}{dt} = \frac{1}{132}(-264) = -2 \text{ radians/sec..}$$

Example An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/h, how fast is the area of the patch increasing when the patch has a radius of 100 m?

Note here that the variables A = area of patch is related to the variable r = radius of the patch. Both of the variables A and r depend on a third variable, t = time. We need to find the rate of change of area with respect to time, $\frac{dA}{dt}$, for that value of t for which $r = 100$.

Solution:

Given: Circular patch

$$\dot{r}(t) = \frac{dr}{dt} = 30\,m/hr$$

Want: $\frac{dA}{dt}$ when $r = 100m$.

$$A = \pi r^2 = \pi (r(t))^2$$

Given

$$\frac{dA}{dt} = 2\pi r(t) \underbrace{\dot{r}(t)} = 2\pi r(t)(30)$$

↑ Chain rule!

r depends on t!

Hence $\left.\frac{dA}{dt}\right|_{r=100} = 2\pi(100)(30) = 6000\pi\ m^2/hr$

Method 2:

Interpret the radius increasing at a rate of 30 m/hr as

$$* \quad r(t) = 30t$$

Then $A = \pi r^2 = \pi(30t)^2 = 900\pi t^2$

$$\Rightarrow \frac{dA}{dt} = 1800\pi t$$

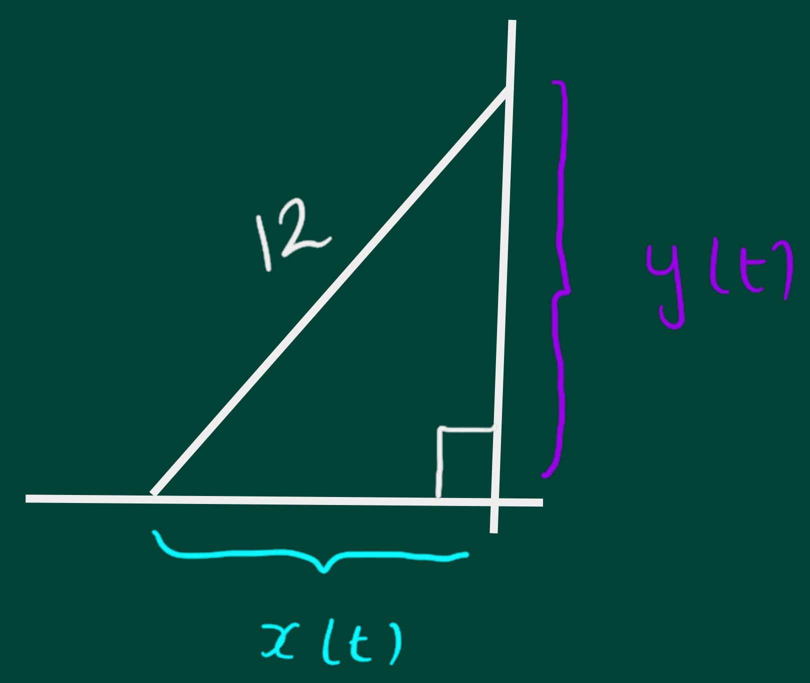
$r = 100 \Rightarrow 30t_* = 100 \Rightarrow t_* = \frac{10}{3}$

particular time

$$\left.\frac{dA}{dt}\right|_{r=100} = \left.\frac{dA}{dt}\right|_{t_* = \frac{10}{3}} = 1800\pi\left(\frac{10}{3}\right) = 6000\pi \ m^2/hr$$

Example A 12 ft. ladder is leaning against a vertical wall. Jack pulls the base of the ladder away from the wall at a rate of 0.2 ft/s. How fast is the top of the ladder sliding down the wall when the base of the ladder is 5 ft. from the wall?

