# §10. Chain Rule:

**Motivation:** When dealing with derivatives, we are dealing with the rate of change of one object with respect to another. These relationships can be quite complicated.

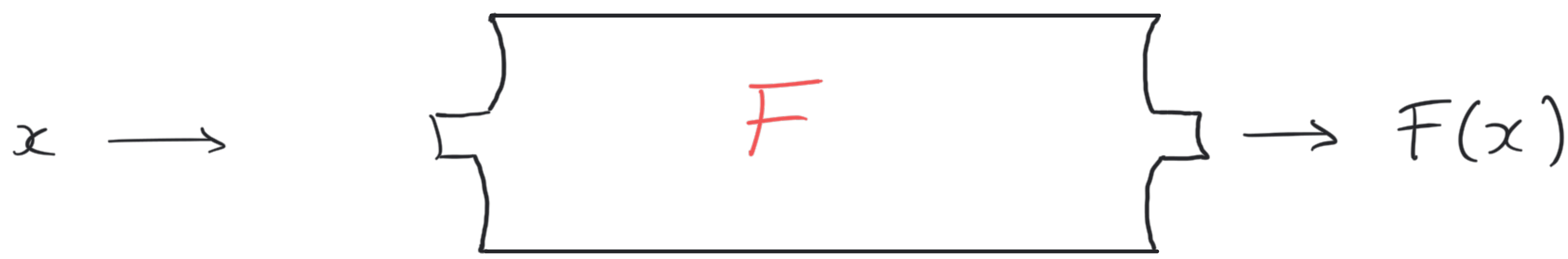
**Examples:**

1) The relationship between the popularity of cooking shows and the amount of people with obscure spices in their home that they'll only use once.

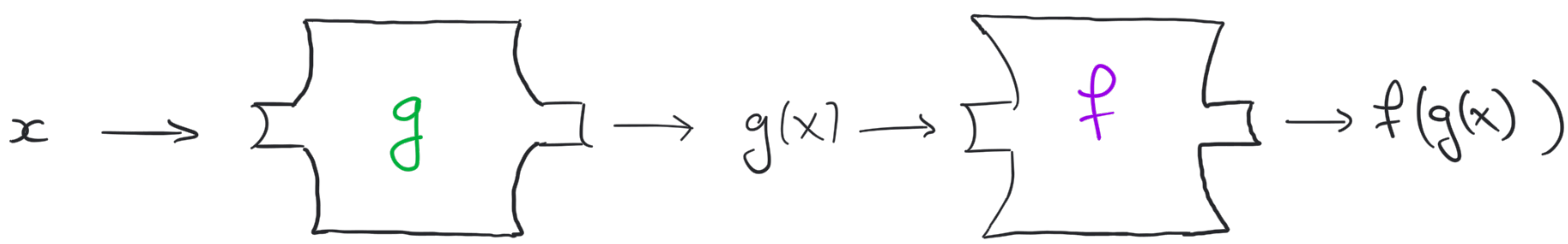
2)

**Remark:** We can make complicated relationships easier to understand by breaking them up into a chain of simpler relationships.

i.e. we break a complicated process:




into a chain of simpler processes:




such that $F(x) = f(g(x))$

i.e. $F = f \circ g$

So we need to figure out how to differentiate a composition.

**Theorem:** (Chain Rule)

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then $f \circ g$ is differentiable at $x$ and

$$\boxed{(f \circ g)'(x) = f'(g(x))\, g'(x)}$$

or, in Leibniz notation:

$$u : x \longmapsto u(x)$$
$$v : u \longmapsto v(u)$$

$$\boxed{\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}}$$

## Examples:

1) Find the derivative of $F(x) = \sin(2x+1)$.

**Step 1:** Write $F$ as a composition:

$$F = f \circ g \quad \text{where} \quad \begin{aligned} f(x) &= \sin(x) \\ g(x) &= 2x+1 \end{aligned}$$

**Step 2:** Differentiate $f$ and $g$:

$$f'(x) = \cos(x)$$

$$g'(x) = 2$$

**Step 3:** $F'(x) = f'(g(x))\, g'(x)$

$$F'(x) = \cos(g(x)) \cdot (2) = 2\cos(2x+1)$$

2) Find the derivative of $G(x) = \sqrt{(x^3+x^2+1)^3}$

# The Chain Rule and Power Rule Combined:

Say $F(x) = (g(x))^n$, for some $n \in \mathbb{R}$.

Then, we may write $F = f \circ g$, where $f(x) = x^n$.

So $f'(x) = n x^{n-1}$, by the Power Rule.