Given: 1) Picture:



2) $\dot{x}(t) = 0.2 \text{ ft/s}$

Want: $\left.\dfrac{dy}{dt}\right|_{t=t_*}$ @ $x(t_*) = 5$

$// \ \dot{y}(t_*)$

particular instant

Solution: $x(t)^2 + y(t)^2 = (12)^2 = 144$

$$\xRightarrow{\frac{d}{dt}} \quad 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t) = 0$$

divide by 2 and fill in $t = t_*$

$$x(t_*)\dot{x}(t_*) + y(t_*)\dot{y}(t_*) = 0$$

We know $x(t_*) = 5$

$\dot{x}(t_*) = 0.2$

and

$$5^2 + y(t_*)^2 = 144$$

$$\Rightarrow \quad y(t_*) = \sqrt{119}$$

Filling in:

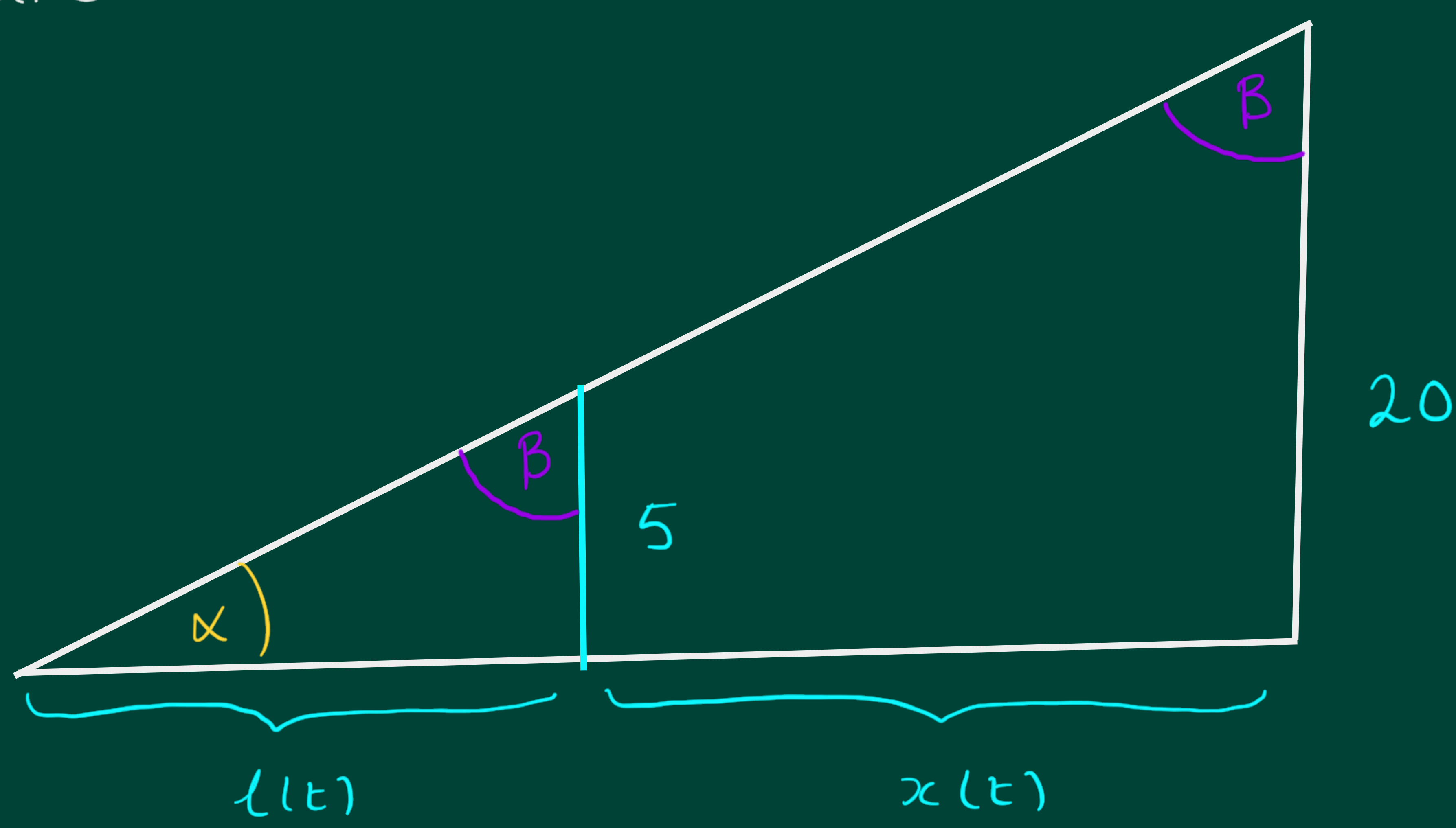
$$5(0.2) + \sqrt{119}\,\dot{y}(t_*) = 0$$