Hence, by the Chain Rule:

$$F'(x) = f'(g(x)) \cdot g'(x) = n\,(g(x))^{n-1} g'(x)$$

**Examples:** Find the derivatives of the following functions:

1) $f(x) = \sin^{100}(x) = (\sin(x))^{100}$

$f'(x) = 100\,(\sin(x))^{99} \cdot \cos(x)$

2) $h(x) = (x+1)^2 \sin(x)$ (This requires another rule)

3) $K(x) = \dfrac{(x^3+1)^{100}}{x^2+2x+5}$ (This requires another rule)

## Iterated Chain Rule:

If $h$ is differentiable at $x$, $g$ is differentiable at $h(x)$ and $f$ is differentiable at $g(h(x))$, then the function:

$$G = f \circ g \circ h$$

is differentiable at $x$ and:

$$G'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

**Example:** Find the derivative of $F(x) = \cos(\sin(x^2 + \pi))$

Find the tangent line to $F$ at $x = 0$.

## More Examples

**Example**(Old Exam Question Fall 2007) Find the derivative of

$$h(x) = x^2 \cos(\sqrt{x^3 - 1} + 2).$$

**Example** Find the derivative of

$$F(x) = \frac{1}{\sqrt{x^2 + x + 1}}$$

$$F(x) = \frac{1}{\sqrt{x^2 + x + 1}} = (x^2 + x + 1)^{-1/2}.$$

By the chain rule,

$$F'(x) = \frac{-1}{2}(x^2 + x + 1)^{-3/2}(2x + 1) = \frac{-(2x + 1)}{2(x^2 + x + 1)^{3/2}}.$$

**Example** Find the derivative of $L(x) = \sqrt{\frac{x-1}{x+2}}$.

Here we use the chain rule followed by the quotient rule. We have

$$L(x) = \sqrt{\frac{x-1}{x+2}} = \left(\frac{x-1}{x+2}\right)^{1/2}.$$

Using the chain rule, we get

$$L'(x) = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \frac{d}{dx}\left(\frac{x-1}{x+2}\right).$$

Using the quotient rule for the derivative on the right, we get

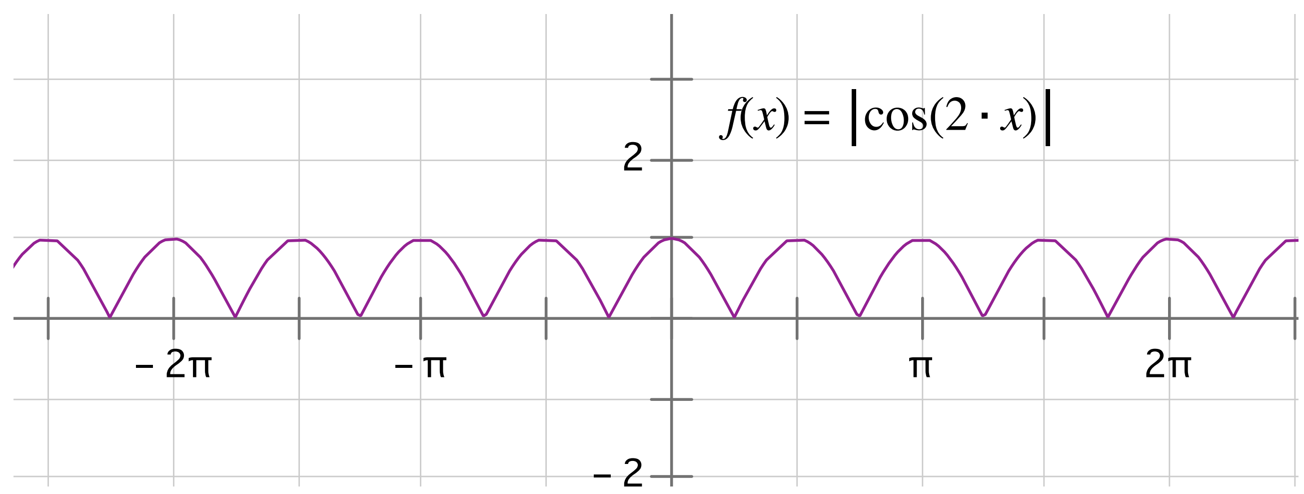
$$L'(x) = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \left[\frac{(x+2) - (x-1)}{(x+2)^2}\right] = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \left[\frac{3}{(x+2)^2}\right].$$

**Example**

wherever it exists

$$f(x) = \sqrt{\cos^2(2x)}$$

Find $f'(x)$. (Note that this is an interesting function, in fact $f(x) = |\cos(2x)|$ which you can graph by sketching the graph of $\cos(2x)$ and then flipping the negative parts over the x-axis. Note that the graph has many sharp points.




Using the chain rule with the chain
$y = f(x) = \sqrt{\cos^2(2x)} = \sqrt{u}$, $\quad u = \cos^2(2x) = (v)^2$, $\quad v = \cos(2x) = \cos(w)$, $\quad w = 2x$, we get

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx} = \frac{1}{2}u^{-1/2} \cdot 2v \cdot [-\sin(w)] \cdot 2 =$$

$$\frac{1}{2\sqrt{\cos^2(2x)}} \cdot 2\cos(2x) \cdot [-\sin(2x)] \cdot 2 = \frac{-2\cos(2x)\sin(2x)}{\sqrt{\cos^2(2x)}}.$$