$$\Rightarrow \quad \dot{y}(t_*) = \frac{-1}{\sqrt{119}} \text{ ft/s}$$

Example A person who is 5 ft. tall walks at 9 ft/s towards a street light that is 20 ft above the ground. What is the rate of change of the length of her shadow when she is 15 ft from the streetlight?

Given:

1) Picture:



$\ell(t)$ = length of shadow @ time t

$x(t)$ = Distance person is from street light @ time t

2) $\dot{x}(t) = -9 \text{ ft/sec}$

↑ negative sign because x is decreasing!

Want: $\dot{\ell}(t_*)$ when $x(t_*) = 15$ ft.

Solution:

From the picture (similar triangles):

$$\frac{\ell(t) + x(t)}{20} = \frac{\ell(t)}{5}$$

multiply both sides by 20

$$\ell(t) + x(t) = 4\ell(t)$$

$$x(t) = 3\ell(t)$$

$\frac{d}{dt}$

$$\dot{x}(t) = 3\dot{\ell}(t)$$

$\dot{x}(t) = -9$

$$-9 = 3\dot{\ell}(t)$$

$$-3 = \dot{\ell}(t)$$

for all time!

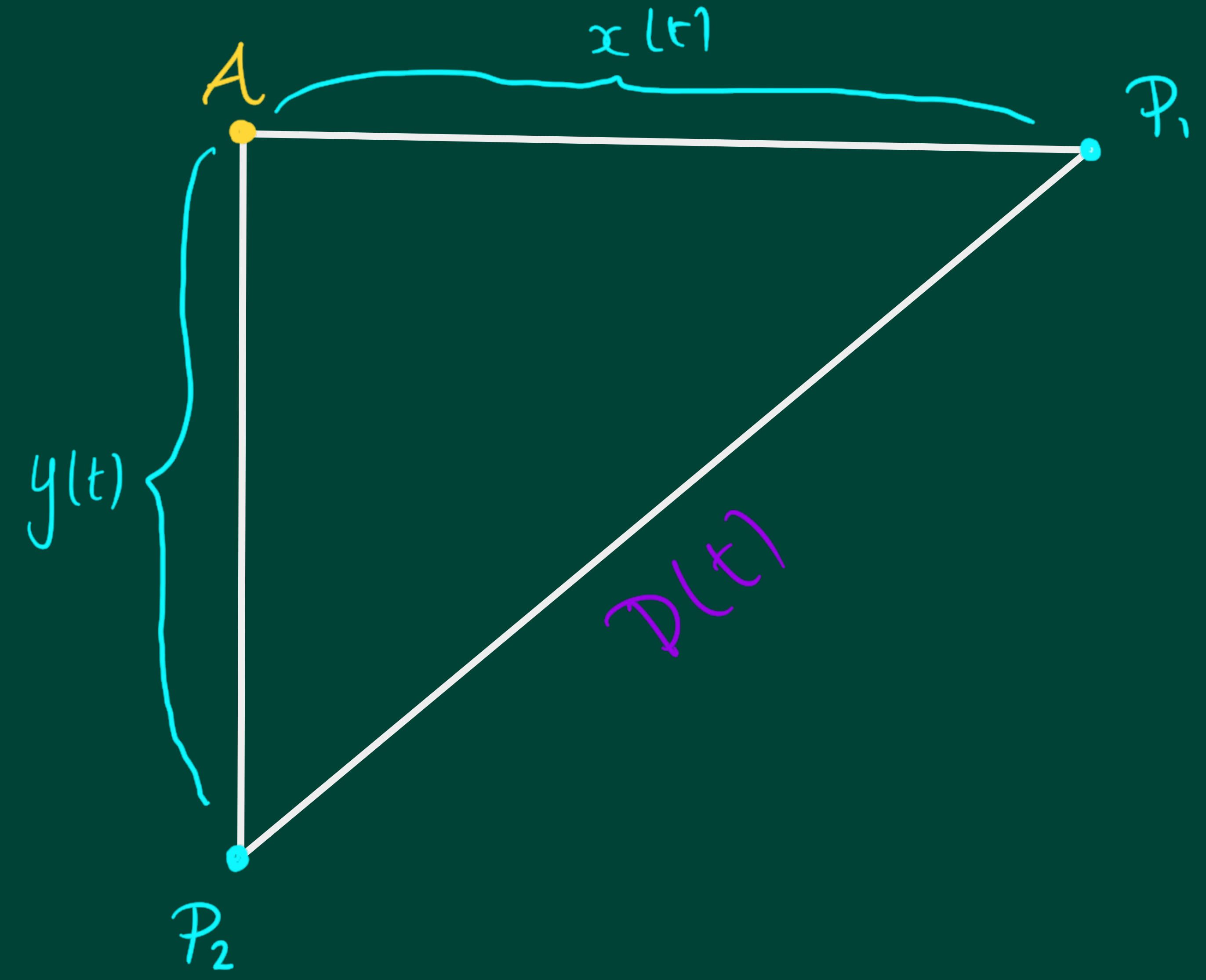
Hence $\dot{\ell}(t_*) = -3 \text{ ft/s}$

specific time

Example Two small planes approach an airport, one flying due west at 100 mi/hr and the other flying due north at 120 mi / hr. Assume that they are flying at the same constant elevation. how fast is the distance between the planes changing when the westbound plane is 180 mi. from the airport and the northbound plane is 200 mi. from the airport?

Given:

1) Picture



2) $\dot{x}(t) = -100$ mph

3) $\dot{y}(t) = -120$ mph

Want: $\dot{D}(t_*)$ @ $x(t_*) = 180$ mi.

$y(t_*) = 200$ mi.

Solution:

From the picture:

$$D(t)^2 = x(t)^2 + y(t)^2 \quad \Big\downarrow \frac{d}{dt}$$

$$2D(t)\dot{D}(t) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t)$$

Divide by 2 and focus on $t = t*$

$$D(t*)\dot{D}(t*) = x(t*)\dot{x}(t*) + y(t*)\dot{y}(t*)$$

Fill in

$$D(t*)\dot{D}(t*) = (180)(-100) + (200)(-120)$$

Find $D(t*)$:

$$D(t*)^2 = (180)^2 + (200)^2 = (20)^2[9^2 + 10^2]$$

$$= (20)^2(181)$$

$$\Rightarrow D(t*) = 20\sqrt{181}$$

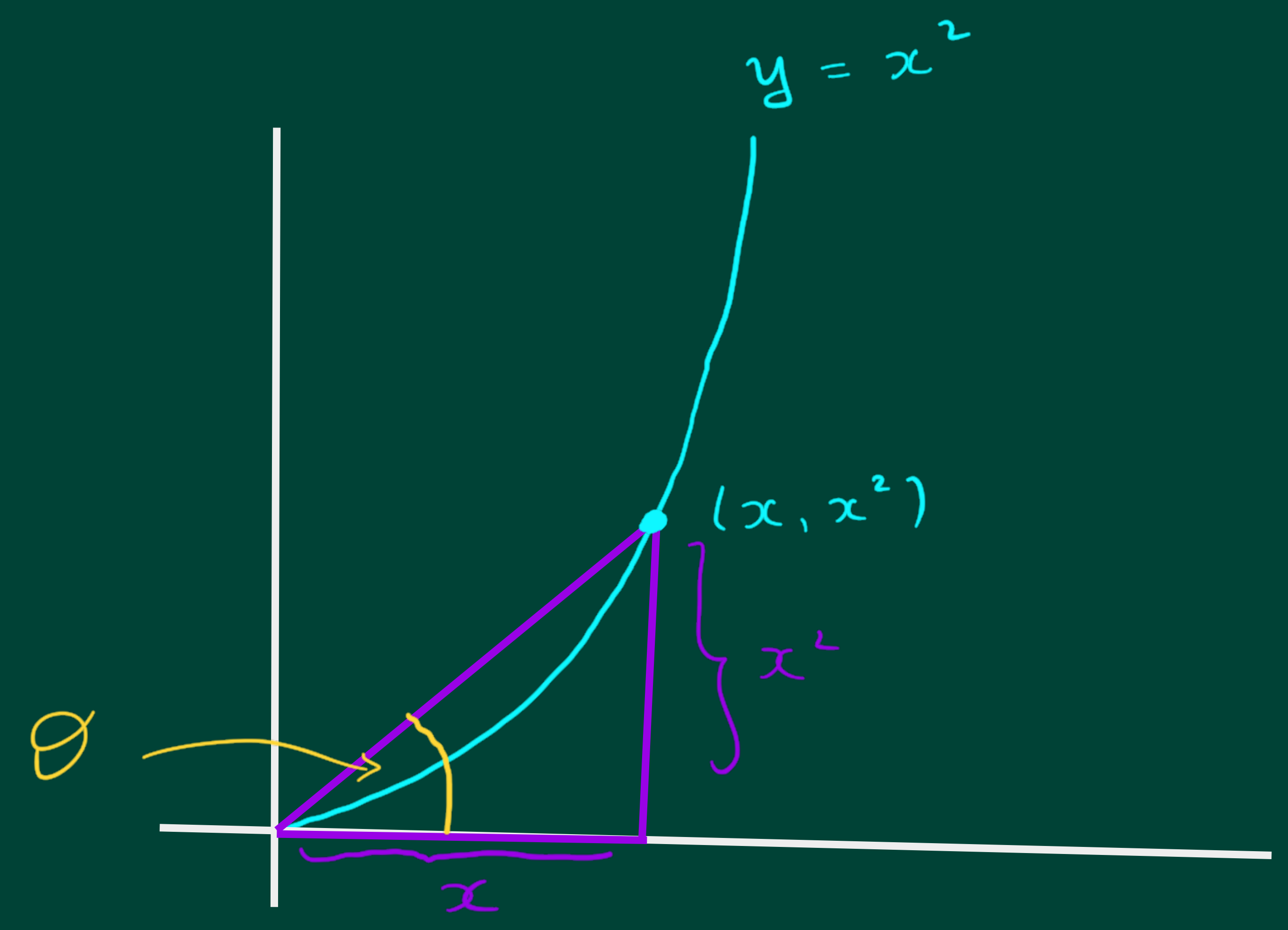
Hence: $20\sqrt{181}\,\dot{D}(t*) = -(180)(100) - (200)(120)$

$$\Rightarrow \dot{D}(t*) = \frac{-2100}{\sqrt{181}} \text{ mph}$$

Example A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-co-ordinate (measured in meters) increases at a steady rate of 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m. ?

Given :

1) Picture :



$$\tan(\theta(t)) = \frac{x^2}{x} = x(t)$$

depends on t

depends on t

2) $\dot{x}(t) = 10$ m/s

Want : $\dot{\theta}$ @ $x = 3$

Solution:

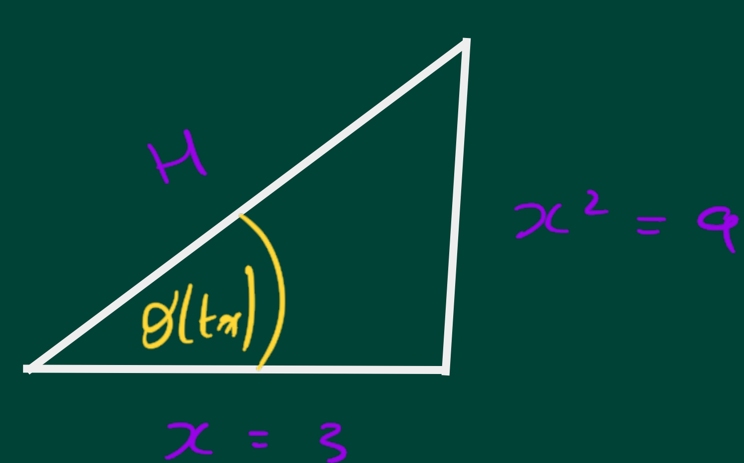
$$\tan(\theta(t)) = x(t)$$

$$\Big\downarrow \frac{d}{dt}$$

$$\sec^2(\theta(t)) \cdot \dot{\theta}(t) = \dot{x}(t)$$

$$\sec^2(\theta(t_*))\,\dot{\theta}(t_*) = 10$$

Find $\sec^2(\theta(t_*))$:



$$H = \sqrt{9^2 + 3^2}$$
$$= \sqrt{90}$$
$$= 3\sqrt{10}$$

$$\sec(\theta(t_*)) = \frac{1}{\cos(\theta(t_*))} = \frac{H}{9} = \frac{3\sqrt{10}}{3} = \sqrt{10}$$

$$\Rightarrow \sec^2(\theta(t_*)) = 10$$

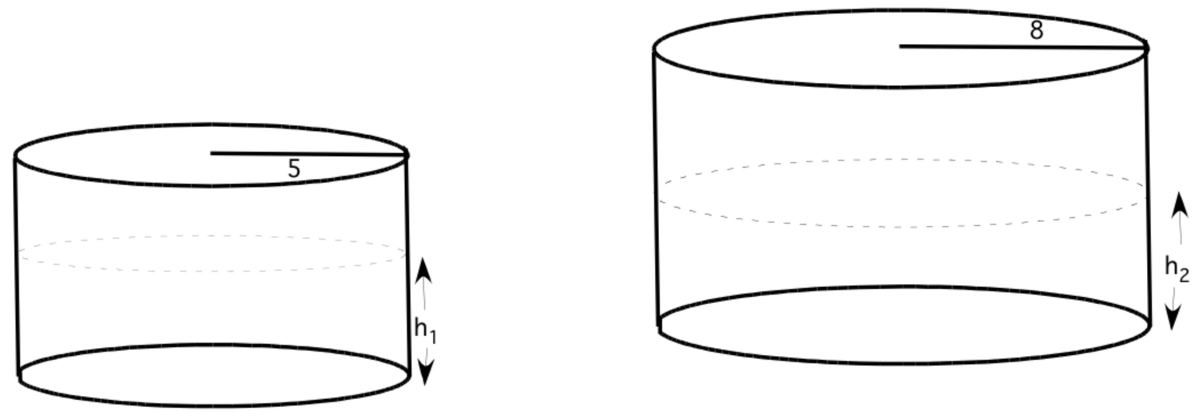
Hence,

$$10\,\dot{\theta}(t_*) = 10$$

$$\Rightarrow \dot{\theta}(t_*) = 1 \text{ radian/s}$$

Extra problems (Please attempt before looking at the solutions)

Example Two cylindrical swimming pools are being filled simultaneously at the same rate in m^3/min. The smaller pool has a radius of 5 m, and the water level rises at a rate of 0.5 m/min. The larger pool has a radius of 8 m. How fast is the water level rising in the larger pool?



V_1 = vol. of water in small pool
h_1 = water level in small pool
$V_1 = f(h_1) =$

V_2 = vol. of water in large pool
h_2 = water level in large pool
$V_2 = g(h_2) =$

(a) What is the relationship between $\frac{dV_1}{dt}$ and $\frac{dV_2}{dt}$?

(b) What rates are you given in the statement of the problem and what rates are you asked to find?

(c) Solve.

Given:

1) "Filled at same rate" $\Rightarrow \frac{dV_1}{dt} = \frac{dV_2}{dt}$

2) $r_1 = 5m$

3) $\frac{dh_1}{dt} = 0.5 \, m/min = \frac{1}{2} \, m/min$

4) $r_2 = 8m$

Want: $\dfrac{dh_2}{dt}$

Solution:

$$V_1(t) = \pi r_1^2 h_1(t) = 25\pi\, h_1(t)$$

$$V_2(t) = \pi r_2^2 h_2(t)^2 = 64\pi\, h_2(t)$$

$$\Big\downarrow \frac{d}{dt}$$

$$\dot{V}_1(t) = 25\pi\, \dot{h}_1(t) = 25\pi\left(\frac{1}{2}\right) = \frac{25}{2}\pi \quad m^3/min$$

$$\dot{V}_2(t) = 64\pi\, \dot{h}_2(t)$$

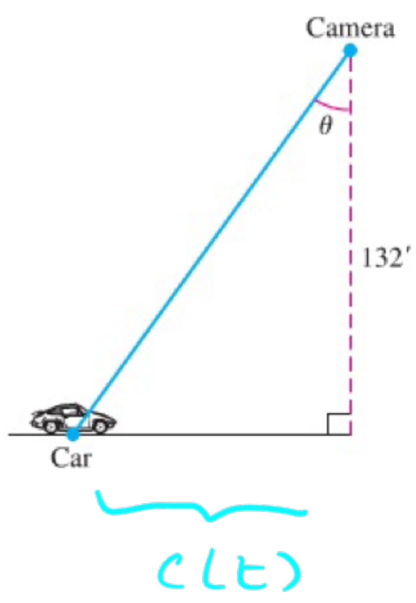
1) $\Rightarrow \quad \dot{V}_1 = \dot{V}_2$

$$\Rightarrow \frac{25\pi}{2} = 64\pi\, \dot{h}_2(t)$$

$$\Rightarrow \frac{25\pi}{2(64\pi)} = \dot{h}_2(t)$$

$$\Rightarrow \dot{h}_2(t) = \frac{25}{128} \quad m/min$$

Example: Videotaping a moving car You are videotaping a race from a stand 132 ft. from the track, following a car that is moving at 180 mi/h (264 ft/sec) as shown below. How fast will your camera angle θ be changing when the car is right in front of you?



Given:

1) Picture above.

2) $\dot{c}(t) = -264$ ft/s

Want: $\dot{\theta}$ when $c = 0$

Solution: $\tan(\theta(t)) = \dfrac{c(t)}{132}$ $\quad \dfrac{d}{dt}$

$$\Rightarrow \sec^2(\theta(t)) \cdot \dot{\theta}(t) = \frac{\dot{c}(t)}{132} = \frac{-264}{132} = -2$$

$c = 0 \Rightarrow \theta = 0$ ← Look at picture!

Hence, $\sec^2(\theta(t*)) = \sec^2(0) = 1$

So,

(1) $\dot{\theta}(t_*) = -2$

$\Rightarrow \dot{\theta}(t_*) = -2 \text{ rad}/s$